Test 3  Tuesday, November 24, 2015. Details on material for which you will be responsible were sent by email after class the preceding Thursday. Again, I ask all of you to study hard. Experience shows that the closing portion of this course has most content. The concepts and techniques will have lasting value.
Problem Let $r(n)$ denote the number of regions determined by $n$ lines that intersect in general position.

Solution

$$r(1) = 2$$
$$r(n + 1) = r(n) + n+1 \quad \text{when } n \geq 0$$
Problem  Let $s(n)$ denote the number of regions determined by $n$ circles that intersect in general position.

Solution

$s(1) = 2$
$s(n+1) = s(n) + 2n$  when  $n \geq 0$. 
Problem  Let $t(n)$ denote the number of ways to tile a $2 \times n$ grid with dominoes of size $1 \times 2$ and $2 \times 1$.

Solution  
$t(1) = 1$
$t(2) = 2$
$t(n+2) = t(n+1) + t(n)$ when $n \geq 0$. 
Problem  Let \( u(n) \) denote the number of ternary sequences that do not contain 01 in consecutive positions.

Solution
\[
\begin{align*}
  u(1) &= 3 \\
  u(2) &= 8 \\
  u(n+2) &= 3u(n+1) - u(n)
\end{align*}
\]
Summary The recurrence equations in the last four examples are:

\[ r(n+1) - r(n) = n+1 \]
\[ s(n+1) - s(n) = 2n \]
\[ t(n+2) - t(n+1) - t(n) = 0 \]
\[ u(n+2) - 3u(n+1) + u(n) = 0 \]
Observation  We consider the family $V$ of all functions which map the set $\mathbb{Z}$ of all integers (positive, negative and zero) to the set $\mathbb{C}$ of complex numbers. This is a more general framework than we first studied, but as will become clear, we need this additional structure to make the form of general solutions relatively easy to obtain.

Note  Each of the four examples presented above have involved functions with range and domain being the set $\mathbb{N}$ of positive integers, so $V$ is a more general setup.
Fact  The family $V$ is an infinite dimensional vector space over the field $C$ of complex numbers, with 
$(f + g)(n) = f(n) + g(n)$ and $(\alpha f)(n) = \alpha(f(n))$.

Note  Students should spot the “operator overloading” in these two equations, even when one of the two operators (multiplication) is indicated simply by adjacent symbols, one a scalar and the other a vector.

Note  The “zero” of $V$ is the constant function which maps all integers to the “zero” in $C$. 
We will first focus on homogeneous linear recurrence equations. These have the following form:

\[ a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + ... + a_{d-1} f(n+1) + a_d f(n) = 0 \]

The coefficients \( a_0, a_1, a_2, ..., a_d \) are complex numbers. Without loss of generality \( a_0 \neq 0 \). For the time being, we will also assume that \( a_d \neq 0 \).
Example A homogeneous equation:

\[(2+3i)g(n+3) - (8-7i)g(n+2) + 42g(n+1) - (5i)g(n) = 0\]

Example A non-homogeneous equation:

\[(2+3i)g(n+3) - (8-7i)g(n+2) + 42g(n+1) - (5i)g(n) = (2-i)(3+i)^n + 12n^3\]

Remark In order to fully understand the homogeneous case, we will need to discuss the non-homogeneous case concurrently.
Alternate Notation  We define the advancement operator $A$ on the vector space $V$ by the rule $A f(n) = f(n+1)$. Note that $A^2 f(n) = f(n+2)$, $A^3 f(n) = f(n+3)$, etc. So our linear homogeneous equation

$$a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \ldots + a_{d-1} f(n+1) + a_d f(n) = 0$$

can then be rewritten as:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \ldots + a_{d-1} A + a_d) f(n) = 0.$$ 

Remark  The "polynomial form" is significant!
The set $S$ of all solutions to a homogeneous linear recurrence equation of the form:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + \ldots + a_{d-1}A + a_d) f_n = 0$$

is a $d$-dimensional subspace of $V$ provided both $a_0$ and $a_d$ are non-zero.

**Conclusion** The solution space can be specified entirely just by providing a basis for the subspace $S$. 
The Case \( d = 1 \)

**Theorem** Let \( a_0 \) and \( a_1 \) be non-zero complex numbers, and set \( r = (-a_1/a_0) \). Then the solution space \( S \) of the advancement operator equation \((a_0A + a_1)f(n) = 0\) is a 1-dimensional subspace of \( V \) and the function \( r^n \) is a basis, i.e., every solution is of the form \( f(n) = c_1 r^n \) where \( c_1 \) is a constant.

**Proof** Let \( f \) be any solution to \((a_0A + a_1)f(n) = 0\), and let \( c_1 = f(0) \). We show that \( f(n) = c_1 r^n \) for all integers \( n \). We first show that \( f(n) = c_1 r^n \) when \( n \geq 0 \). We do this by induction on \( n \).
The Case  $d = 1$  (Part 2)

**Base Case**  The base case is  $n = 0$, where the left hand side is  $f(0) = c_1$, and the right hand side is  $c_1 \cdot r^0$. But since  $r \neq 0$, the right hand side is  $c_1 \cdot 1 = c_1$. So the statement holds when  $n = 0$.

**The Inductive Step**  Now assume that  $f(k) = c_1 \cdot r^k$  for some  $k \geq 0$. Since  $(a_0A + a_1)f(n) = 0$  for all integers  $n$, we know that:

- $(a_0A + a_1)f(k) = 0$
- $(a_0f(k+1) + a_1f(k)) = 0$
- $f(k+1) = (-a_1/a_0)c_1r^k$
- $f(k+1) = r \cdot c_1 \cdot r^k$
- $f(k+1) = c_1r^{k+1}$
The Case $d = 1$ (Part 3)

**Negative Integers**  It remains only to show that $f(n) = c_1 r^n$ for all integers $n \leq 0$. This is equivalent to showing that $f(-n) = c_1 r^{-n}$ for all $n \geq 0$. This is done by induction and the argument is a trivial modification of what we have just done.

**Conclusion**  We have verified the assertion that the solution space to the homogeneous equation

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + ... + a_{d-1} A + a_d) f(n) = 0$$

with $a_0$ and $a_d$ non-zero is a $d$-dimensional subspace of $V$ when $d = 1$. 
Exercise  Note that

\[ A^2 + 2A - 35 = (A + 7)(A - 5) \]

Example  The functions \((-7)^n\) and \(5^n\) are solutions to the equation:

\[(A^2 + 2A - 35) f(n) = 0.\]

Observation  If \(r \neq 0\) and \(r\) is a root of the advancement operator polynomial, then \(r^n\) is a solution.
Exercise  Show that
\[ A^2 + (-12 + i)A + 41 - i = (A - 5 - 2i)(A - 7 + 3i) \]

Example  The functions \((5 + 2i)^n\) and \((7 - 3i)^n\) are solutions to the equation:
\[(A^2 + (-12 + i)A + 41 - i)f(n) = 0.\]

Observation  If \(r \neq 0\) and \(r\) is a root of the advancement operator polynomial, then \(r^n\) is a solution.
Example  Note that $A^2 - 10A + 25 = (A - 5)^2$.

Also note that the functions $5^n$ and $n5^n$ are solutions to the equation:

$$(A - 5)^2 f(n) = 0.$$  

Observation  If $r \neq 0$ and $r$ is a root of multiplicity 2, then $r^n$ and $n r^n$ are solutions.
Example  The functions \((5 - 2i)^n\) and \(n(5 - 2i)^n\) are solutions to the equation:

\[(A - 5 + 2i)^2 f(n) = 0.\]

Observation  If \(r \neq 0\) and \(r\) is a root of multiplicity 2, then \(r^n\) and \(n r^n\) are solutions.
Lemma  If \( p(A) \) is a polynomial in the advancement operator \( A \), \( r \neq 0 \) and \( r \) is a root of multiplicity \( m \), then each of the following functions is a solution of the equation: \( p(A) f(n) = 0 \)

\[
r^n \quad n \, r^n \quad n^2 \, r^n \quad n^3 \, r^n \quad n^4 \, r^n \quad \ldots \quad n^{m-1} \, r^n
\]

Proof  We will outline the proof in Thursday's lecture.
Example  The general solution to

\[ ((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2) f(n) = 0 \]

is:

\[ f(n) = c_1 3^n + c_2 n3^n + c_3 n^23^n + c_4 n^33^n \\
+ c_5(7 - 2i)^n + c_6 n(7 - 2i)^n + c_7 n^2(7 - 2i)^n \\
+ c_8(-5 + 8i)^n + c_9 n(-5 + 8i)^n \]
Towards the General Case (7)

Example  The solution space to:

\[((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2) f(n) = 0\]

is a 9-dimensional subspace of \( V \) and the following functions are a basis:

\[3^n, n3^n, n^23^n, n^33^n, (7 - 2i)^n, n(7 - 2i)^n, n^2(7 - 2i)^n, (-5 + 8i)^n, n(-5 + 8i)^n\]
Analyses with Partial Fractions

Example  Given a proper rational function \( p(x)/q(x) \) whose denominator polynomial \( q(x) \) can be factored as

\[
q(x) = (x - 3)^3 x^2(x^2 + 2x + 9)
\]

there are constants \( c_1, c_2, c_3, c_4, c_5, c_6 \) and \( c_7 \) so that

\[
p(x)/q(x) = c_1/(x - 3) + c_2/(x - 3)^2 + c_3/(x - 3)^3 \\
+ c_4/x + c_5/x^2 \\
+ c_6/(x^2 + 2x + 9) + c_7 x/(x^2 + 2x + 9)
\]
Analogies with Differential Equations

**Example**  Let $D$ be the differential operator, i.e., $D f$ is the derivative of $f$. Then the solution to the equation:

$$(D - 3)^3 D^2(D^2 + 16) f = 0$$

has the form:

$$f(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} + c_4 + c_5 x + c_6 e^{4i} + c_7 e^{-4i}$$

where $c_6$ and $c_7$ are complex conjugates.
**The Non-Homogeneous Case**

**Theorem**  Let $p(A)f = g$ be a non-homogeneous equation. If $h_0$ is any solution to this equation, then the general solution is $h_0 + f$ where $f$ is a solution to the associated homogeneous equation $p(A)f = 0$.

**Note**  The proof of this theorem is relatively straightforward.

**Terminology**  The function $h_0$ is referred to as a particular solution to $p(A)f = g$. 
**Example**  For the non-homogeneous equation \((A - 3) f(n) = 8 \cdot 5^n\), the function \(h_0 = 4 \cdot 5^n\) is a particular solution. Accordingly, the general solution has the form:

\[
f(n) = c_1 3^n + 4 \cdot 5^n
\]