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15 - Inclusion/Exclusion

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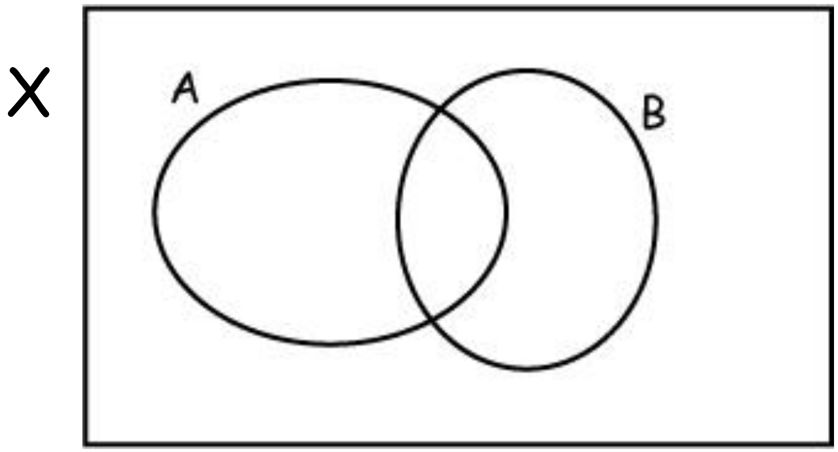
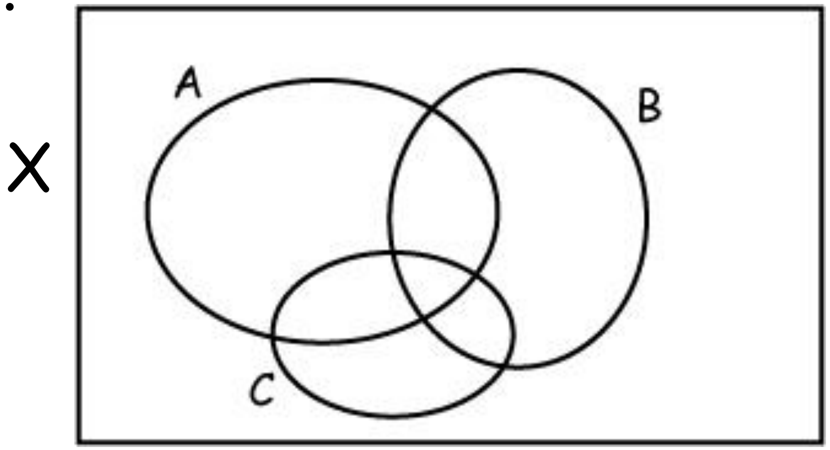
Inclusion/Exclusion - Prelude

Question In the "Venn Diagram" shown below, the universe X contains 23 elements. There are 8 in the set A and 11 in B. If there are 5 in $A \cap B$, then how many elements of X belong to neither A nor B?



Inclusion/Exclusion – Prelude (2)

Question In the "Venn Diagram" shown below, the universe X contains 2307 elements. We want to determine the number of elements of X that don't belong to any of A, B and C. If we know the number of elements in the following sets, can we do this? A, B, C, $A \cap B$, $B \cap C$, $A \cap C$, $A \cap B \cap C$.



Inclusion/Exclusion (1)

Notation Let X be a set of objects and suppose that for every element i in $\{1, 2, \ldots, n\}$, we have a property P_i so that for all x in X, the statement "x satisfies property P_i" is either true or false … but never ambiguous. Then for a subset S of $\{1, 2, \ldots, n\}$, let $N(S)$ be the subset of X consisting of all x in X which satisfy property P_i for all i in S. Note that $N(\varnothing) = X$.

Notation Let N_0 be the subset of X consisting of those objects which satisfy none of the properties.

Inclusion/Exclusion (2)

Theorem Let X be a set of objects and let P_i be a property for X for each $i = 1, 2, \ldots, n$. Then:

$$N_0 = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S)$$

Example When $n = 2$,

$$N_0 = N(\emptyset) - N(1) - N(2) + N(1).$$

Inclusion/Exclusion (3)

Theorem Let X be a set of objects and let P_i be a property for X for each $i = 1, 2, \ldots, n$. Then:

$$N_0 = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S)$$

Example When $n = 3$,

$$N_0 = N(\emptyset)$$
$$- N(1) - N(2) - N(3)$$
$$+ N(12) + N(13) + N(23)$$
$$- N(123).$$

Inclusion/Exclusion (4)

Example When $n = 4$,

$$
\begin{aligned}
N_0 = N(\varnothing) & \\
& - N(1) - N(2) - N(3) - N(4) \\
& + N(12) + N(13) + N(14) + N(23) + N(24) + N(34) \\
& - N(123) - N(124) - N(134) - N(234) \\
& + N(1234).
\end{aligned}
$$

Inclusion/Exclusion (5)

Observation In general, there are 2^n terms in the inclusion/exclusion formula. How can this possibly be of use?

Conclusion Inclusion/Exclusion may be of value when $|N(S)|$ depends only on $|S|$. Also, it may be of value when there is some other form of "collapsing" among the exponentially many terms in the formula.

Derangements

Definition A permutation σ of $\{1, 2, \ldots, n\}$ is called a derangement if $\sigma(i) \neq i$ for all $i = 1, 2, \ldots, n$.

Example 38754126 and 21436587 are derangements but 57314682 and 75318642 are not.

Exercise Write all derangements of $\{1,2,3,4,5\}$.

Notation Let d_n denote the number of derangements of $\{1, 2, \ldots, n\}$.

Derangements (2)

Inclusion/Exclusion Formula for Derangements

$$d_n = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S)$$

$$= \sum_{0 \leq k \leq n} (-1)^k C(n,k)\,(n - k)!$$

Explanation When S is a subset of $\{1, 2, \ldots, n\}$ and $|S| = k$, $|N(S)| = (n - k)!$ To see this, note that if σ satisfies P_i and i belongs to S, then $\sigma(i) = i$. So the positions corresponding to elements of S are determined, and the other $n - k$ positions are an arbitrary permutation of the remaining elements.

Surjections (1)

Notation For an integer n, let $[n]$ denote $\{1, 2, \ldots, n\}$. Also, let $S(n, m)$ denote the number of surjections from $[n]$ to $[m]$.

Exercise Determine $S(5, 3)$ by hand.

Surjections (2)

Inclusion/Exclusion Formula for Surjections

$$d_n = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S)$$

$$= \sum_{0 \leq k \leq m} (-1)^k C(m, k) (m - k)^n$$

Explanation When S is a subset of $\{1, 2, \ldots, m\}$ and $|S| = k$, $|N(S)| = (m - k)^n$. To see this, note that if f satisfies P_i and i belongs to S, then i is not in the range of f. In other words, f is an function whose domain is $[n]$ and whose range is a set of size $m - k$.

The Euler φ- function

Notation For an integer $n \geq 2$, let $\varphi(n)$ denote the number of elements in $[n]$ which are relatively prime to n.

Example $\varphi(12) = 4$ since 1, 5, 7 and 11 are relatively prime to 12.

Exercise Compute $\varphi(144)$.

Exercise Compute $\varphi(324481700624)$.

The Euler φ- function

Inclusion/Exclusion Formula for Euler φ-Function

Suppose the prime factors of n are: $p_1, p_2, \ldots, p_k$.

Then

$$\varphi(n) = n\,(1 - 1/p_1)(1 - 1/p_2)\ldots(1-1/p_k)$$

Explanation When m has the common prime factors p_3, p_7 and p_8 with n, then the number of such m is $n/p_3p_7p_8$.

The Euler φ- function

Example Compute φ(324481700624)

Maple reports that

$324481700624 = 2^4(109)(727)(255923)$

Therefore

$$\varphi(324481700624) = 324481700624(1-1/2)(1-1/109)(1-1/727)(1-1/255923)$$

$$= 2^3(108)(726)(255922)$$

$$= 160530657408.$$