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Math 3012 – Applied Combinatorics Lecture 11

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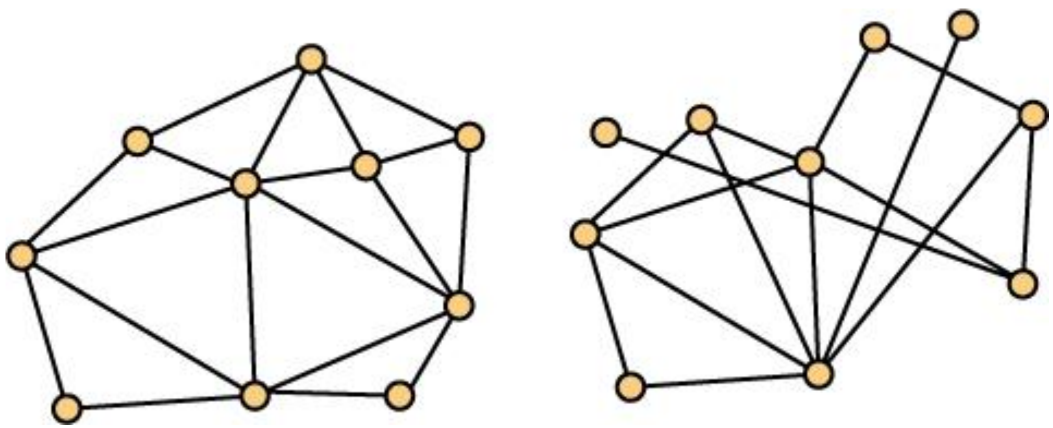
Planar Graphs

Definition A graph G is planar if it can be drawn in the plane with no edge crossings.

Exercise The two graphs shown below are both planar. Explain why.



Problems for Planar Graphs

Question Do planar graphs have any interesting properties? What can be said about maximum clique size and chromatic number? Do plane drawings exhibit any interesting properties that we should note?

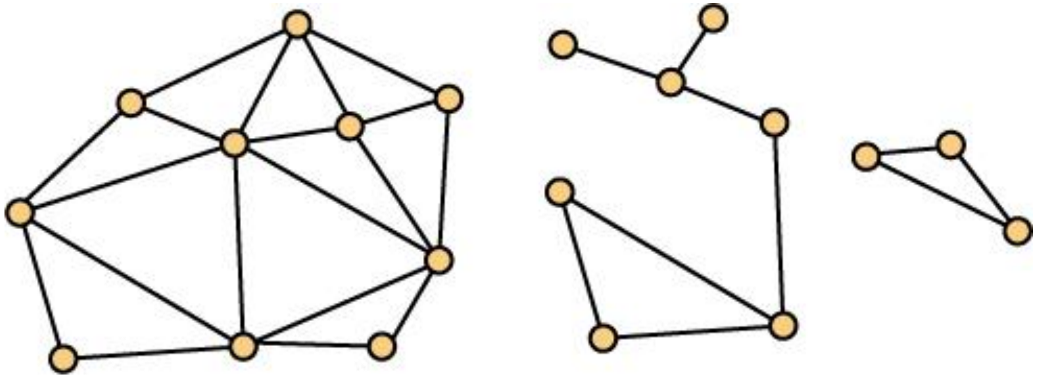
Question Given the data for a graph G, can you efficiently test whether G is planar? When the answer is yes, can you provide a "nice" plane drawing (a drawing with no crossings, straight line segments for edges and vertices positioned at points from a small grid). If no, can you provide a certificate to justify this negative response.

Euler's Formula

Theorem If n, q and f denote respectively, the number of vertices, edges and faces in a plane drawing of a planar graph with t components, then

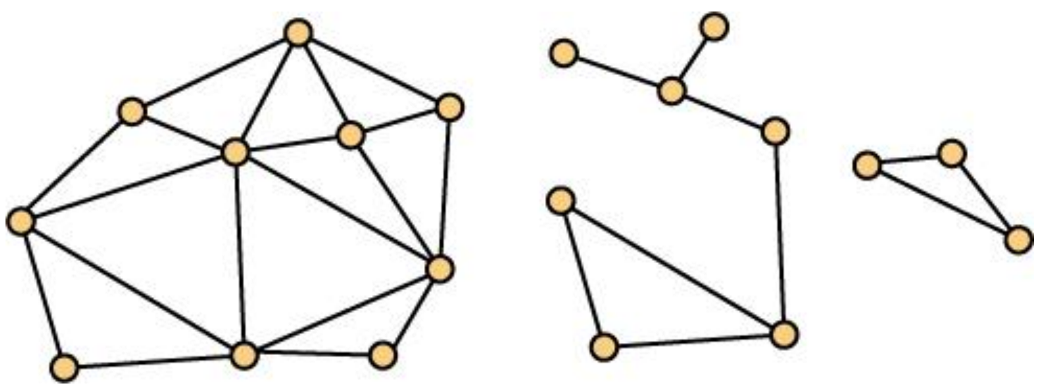
$$n - q + f = 1 + t$$

Exercise For the plane drawing shown below, $t = 3$. Determine n, q and f and show that the formula holds.



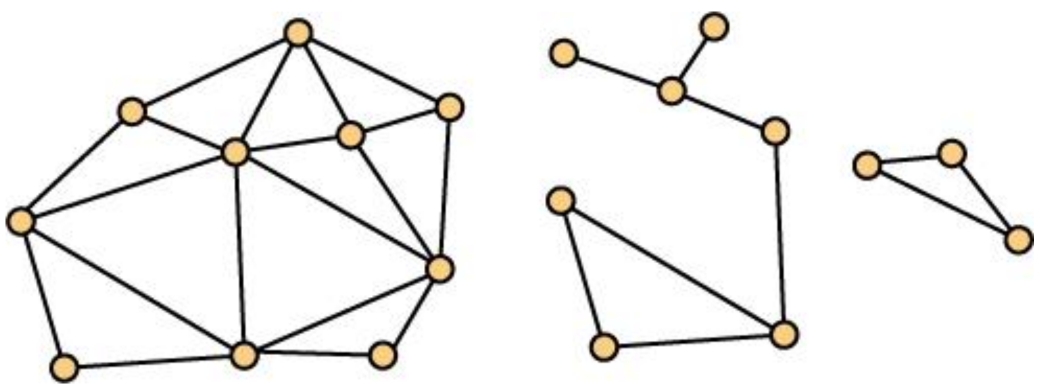
Bridges in Graphs

Definition An edge e in a graph G with t components is called a bridge when the removal of e leaves a subgraph with $t + 1$ components. In the graph shown below, $t = 3$ and there are four bridges.



2-Connected Graphs

Definition A connected graph G is said to be 2-connected when it has no bridges. In the graph shown below, there are three components. The component on the left is a 2-connected graph. So is the component on the right. But the component in the middle is not 2-connected.



Proof of Euler's Formula

Proof Fix the value of n. Then proceed by induction on q. Base case is $q = 0$. In this case, $f = 1$ and $t = n$. Check!

Inductive Step Suppose it holds when $q = k$ for some $k \geq 0$. Then suppose $q = k + 1$ edges.

Case 1 Suppose G has an edge e which is a bridge.

Case 2 G has no bridges.

Maximum Number of Edges

Theorem If G is a planar graph with $n \geq 3$ vertices and q edges, then $q \leq 3n - 6$.

Proof Fix the value of n and consider a plane drawing of a planar graph G on n vertices having the maximum number of edges. Clearly, G is connected and has no bridges so that every edge of G belongs to exactly two faces.

For each $m \geq 3$, let f_m be the number of faces whose boundary is a cycle of size m.

Maximum Number of Edges (2)

$2q = 3f3 + 4f4 + 5f5 + \ldots$

$\geq 3f3 + 3f4 + 3f5 + \ldots$

$= 3(f3 + f4 + f5 + \ldots)$

$= 3f$ So

$3f \leq 2q$.

Multiply Euler by 3 to obtain $3n - 3q + 3f = 6$.

This implies

$6 = 3n - 3q + 3f \leq 3n - 3q + 2q = 3n - q$, so

$q \leq 3n - 6$

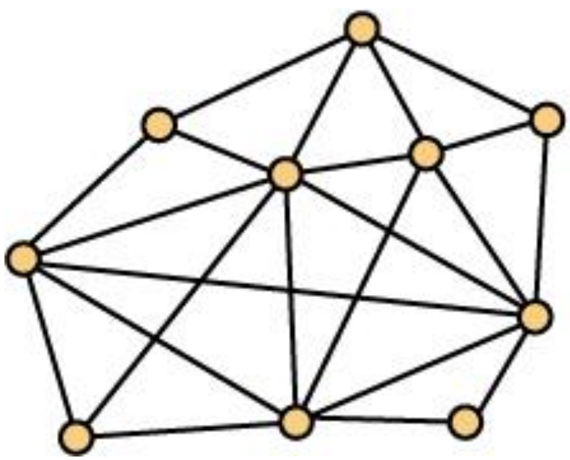
Using Euler to Determine Non-Planarity

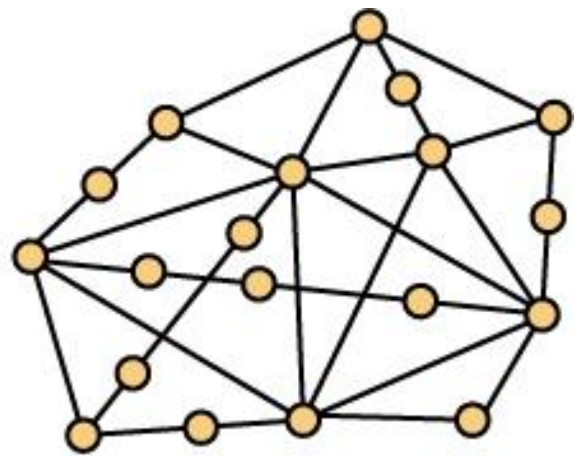
Theorem The complete graph K_5 is non-planar.

Proof The complete graph K_5 has $n = 5$ vertices and $q = 10 = C(5, 2)$ edges. Since $10 > 3 \cdot 5 - 6 = 15 - 6 = 9$, K_5 cannot be planar.

Homeomorphs of a Graph

Definition A graph H is a homeomorph of a graph G if H is obtained by "inserting" one or more vertices on some of the edges of G. The graph on the right is a homeomorph of the graph on the left.





Homeomorphs and Planarity

Observation If a graph G is planar, then any subgraph of G is planar.

Observation If a graph H is a homeomorph of a graph G, then H is planar if and only if G is planar.

Consequence A graph is non-planar if it contains a homeomorph of the complete graph K_5 as a subgraph.