Math 3012 - Applied Combinatorics
Lecture 21

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**Observation** We consider the family $V$ of all functions which map the set $\mathbb{Z}$ of all integers (positive, negative and zero) to the set $\mathbb{C}$ of complex numbers. This is a more general framework than we first studied, but as will become clear, we need this additional structure to make the form of general solutions relatively easy to obtain.

**Remark** The family $V$ is an infinite dimensional vector space over the field $\mathbb{C}$ of complex numbers, with $(f + g)(n) = f(n) + g(n)$ and $(\alpha f)(n) = \alpha(f(n))$. 
Linear Recurrence Equations

**Observation**  We will first focus on homogeneous linear recurrence equations. These have the following form:

\[ a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \ldots + a_{d-1} f(n+1) + a_d f(n) = 0 \]

**Note**  The coefficients \( a_0, a_1, a_2, \ldots, a_d \) are complex numbers. Without loss of generality \( a_0 \neq 0 \).
The Advancement Operator

Alternate Notation  Our linear homogeneous equation

\[
a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \ldots \\
+ a_{d-1} f(n+1) + a_d f(n) = 0
\]

can then be rewritten as:

\[
(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \ldots + a_{d-1} A + a_d) f(n) = 0.
\]

Remark  The “polynomial form” of this advancement operator equation is significant!
The General Theorem

**Theorem**  The solution space $S$ of the advancement operator equation:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + ... + a_{d-1}A + a_d) f(n) = 0$$

is a $d$-dimensional subspace of $V$, provided both $a_0$ and $a_d$ are non-zero. Furthermore, a basis for $S$ can be formed by taking functions of the form $n^i r^n$ where $r \neq 0$ is a root of the associated polynomial and $0 \leq i < m$, with $m$ the multiplicity of $r$. 
Applying the Theorem

Example  The general solution to

\[ ((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2(A - 1)^5) f(n) = 0 \]

is:

\[ f(n) = c_1 3^n + c_2 n 3^n + c_3 n^2 3^n + c_4 n^3 3^n + c_5 (7 - 2i)^n + c_6 n(7 - 2i)^n + c_7 n^2(7 - 2i)^n + c_8(-5 + 8i)^n + c_9 n(-5 + 8i)^n + c_{10} + c_{11} n + c_{12} n^2 + c_{13} n^3 + c_{14} n^4 \]
**Using Initial Conditions**

**Example** Find the solution to \((A^2 - 7A + 10) f(n) = 0\) with \(f(0) = 9\) and \(f(1) = 27\).

**Solution** The general solution is \(f(n) = c_1 2^n + c_2 5^n\). So our constraints become:

\[
\begin{align*}
c_1 + c_2 &= 9 \\
2c_1 + 5c_2 &= 27
\end{align*}
\]

This forces \(c_1 = 6\) and \(c_2 = 3\), so the answer is

\(f(n) = 6 \cdot 2^n + 3 \cdot 5^n\)
Example  For the non-homogeneous equation 
\[(A - 3) f(n) = 8 \cdot (5)^n,\] 
the function \[h_0 = 4 \cdot 5^n\] 
is a particular solution. Accordingly, the general solution has the form:

\[f(n) = c_1 3^n + 4 \cdot 5^n\]

Exercise  Find the solution to \[(A - 3) f(n) = 8 \cdot (5)^n\]
subject to the requirement that \[f(3) = 118.\] 
This requires \[118 = 9c_1 + 100,\] so \[c_1 = 2\] and the answer is \[f(n) = 2 \cdot 3^n + 4 \cdot 5^n\]
When 0 is a root

**Observation** Consider the equation $A^m f(n) = 0$. A solution must satisfy $f(n+m) = 0$ for all integers $n$. This forces $f(n) = 0$ for all $n$, i.e., the only solution is the zero function.

**Consequence** If $p(A) = A^m q(A)$ where $q(A)$ is a polynomial of degree $d \geq 1$, then the solution space of the equation $p(A) f(n) = 0$ will be a $d$-dimensional subspace of $V$.

**Remark** This explains why we have focused on the form:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + ... + a_{d-1} A + a_d) f(n) = 0$$

with both $a_0$ and $a_d$ non-zero.
The Non-Homogeneous Case

**Theorem** Let \( p(A) f = g \) be a non-homogeneous equation. If \( h_0 \) is any solution to this equation, then the general solution is \( h_0 + f \) where \( f \) is a solution to the associated homogeneous equation \( p(A) f = 0 \).

**Note** The proof of this theorem is relatively straightforward.

**Terminology** The function \( h_0 \) is referred to as a particular solution to \( p(A) f = g \).
Theorem  The solution space $S$ of the operator equation:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + \ldots + a_{d-1}A + a_d) f(n) = 0$$

is a $d$-dimensional subspace of $V$, provided both $a_0$ and $a_d$ are non-zero. Furthermore, a basis for $S$ can be formed by taking functions of the form $n^i r^n$ where $r \neq 0$ is a root of the associated polynomial and $0 \leq i < m$, with $m$ the multiplicity of $r$. 
A Key Lemma

**Theorem** Let $d \geq 1$, let $r, s \neq 0$. Then let

$$p(n) = a_0 n^d + a_1 n^{d-1} + a_2 n^{d-2} + \ldots + a_{d-1} n + a_d$$

be a complex polynomial of degree $d$, i.e., the leading coefficient $a_0 \neq 0$. Then $(A - r)p(n)r^n = q(n)r^n$ for some polynomial $q(n)$ of degree $d - 1$.

Furthermore, if $s \neq r$, then $(A - s)p(n)r^n = q'(n)r^n$ for some polynomial $q'(n)$ of degree $d$. 
Corollary \hspace{1cm} Let \( d \geq 0 \) and let

\[ p(n) = a_0 n^d + a_1 n^{d-1} + a_2 n^{d-2} + \ldots + a_{d-1} n + a_d \]

be a complex polynomial of degree \( d \), i.e., the leading coefficient \( a_0 \neq 0 \). Then there is a uniquely determined polynomial \( q(n) \) of degree \( d + 1 \) so that

\[ (A - r) q(n) r^n = p(n) r^n \]
Theorem  Let $m \geq 1$ and let $r \neq 0$. Then the solution space $S$ of the equation $(A - r)^m f(n)$ is an $m$-dimensional subspace of $V$ and the following functions form a basis for $S$:

$$r^n \quad n \quad n^2 \quad r^n \quad n^3 \quad r^n \quad n^4 \quad r^n \quad \ldots \quad n^{m-1} \quad r^n$$

Remark  There are three parts to the proof. First is showing that each of these functions is a solution. Second is showing that every solution is a linear combination of these functions. Third is showing that they are linearly independent. We will sketch these arguments in class.
Consider the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Is the 1000\(^{th}\) term more or less than 10\(^{300}\)?

Does the ratio \( f(n+1)/f(n) \) tend to a limit.

The equation is \((A^2 - A - 1) f(n) = 0\). There are two roots: \((1 + \sqrt{5})/2\) and \((1 - \sqrt{5})/2\) and the initial conditions are \( f(0) = f(1) = 1 \).