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Math 3012 – Applied Combinatorics Lecture 21

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Vector Space of Functions

Observation We consider the family **V** of all functions which map the set **Z** of all integers (positive, negative and zero) to the set ***C*** of complex numbers. This is a more general framework than we first studied, but as will become clear, we need this additional structure to make the form of general solutions relatively easy to obtain.

Remark The family **V** is an infinite dimensional vector space over the field ***C*** of complex numbers, with $(f + g)(n) = f(n) + g(n)$ and $(a f)(n) = a(f(n))$.

Linear Recurrence Equations

Observation We will first focus on homogeneous linear recurrence equations. These have the following form:

$$a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \ldots + a_{d-1} f(n+1) + a_d f(n) = 0$$

Note The coefficients $a_0, a_1, a_2, \ldots, a_d$ are complex numbers. Without loss of generality $a_0 \neq 0$.

The Advancement Operator

Alternate Notation Our linear homogeneous equation

$$a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \dots + a_{d-1} f(n+1) + a_d f(n) = 0$$

can then be rewritten as:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \dots + a_{d-1} A + a_d)\, f(n) = 0.$$

Remark The "polynomial form" of this advancement operator equation is significant!

The General Theorem

Theorem The solution space S of the advancement operator equation:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + \ldots + a_{d-1}A + a_d)\, f(n) = 0$$

is a d-dimensional subspace of **V**, provided both a_0 and a_d are non-zero. Furthermore, a basis for S can be formed by taking functions of the form $n^i r^n$ where $r \neq 0$ is a root of the associated polynomial and $0 \leq i < m$, with m the multiplicity of r.

Applying the Theorem

Example The general solution to

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2(A - 1)^5)\, f(n) = 0$$

is:

$$\begin{aligned} f(n) = {} & c_1 3^n + c_2 n 3^n + c_3 n^2 3^n + c_4 n^3 3^n \\ & + c_5(7 - 2i)^n + c_6 n(7 - 2i)^n + c_7 n^2(7 - 2i)^n \\ & + c_8(-5 + 8i)^n + c_9 n(-5 + 8i)^n \\ & + c_{10} + c_{11} n + c_{12} n^2 + c_{13} n^3 + c_{14} n^4 \end{aligned}$$

Using Initial Conditions

Example Find the solution to $(A^2 - 7A + 10)\, f(n) = 0$ with $f(0) = 9$ and $f(1) = 27$.

Solution The general solution is $f(n) = c_1\, 2^n + c_2\, 5^n$.
So our constraints become:

$$c_1 + c_2 = 9$$
$$2c_1 + 5c_2 = 27$$

This forces $c_1 = 6$ and $c_2 = 3$, so the answer is

$$f(n) = 6 \cdot 2^n + 3 \cdot 5^n$$

Using Initial Conditions (2)

Example For the non-homogeneous equation $(A - 3)\ f(n) = 8\ (5)^n$, the function $h_0 = 4 \cdot 5^n$ is a particular solution. Accordingly, the general solution has the form:

$$f(n) = c_1 3^n + 4 \cdot 5^n$$

Exercise Find the solution to $(A - 3)\ f(n) = 8\ (5)^n$ subject to the requirement that $f(3) = 118$. This requires $118 = 9c_1 + 100$, so $c_1 = 2$ and the answer is $f(n) = 2 \cdot 3^n + 4 \cdot 5^n$

When 0 is a root

Observation Consider the equation $A^m f(n) = 0$. A solution must satisfy $f(n+m) = 0$ for all integers n. This forces $f(n) = 0$ for all n, i.e., the only solution is the zero function.

Consequence If $p(A) = A^m q(A)$ where $q(A)$ is a polynomial of degree $d \geq 1$, then the solution space of the equation $p(A) f(n) = 0$ will be a d-dimensional subspace of $\mathbf{V}$.

Remark This explains why we have focused on the form:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \ldots + a_{d-1} A + a_d) f(n) = 0$$

with both a_0 and a_d non-zero.

The Non-Homogeneous Case

Theorem Let $p(A)\,f = g$ be a non-homogeneous equation. If h_0 is any solution to this equation, then the general solution is $h_0 + f$ where f is a solution to the associated homogeneous equation $p(A)\,f = 0$.

Note The proof of this theorem is relatively straightforward.

Terminology The function h_0 is referred to as a particular solution to $p(A)\,f = g$.

Proof of the General Theorem

Theorem The solution space S of the operator equation:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + \ldots + a_{d-1}A + a_d)\, f(n) = 0$$

is a d-dimensional subspace of **V**, provided both a_0 and a_d are non-zero. Furthermore, a basis for S can be formed by taking functions of the form $n^i r^n$ where $r \neq 0$ is a root of the associated polynomial and $0 \leq i < m$, with m the multiplicity of r.

A Key Lemma

Theorem Let $d \geq 1$, let $r, s \neq 0$. Then let

$$p(n) = a_0 n^d + a_1 n^{d-1} + a_2 n^{d-2} + \ldots + a_{d-1} n + a_d$$

be a complex polynomial of degree d, i.e., the leading coefficient $a_0 \neq 0$. Then $(A - r)\, p(n) r^n = q(n) r^n$ for some polynomial $q(n)$ of degree $d - 1$.

Furthermore, if $s \neq r$, then $(A - s)\, p(n) r^n = q'(n) r^n$ for some polynomial $q'(n)$ of degree d.

A Useful Corollary

Corollary Let $d \geq 0$ and let

$$p(n) = a_0 n^d + a_1 n^{d-1} + a_2 n^{d-2} + \dots + a_{d-1} n + a_d$$

be a complex polynomial of degree d, i.e., the leading coefficient $a_0 \neq 0$. Then there is a uniquely determined polynomial $q(n)$ of degree $d + 1$ so that

$$(A - r)\, q(n) r^n = p(n) r^n$$

Outline of Arguments

Theorem Let $m \geq 1$ and let $r \neq 0$. Then the solution space S of the equation $(A - r)^m f(n)$ is an m-dimensional subspace of **V** and the following functions form a basis for S:

$$r^n \quad n\, r^n \quad n^2\, r^n \quad n^3\, r^n \quad n^4\, r^n \quad \ldots \quad n^{m-1}\, r^n$$

Remark There are three parts to the proof. First is showing that each of these functions is a solution. Second is showing that every solution is a linear combination of these functions. Third is showing that they are linearly independent. We will sketch these arguments in class.

Analysis of Solutions

Questions Consider the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Is the 1000th term more or less than 10^{300} ?

Does the ratio $f(n+1)/f(n)$ tend to a limit.

Answers The equation is $(A^2 - A - 1)\, f(n) = 0$.
There are two roots: $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$
and the initial conditions are $f(0) = f(1) = 1$.