November 14, 2017

15 - Inclusion/Exclusion

William T. Trotter
trotter@math.gatech.edu
Question: In the “Venn Diagram” shown below, the universe $X$ contains 23 elements. There are 8 in the set $A$ and 11 in $B$. If there are 5 in $A \cap B$, then how many elements of $X$ belong to neither $A$ nor $B$?
Question  In the “Venn Diagram” shown below, the universe $X$ contains 2307 elements. We want to determine the number of elements of $X$ that don’t belong to any of $A$, $B$ and $C$. If we know the number of elements in the following sets, can we do this? $A$, $B$, $C$, $A \cap B$, $B \cap C$, $A \cap C$, $A \cap B \cap C$. 

![Venn Diagram]
**Notation** Let $X$ be a set of objects and suppose that for every element $i$ in $\{1, 2, \ldots, n\}$, we have a property $P_i$ so that for all $x$ in $X$, the statement “$x$ satisfies property $P_i$” is either true or false ... but never ambiguous. Then for a subset $S$ of $\{1, 2, \ldots, n\}$, let $N(S)$ be the subset of $X$ consisting of all $x$ in $X$ which satisfy property $P_i$ for all $i$ in $S$. Note that $N(\emptyset) = X$.

**Notation** Let $N_0$ be the subset of $X$ consisting of those objects which satisfy none of the properties.
Theorem  Let $X$ be a set of objects and let $P_i$ be a property for $X$ for each $i = 1, 2, \ldots, n$. Then:

$$N_0 = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S)$$

Example  When $n = 2$,

$$N_0 = N(\emptyset) - N(1) - N(2) + N(1).$$
Theorem  Let $X$ be a set of objects and let $P_i$ be a property for $X$ for each $i = 1, 2, \ldots, n$. Then:

$$N_0 = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S)$$

Example  When $n = 3$,

$$N_0 = N(\emptyset) - N(1) - N(2) - N(3) + N(12) + N(13) + N(23) - N(123).$$
Example  When  \( n = 4 \),

\[
N_0 = N(\emptyset)
- N(1) - N(2) - N(3) - N(4)
- N(123) - N(124) - N(134) - N(234)
+ N(1234).
\]
Observation  In general, there are $2^n$ terms in the inclusion/exclusion formula. How can this possibly be of use?

Conclusion  Inclusion/Exclusion may be of value when $|N(S)|$ depends only on $|S|$. Also, it may be of value when there is some other form of “collapsing” among the exponentially many terms in the formula.
**Derangements**

**Definition**  A permutation $\sigma$ of $\{1, 2, \ldots, n\}$ is called a derangement if $\sigma(i) \neq i$ for all $i = 1, 2, \ldots, n$.

**Example**  38754126 and 21436587 are derangements but 57314682 and 75318642 are not.

**Exercise**  Write all derangements of $\{1, 2, 3, 4, 5\}$.

**Notation**  Let $d_n$ denote the number of derangements of $\{1, 2, \ldots, n\}$. 
Derangements (2)

Inclusion/Exclusion Formula for Derangements

\[ d_n = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S) \]

\[ = \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} (n - k)! \]

**Explanation**  When \( S \) is a subset of \( \{1, 2, \ldots, n\} \) and \(|S| = k\), \(|N(S)| = (n - k)!\). To see this, note that if \( \sigma \) satisfies \( P_i \) and \( i \) belongs to \( S \), then \( \sigma(i) = i \). So the positions corresponding to elements of \( S \) are determined, and the other \( n - k \) positions are an arbitrary permutation of the remaining elements.
Notation  For an integer  \( n \), let \([n]\) denote \(\{1, 2, \ldots, n\}\). Also, let \(S(n, m)\) denote the number of surjections from \([n]\) to \([m]\).

Exercise  Determine \(S(5, 3)\) by hand.
Surjections (2)

Inclusion/Exclusion Formula for Surjections

\[ S(n, m) = \sum_{S \subseteq \{1, 2, \ldots, n\}} (-1)^{|S|} N(S) \]

\[ = \sum_{0 \leq k \leq m} (-1)^k C(m, k) (m - k)^n \]

Explanation  When \( S \) is a subset of \( \{1, 2, \ldots, m\} \) and \(|S| = k\), \(|N(S)| = (m - k)^n\). To see this, note that if \( f \) satisfies \( P_i \) and \( i \) belongs to \( S \), then \( i \) is not in the range of \( f \). In other words, \( f \) is an function whose domain is \( [n] \) and whose range is a set of size \( m - k \).
The Euler φ-function

Notation For an integer $n \geq 2$, let $\varphi(n)$ denote the number of elements in $[n]$ which are relatively prime to $n$.

Example $\varphi(12) = 4$ since $1, 5, 7$ and $11$ are relatively prime to $12$.

Exercise Compute $\varphi(144)$.

Exercise Compute $\varphi(324481700624)$. 
The Euler $\varphi$-function

Inclusion/Exclusion Formula for Euler $\varphi$-Function

Suppose the prime factors of $n$ are: $p_1, p_2, \ldots, p_k$.

Then

$$\varphi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right)$$

Explanation
When $m$ has the common prime factors $p_3, p_7$ and $p_8$ with $n$, then the number of such $m$ is $n/p_3p_7p_8$. 
The Euler $\varphi$-function

**Example** Compute $\varphi(324481700624)$

Maple reports that

$$324481700624 = 2^4(109)(727)(255923)$$

Therefore

$$\varphi(324481700624) = 324481700624(1-1/2)(1-1/109)$$

$$\quad (1 - 1/727)(1 - 1/255923)$$

$$= 2^3(108)(726)(255922)$$

$$= 160530657408.$$