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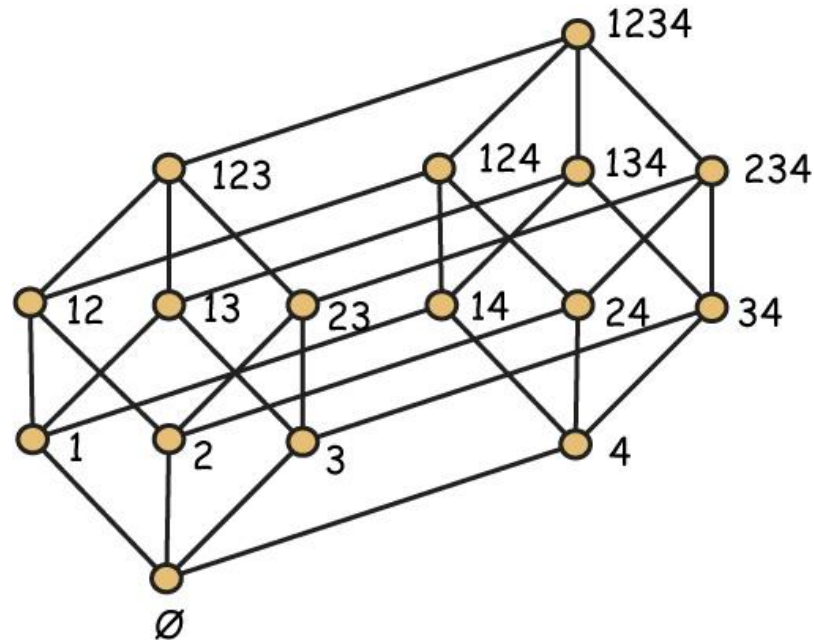


# 14 - Subset Lattices

William T. Trotter  
trotter@math.gatech.edu

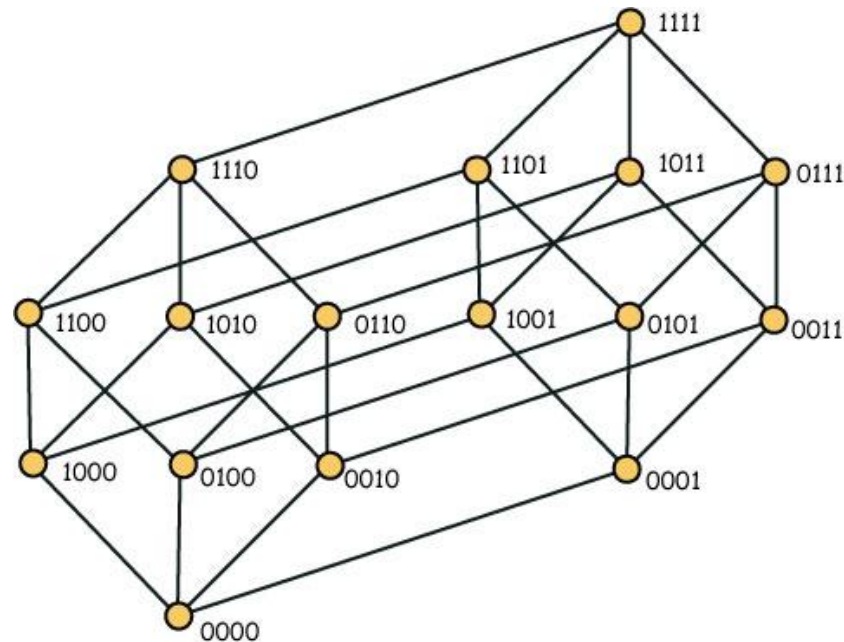
# Subset Lattices

**Definition** For an integer  $n \geq 1$ , the poset consisting of all subsets of  $\{1, 2, \dots, n\}$  ordered by inclusion is called the **subset lattice**. We will denote it as  $2^n$ . Here is a diagram for  $2^4$ .



# Subset Lattices - Cubes

**Remark** Using the alternate notation for subsets as bit strings, subset lattices are also called cubes. Here is the 4-cube.



# Basic Properties of Subset Lattices

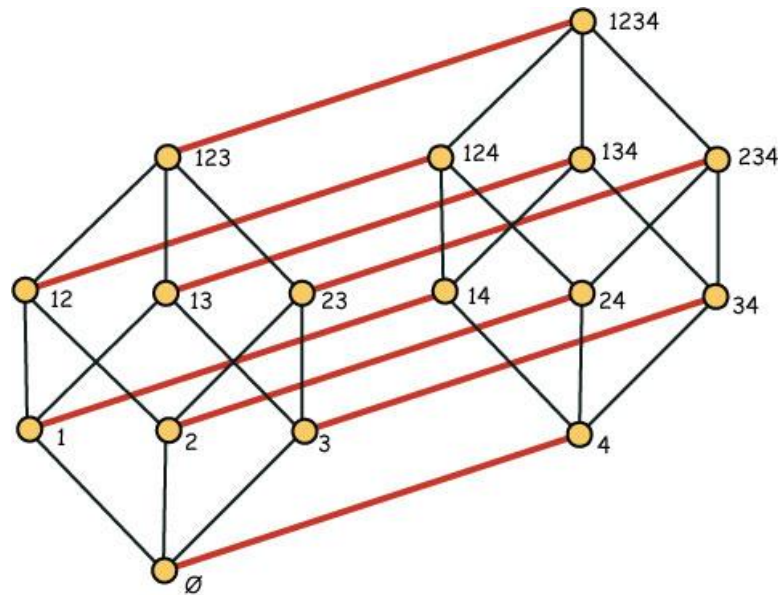
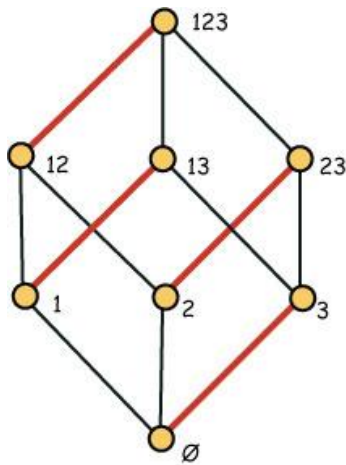
**Fact** The size of  $2^n$  is  $2^n$ .

**Fact** The unique maximal element in  $2^n$  is the set  $\{1, 2, \dots, n\}$  and the unique minimal element is the empty set  $\emptyset$ .

**Fact** The height of  $2^n$  is  $n + 1$ . In fact, all maximal chains are maximum.

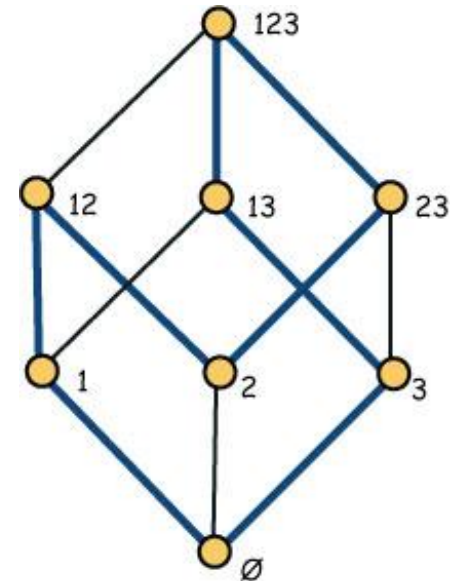
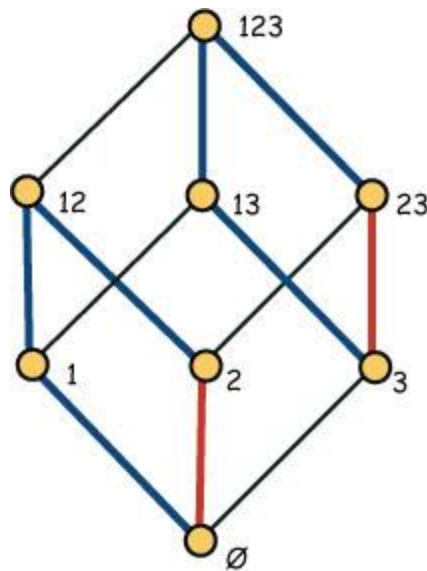
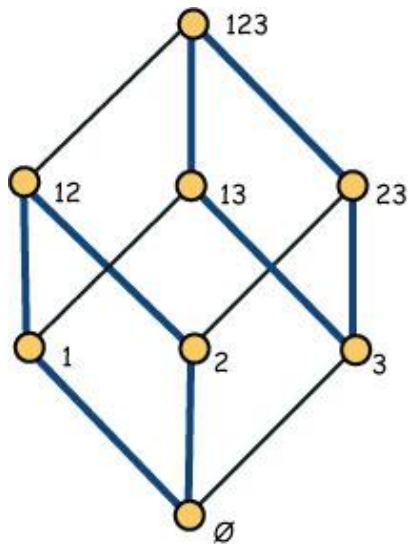
# Inductive Nature of Subset Lattices

**Basic Fact** The subset lattice  $2^{n+1}$  can be viewed as  $2 \times 2^n$ .



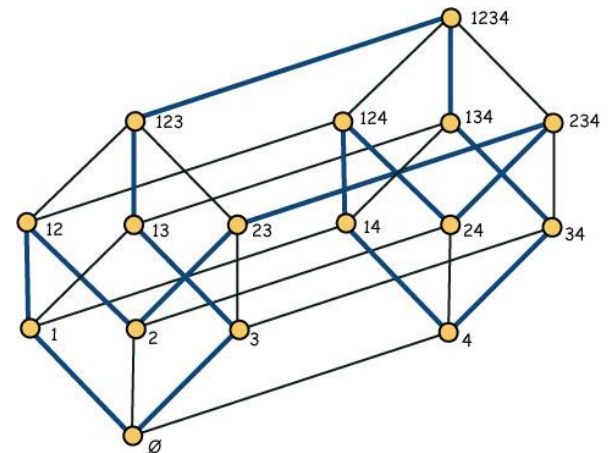
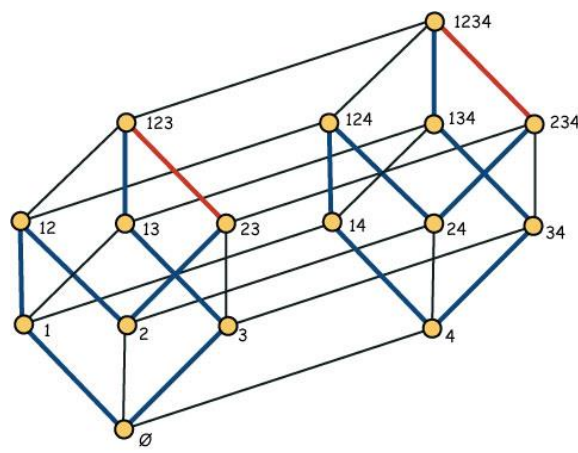
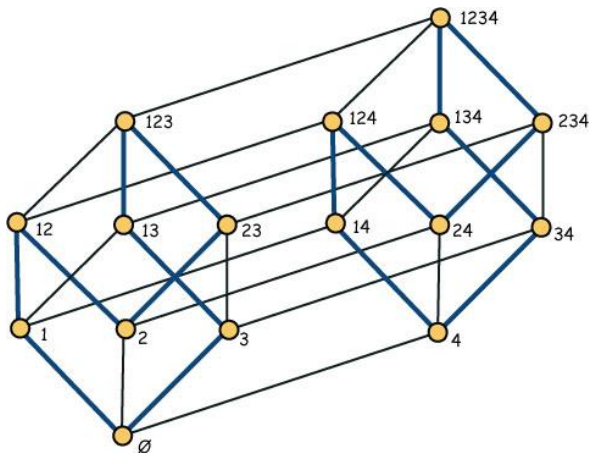
# Hamiltonian Property for Subset Lattices

**Theorem** For  $n \geq 2$ , the  $n$ -cube subset  $2^n$  is Hamiltonian.



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# The Width of Subset Lattices

**Fact** If  $A$  is a set with  $|A| = k$ , then the number of maximal chains in  $2^n$  containing  $A$  is  $k!(n-k)!$

**Fact** The width of  $2^n$  is at least as large as any binomial coefficient  $C(n, k)$ , where  $0 \leq k \leq n$ .

**Fact** The largest binomial coefficient of the form  $C(n, k)$  is when  $k = \lfloor n/2 \rfloor$ . When  $n$  is even, there is just one value of  $k$  for which  $C(n, k)$  is maximum. When  $n$  is odd, there are two. For example, the width of  $2^{13} \geq C(13, 6) = C(13, 7)$  while the width of  $2^{14} \geq C(14, 7)$ .



# The Width of Subset Lattices (2)

**Theorem Fact** (Sperner, '28) The width of the subset lattice  $2^n$  is the binomial coefficient  $C(n, \lfloor n/2 \rfloor)$ .

**Note** We will give two proofs of this result in class. The first proof is the more classical of the two and rests on the following elementary fact.

**Fact** If  $A$  is a subset of  $\{1, 2, \dots, n\}$  and  $|A| = k$ , then the number of maximal chains containing  $A$  is  $k!(n - k)!$ . To see this, consider bit strings. There are  $k!$  ways to add the bits in  $A$  and then another  $(n - k)!$  ways to add the bits in the complement of  $A$ .

# The Width of Subset Lattices (3)

**Proof of Spener's theorem** Let  $\{A_1, A_2, \dots, A_t\}$  be a maximum antichain in  $2^n$ . For each  $i$ , let  $k_i = |A_i|$ . Then

$$\sum_{1 \leq i \leq t} k_i! (n - k_i)! \leq n!$$

$$\sum_{1 \leq i \leq t} [k_i! (n - k_i)! ]/n! \leq 1.$$

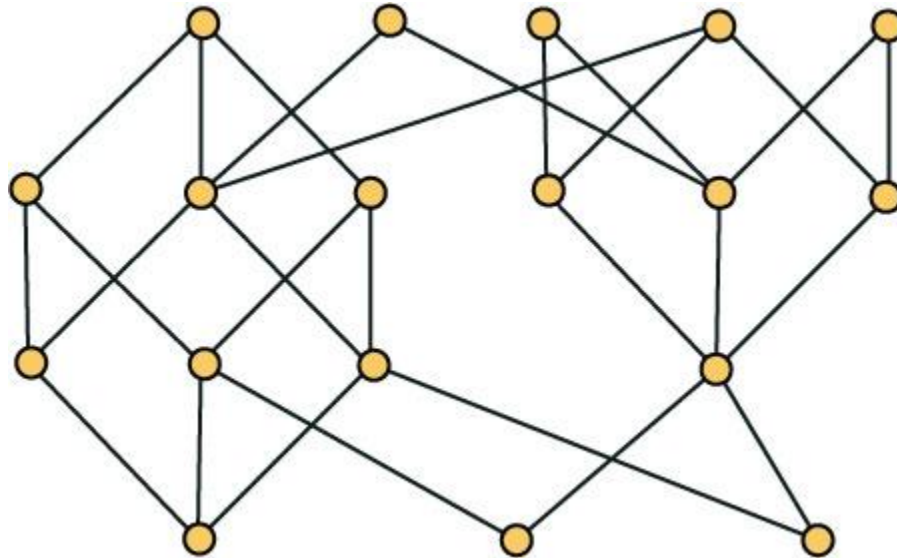
$$\sum_{1 \leq i \leq t} 1/C(n, k_i) \leq 1$$

$$t /C(n, \lfloor n/2 \rfloor) \leq 1$$

$$t \leq C(n, \lfloor n/2 \rfloor)$$

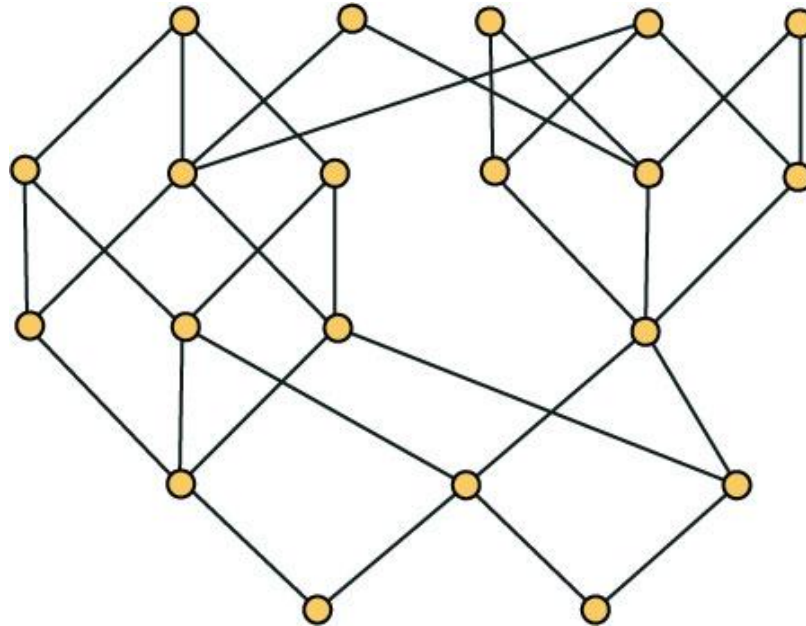
# Ranked Posets - Start of Second Proof

**Definition** A poset is **ranked** if all maximal chains are maximum. Here is a ranked poset of height 4.



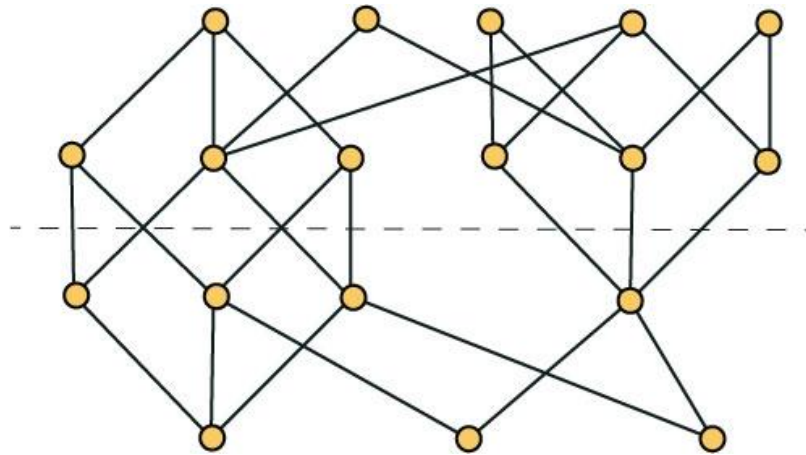
# Ranked Posets (2)

**Definition** Here is a ranked poset of height 5.



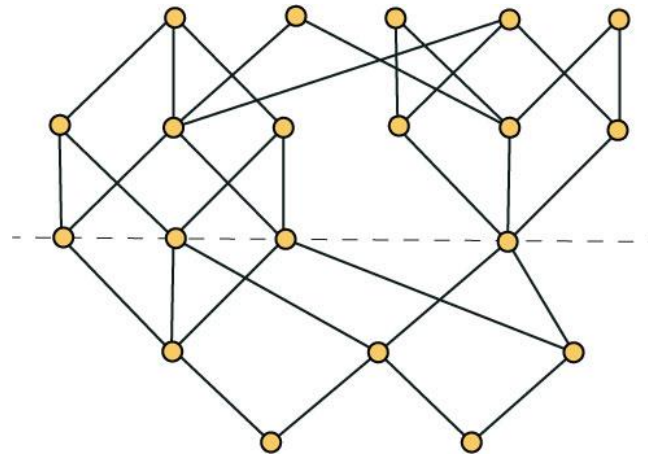
# Middle Level of Ranked Poset

**Observation** Here is the middle level for a ranked poset of height 4.



# Middle Level of Ranked Poset (2)

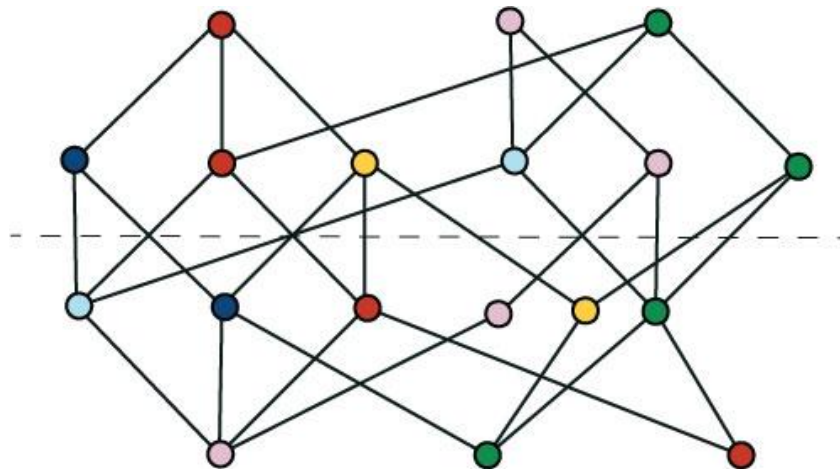
**Observation** Here is the middle level for a ranked poset of height 5.



# Symmetric Chain Partition

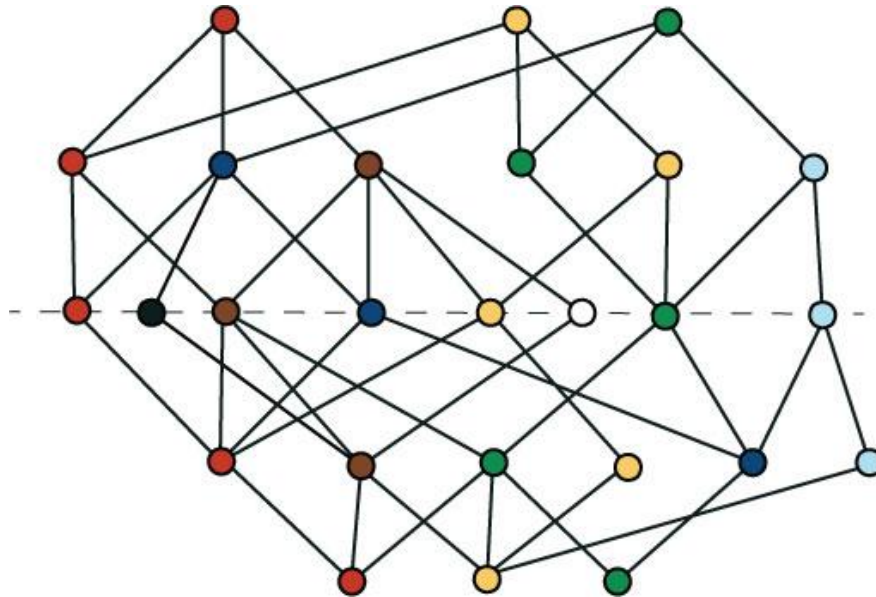
**Definition** A chain in a ranked poset is **symmetric** when it (1) goes the same distance above and below the middle levels and (2) doesn't skip levels.

**Observation** Here is a symmetric chain partition for a ranked poset of height 4.



# Symmetric Chain Partition (2)

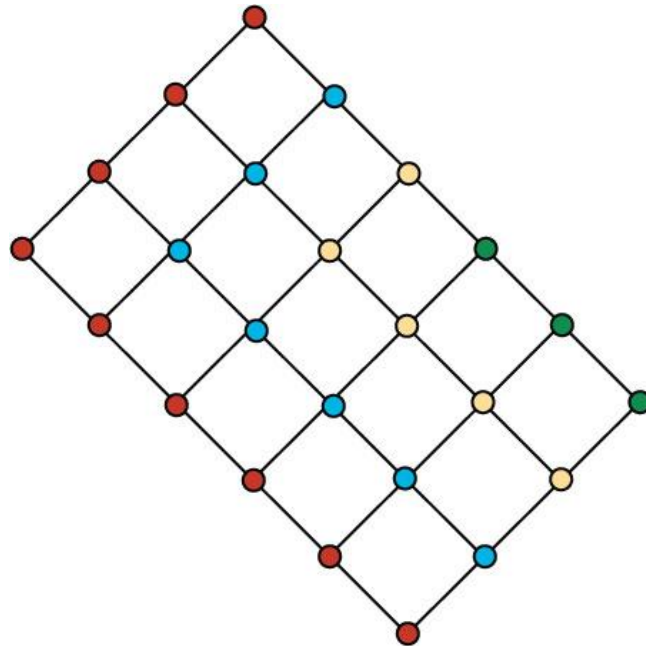
**Observation** Here is a symmetric chain partition for a ranked poset of height 5.





# Symmetric Chain Partition (3)

**Lemma** The Cartesian product of two chains has a symmetric chain partition.



# Symmetric Chain Partition (4)

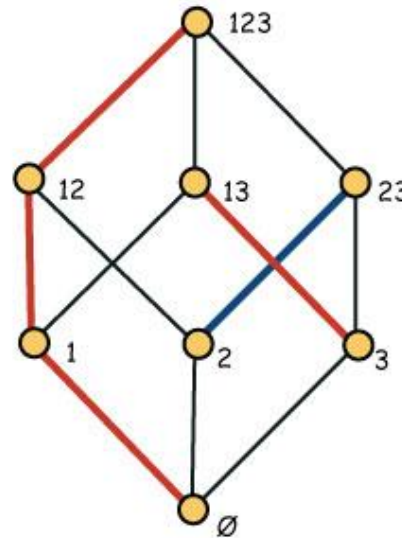
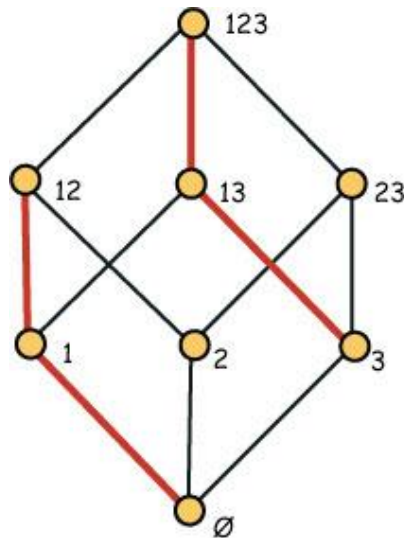
**Theorem** For every  $n \geq 1$ , the subset lattice  $2^n$  has a symmetric chain partition.

**Proof** On the next two slides, we illustrate the inductive construction. In each case, we start with a symmetric chain partition of  $2^n$  and show how to modify two copies to obtain a symmetric chain partition of  $2^{n+1}$ .

Note that when  $n$  is even, we have some 1-element chains in the partition. Each pair becomes a 2-element chain in the next step. But when  $n$  is odd, each pair of chains produces another pair.

# Symmetric Chain Partition (5)

**Example** Using two copies of a symmetric chain partition of  $2^2$  to form a symmetric chain partition of  $2^3$ . Note that the 1-element chains 2 and 23 become a 2-element chain.



# Symmetric Chain Partition (6)

**Example** Using two copies of a symmetric chain partition of  $2^3$  to form a symmetric chain partition of  $2^4$ .

