November 10, 2015

Math 3012 – Applied Combinatorics Lecture 22

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Reminders

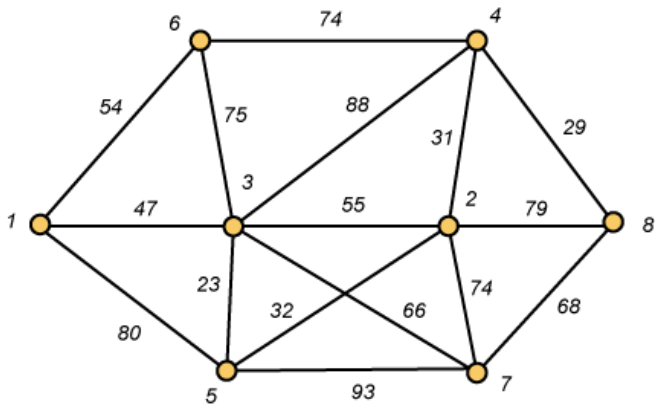
Test 3 Tuesday, November 24, 2015

Final Exam Tuesday, December 8, 2015, 8:05 – 10:55am.

Three-way Option (Full details in email)

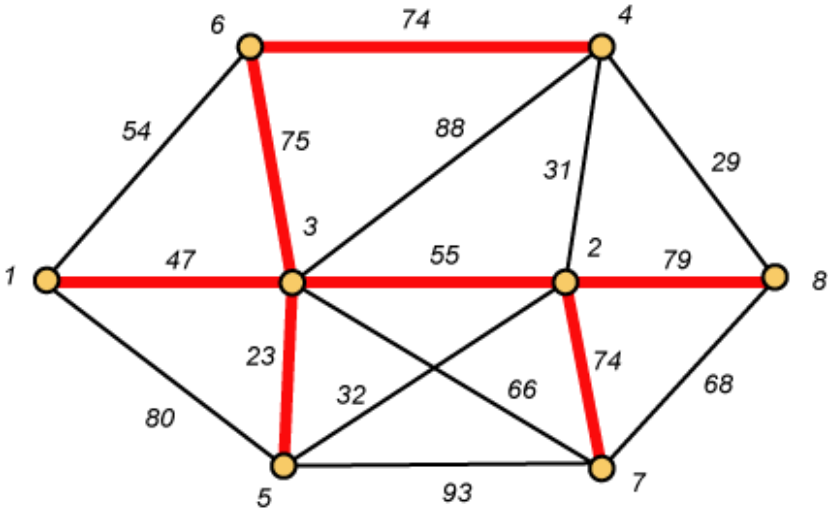
1. Do even numbered problems from assigned set.
2. Obtain/write code for implementing one of the algorithms in our course on my data set.
3. Write 3 – 4 page (typewritten) report on one of the selected math papers, all of which are accessible to undergraduates.

A Networking Problem



Problem The vertices represent 8 regional data centers which need to be connected with high-speed data lines. Feasibility studies show that the links illustrated above are possible, and the cost in millions of dollars is shown next to the link. Which links should be constructed to enable full communication (with relays allowed) and keep the total cost minimal?

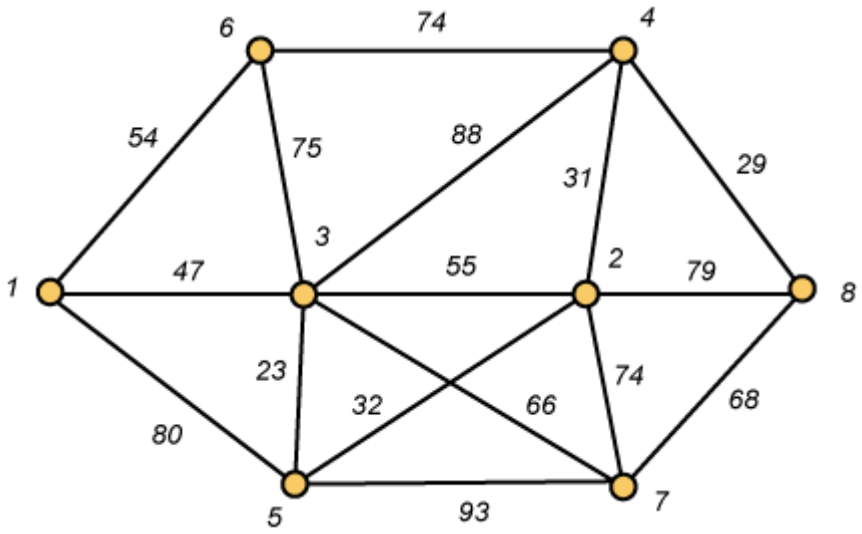
Links Will Form a Spanning Tree



Cost (T) = 47 + 23 + 75 + 74 + 55 + 74 + 79

= 427

Minimum Weight Spanning Trees



Problem Given a connected graph with non-negative weights on the edges, find a spanning tree T for which the sum of the weights on the edges in T is as small as possible.

Why Not Try *All* Possibilities?

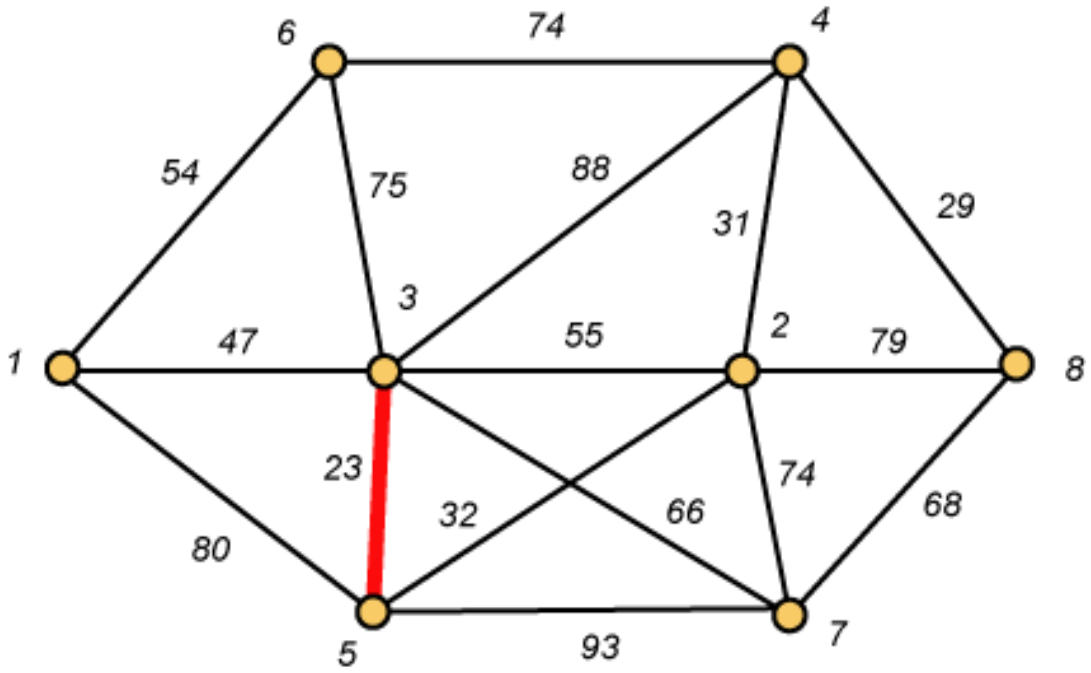
1. Suppose the graph has n vertices. Then the number of possible spanning trees can be as large as n^{n-2}.
2. When $n = 75$, this means that the number of spanning trees can be as large as

75765628046446014790863186515904134648140678330884033924704328101802427997135680470819352194666862487792 96875

Kruskal's Algorithm (Avoid Cycles)

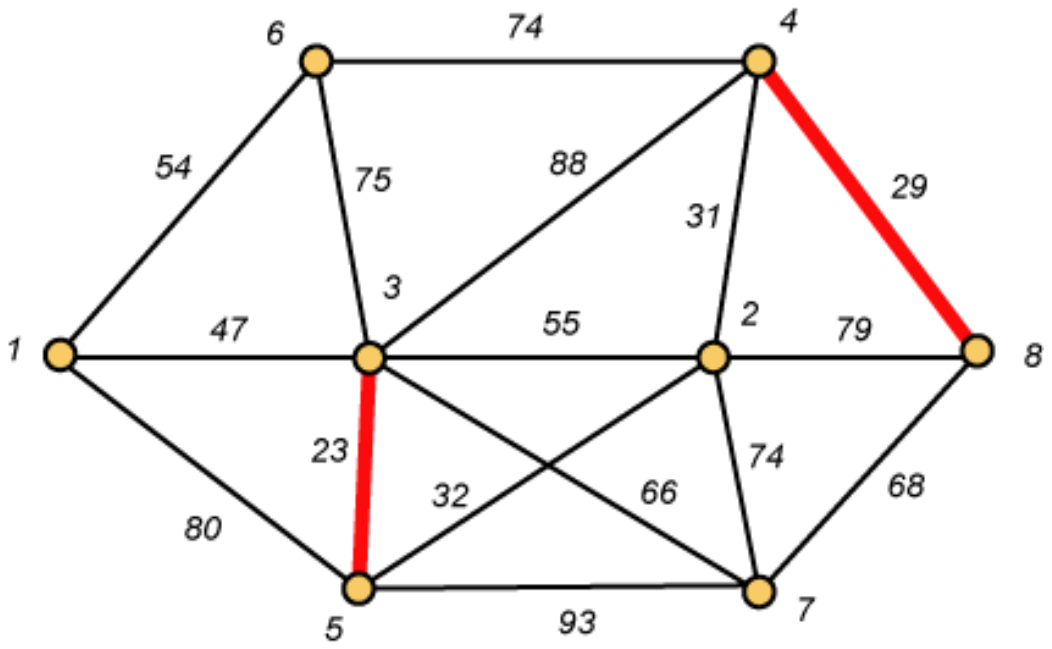
1. Sort the edges by weight.
2. Build a spanning forest (that eventually becomes a tree) by proceeding in a "greedy" manner, adding the edge of minimum weight which when added to those already chosen does not form a cycle.

Kruskal – Step 1



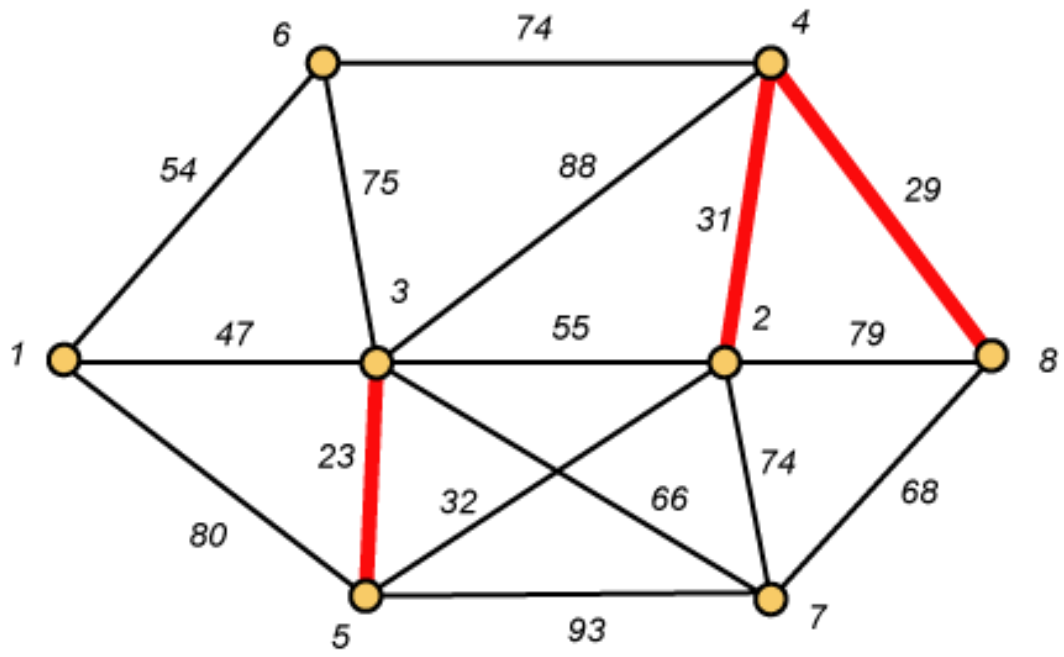
Remark Start with an edge having minimum weight.

Kruskal – Step 2



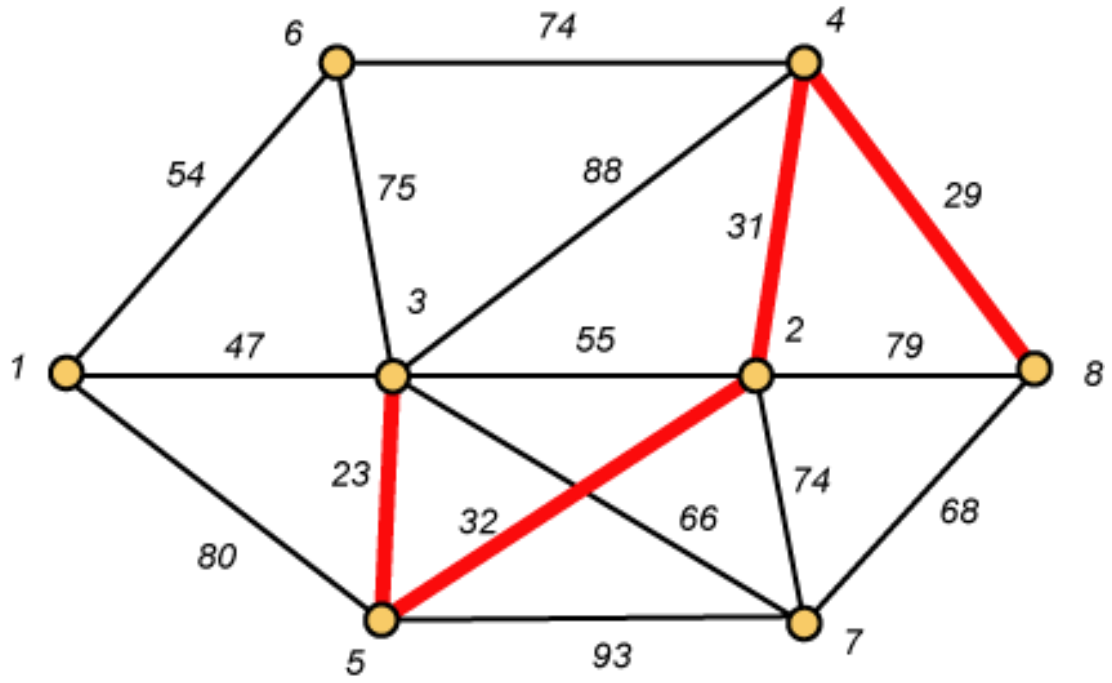
Remark Among the remaining edges, choose one of minimum weight.

Kruskal – Step 3



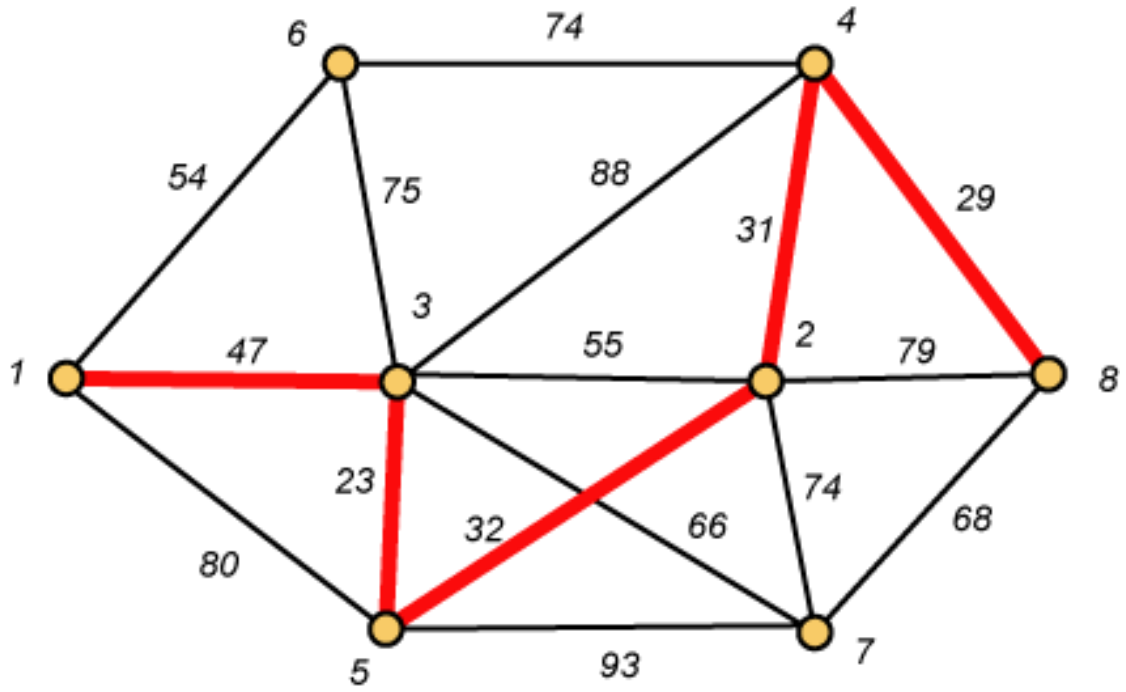
Remark Among the remaining edges, choose one of minimum weight which avoids cycles.

Kruskal – Step 4



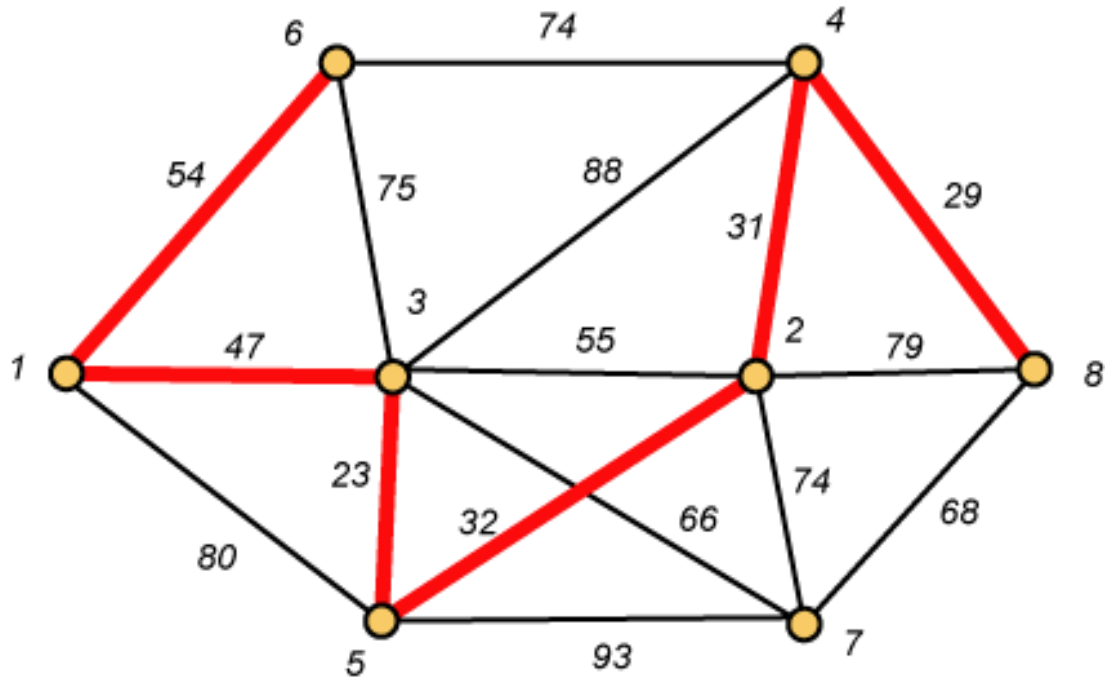
Remark Among the remaining edges, choose one of minimum weight which avoids cycles.

Kruskal – Step 5



Remark Among the remaining edges, choose one of minimum weight which avoids cycles.

Kruskal – Step 6

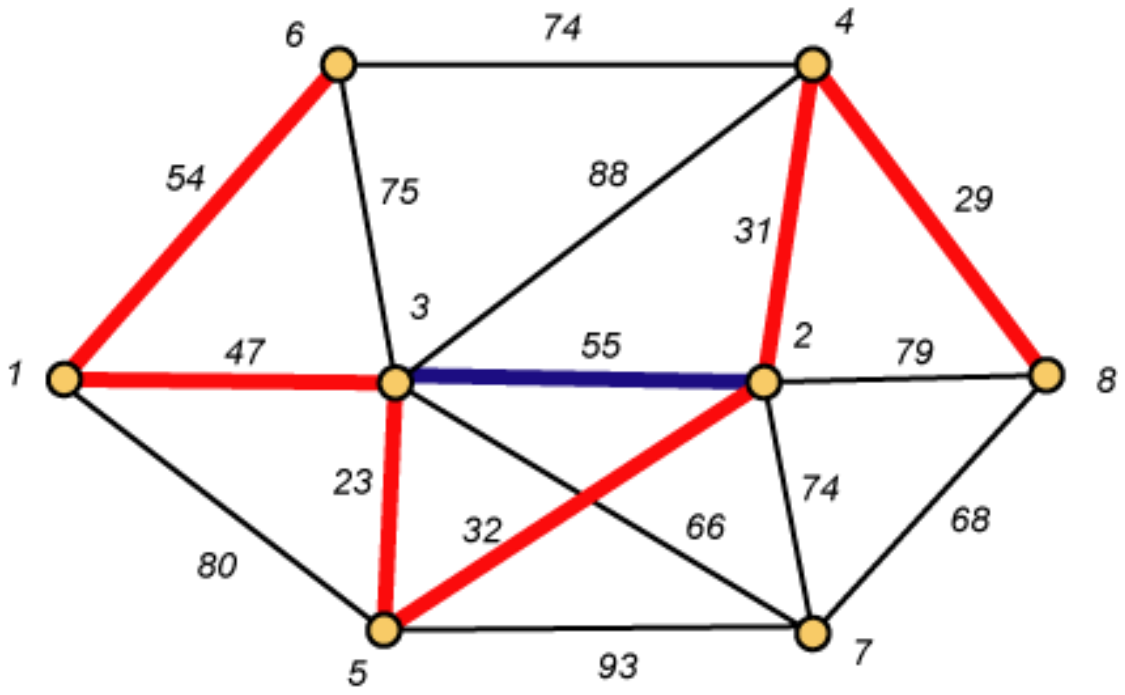


Remark Among the remaining edges, choose one of minimum weight which avoids cycles.

Why Avoiding Cycles Matters

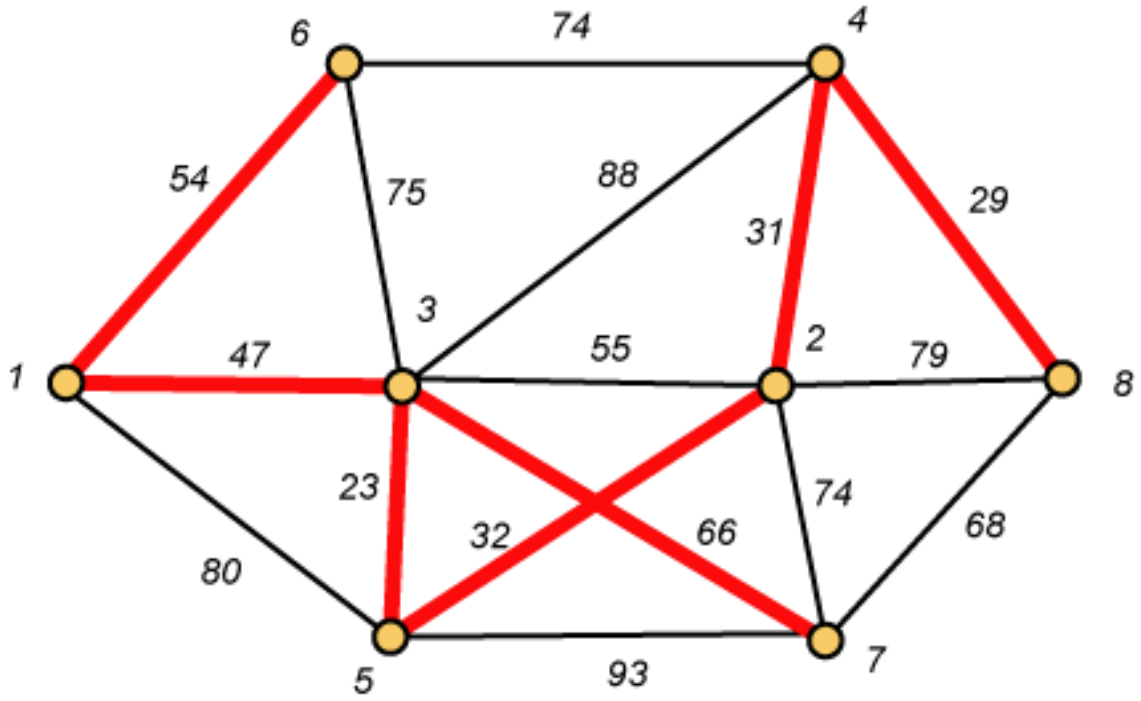
Remark Up to this point, we have simply taken the edges in order of their weight. But now we will have to reject an edge since it forms a cycle when added to those already chosen.

Forms a Cycle



Note We cannot take the blue edge having weight 55, as this would form a cycle.

Kruskal – Step 7 *DONE!!*

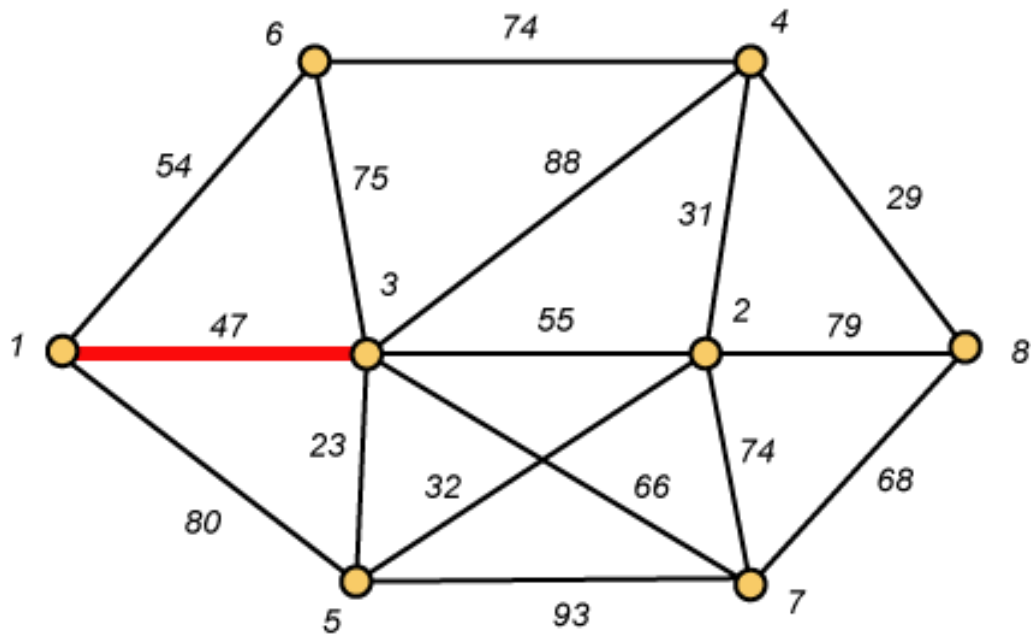


Weight (T) = 23 + 29 + 31 + 32 + 47 + 54 + 66 = **282**

Prim's Algorithm (Build Tree)

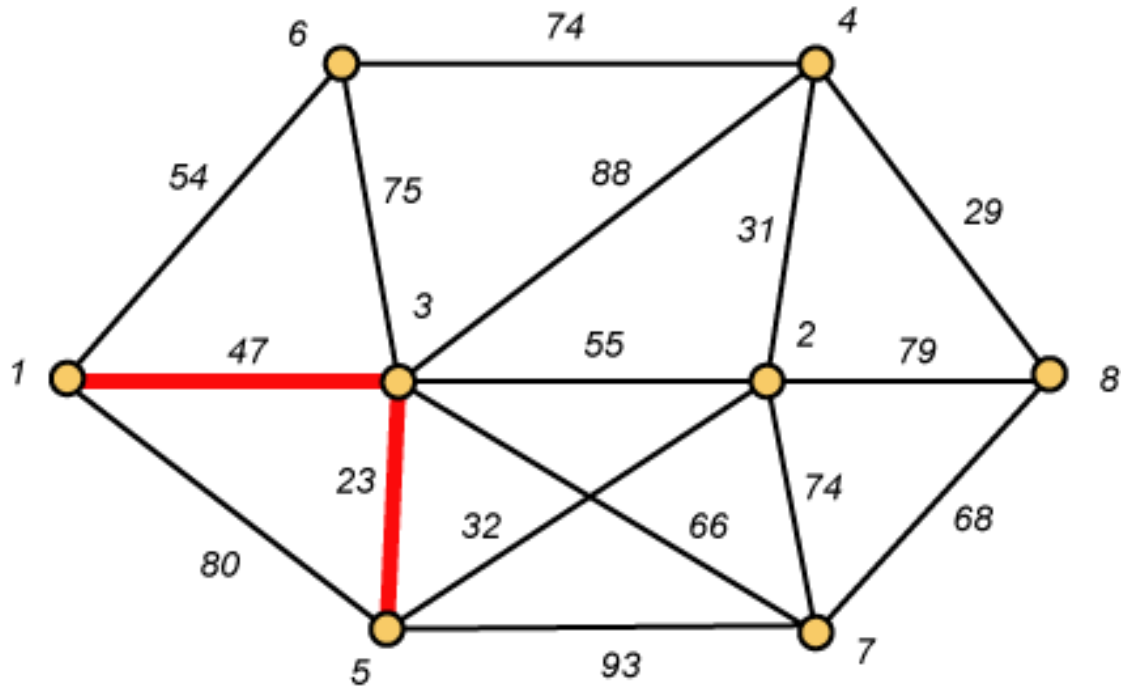
1. Build a tree one vertex at a time.
2. Start with a trivial one point tree T.
3. Then add the edge of minimum weight among those with one endpoint in T and the other not in T.

Prim – Step 1



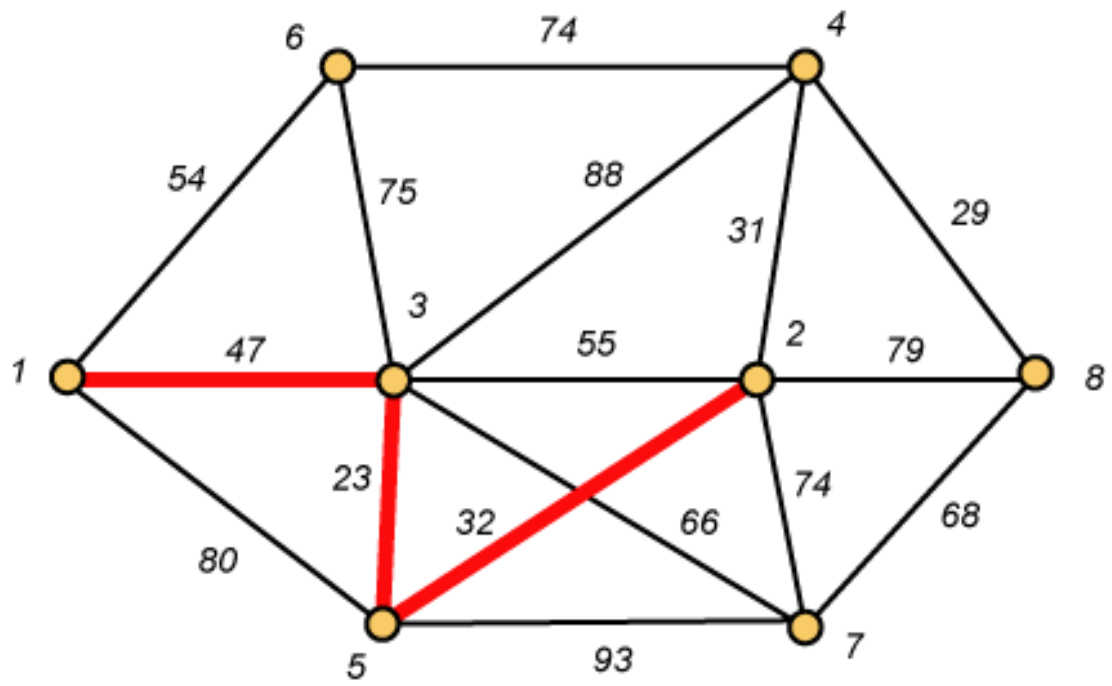
Remark In this example, we take vertex 1 as the "root" and start with the trivial tree consisting of just the root. We illustrate the first step after the initialization step.

Prim – Step 2



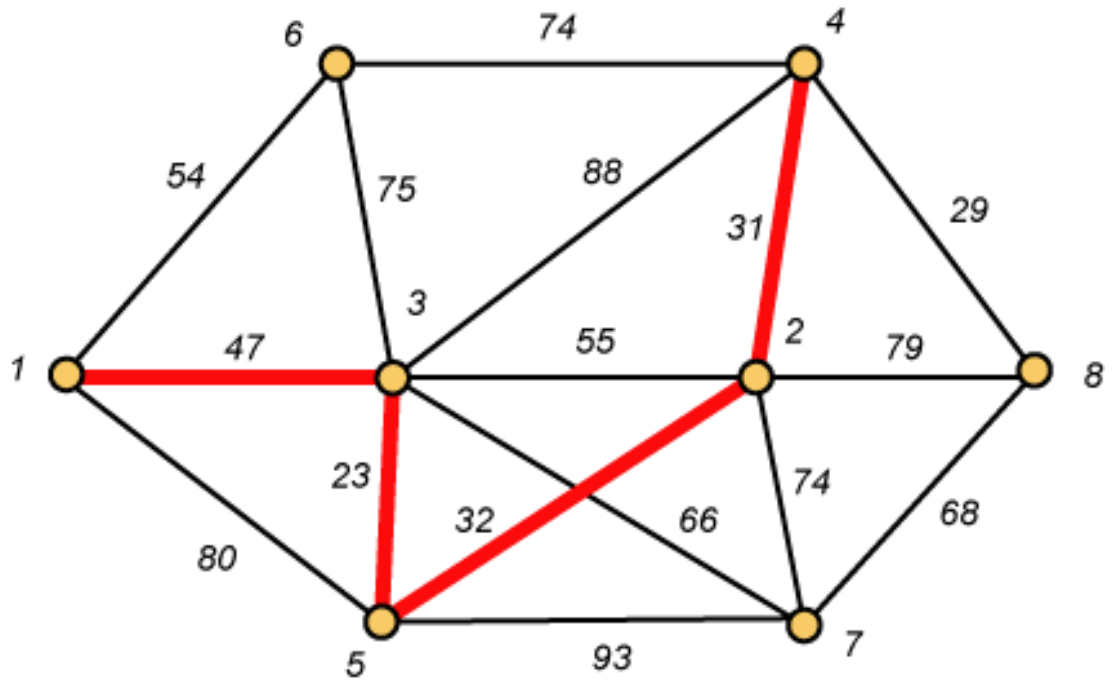
Remark Choose the minimum weight edge which expands T by a single vertex.

Prim – Step 3



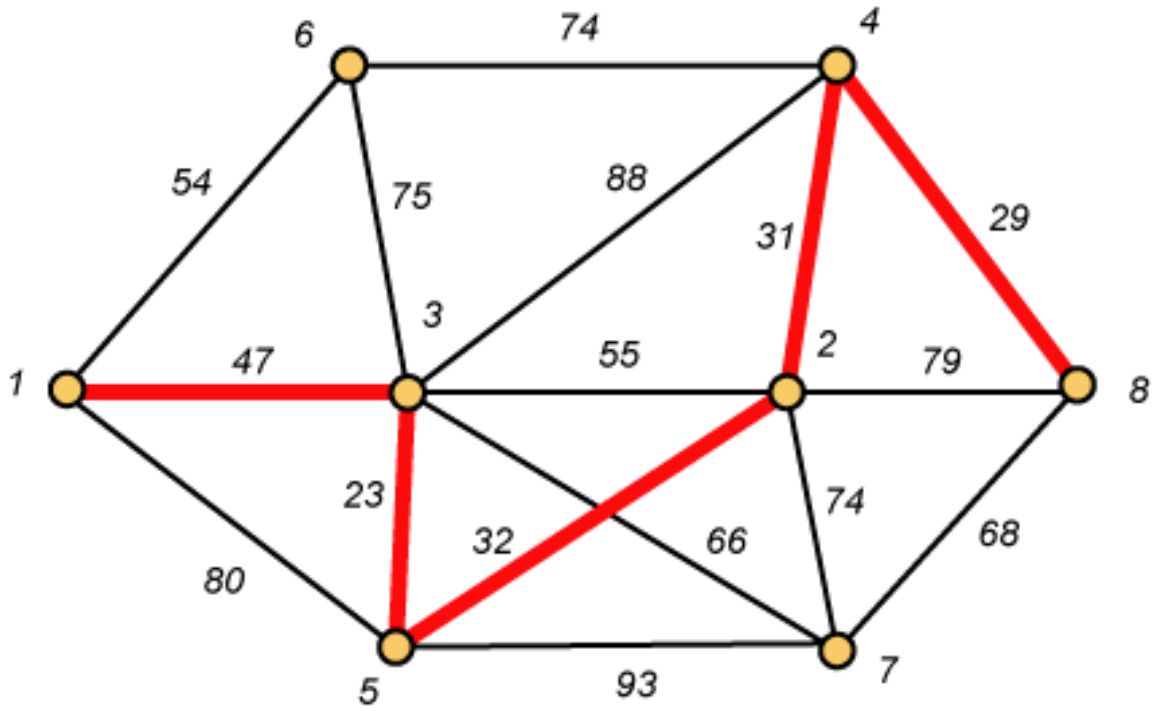
Remark Choose the minimum weight edge which expands T by a single vertex.

Prim – Step 4



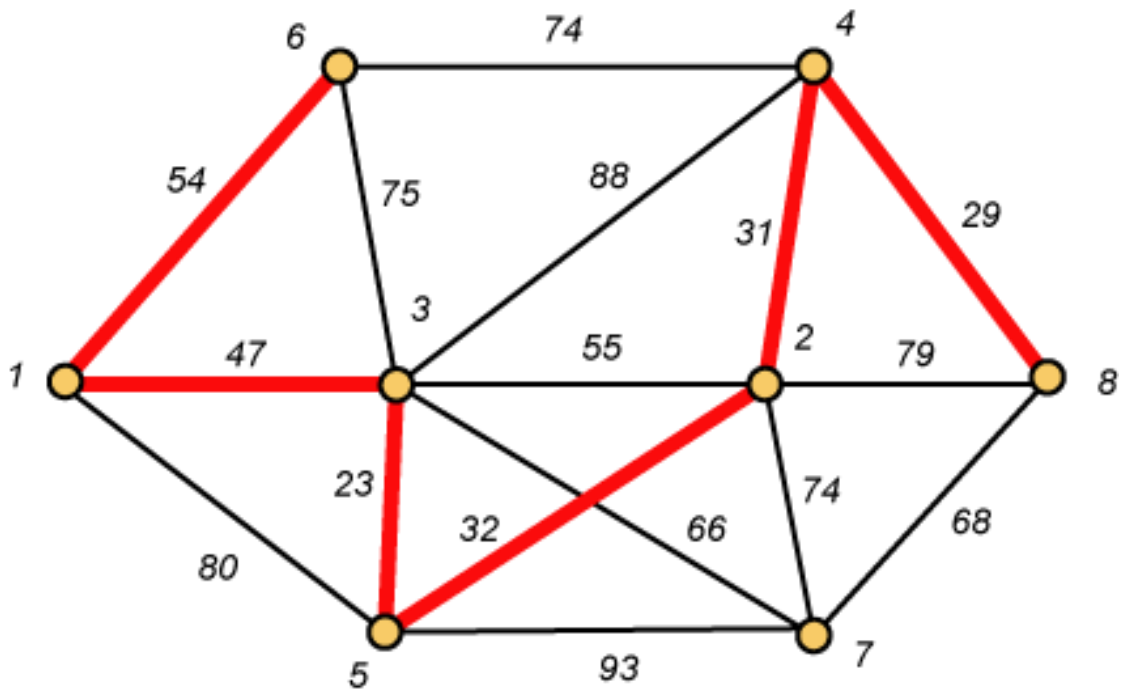
Remark Choose the minimum weight edge which expands T by a single vertex.

Prim – Step 5



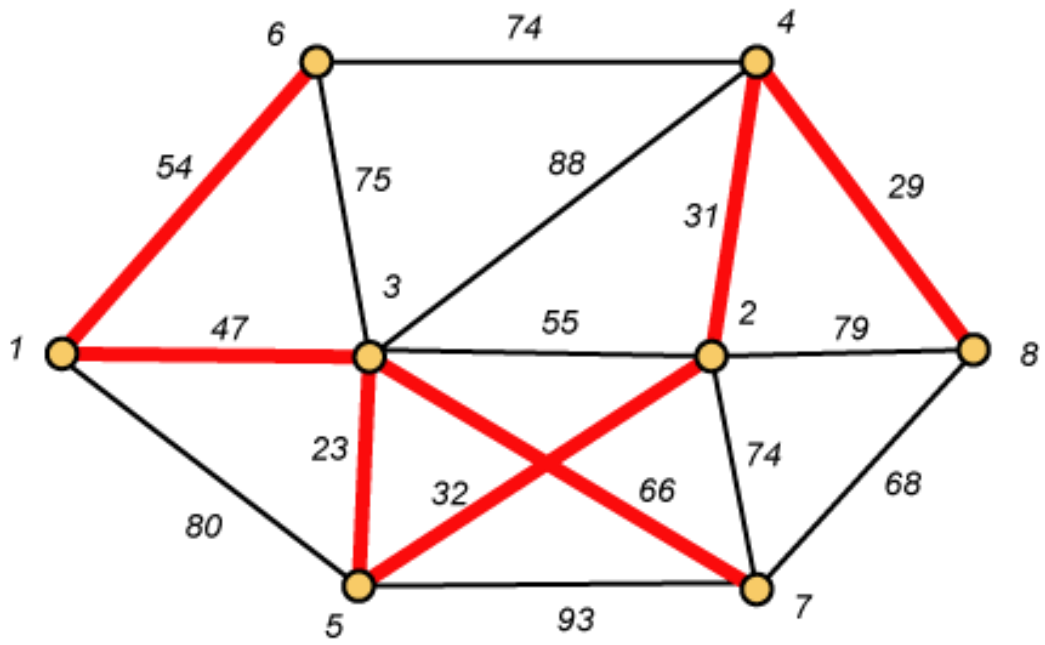
Remark Choose the minimum weight edge which expands T by a single vertex.

Prim – Step 6



Remark Choose the minimum weight edge which expands T by a single vertex.

Prim – Step 7 Done!!

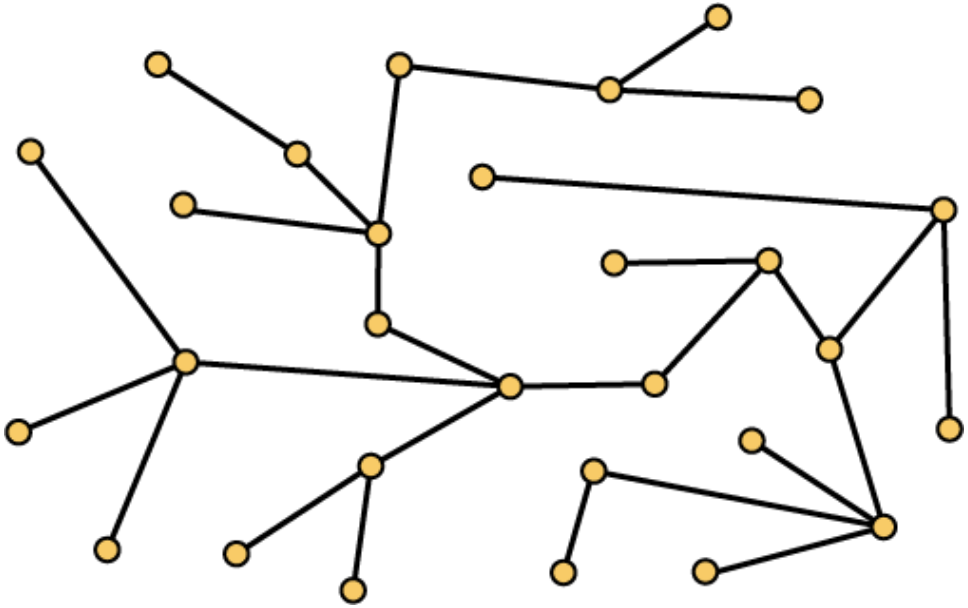


Weight (T) = 23 + 29 + 31 + 32 + 47 + 54 + 66 = **282**

The Mathematics Behind the Algorithms

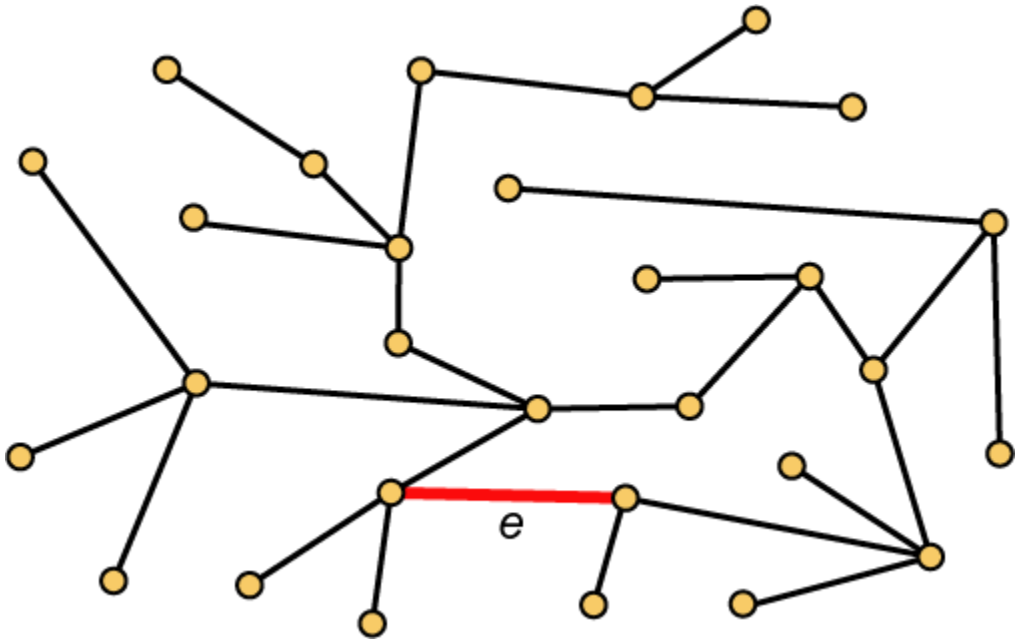
Remark With the next several slides, we will explain why Kruskal (avoid cycles) and Prim (build tree) work as intended. Students may find it interesting that the underlying principles are drawn from linear algebra, but now with a finite field rather than with the more familiar fields of real numbers and complex numbers. In fact, these principles apply to a wide range of discrete optimization problems.

The Exchange Principle (1)



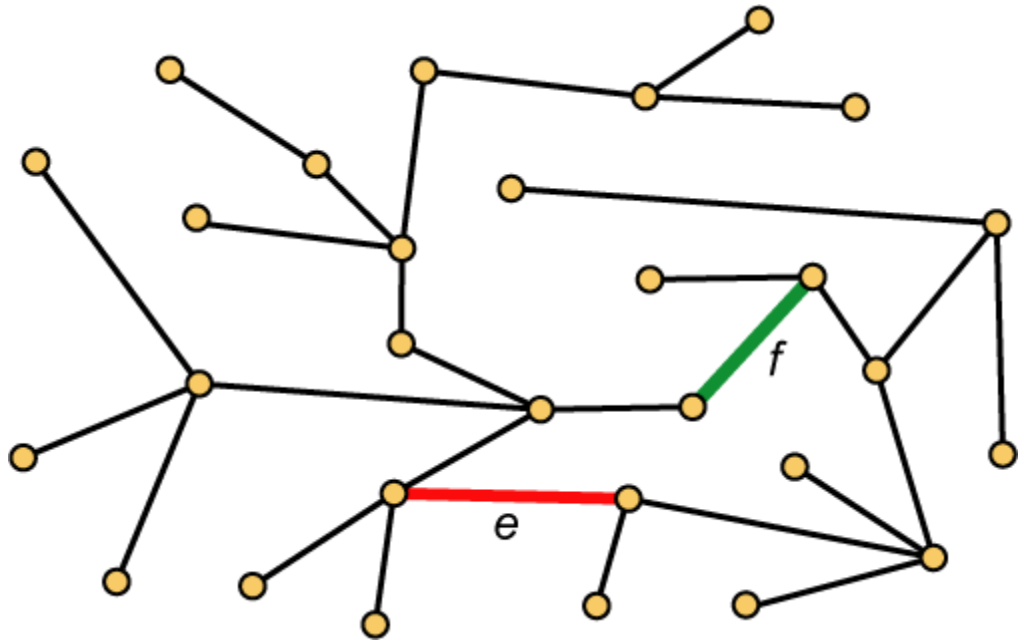
Remark We start by considering a spanning tree T in a graph G.

The Exchange Principle (2)



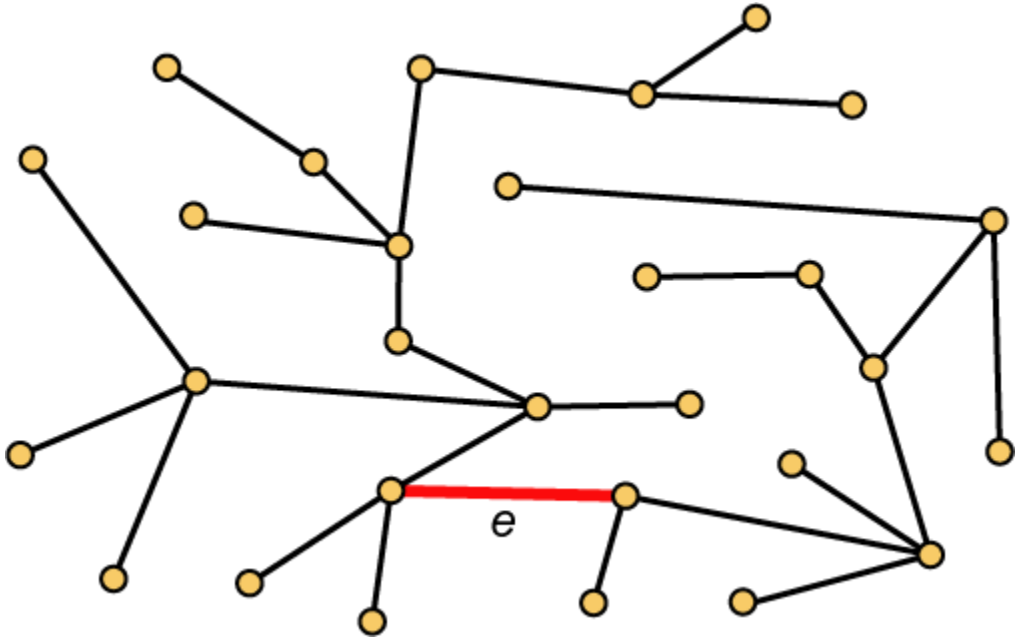
Observation Consider any edge e not in the tree T. Then there is a unique path P in T from one endpoint of e to the other.

The Exchange Principle (3)



Remark Consider any edge f from T which belongs to the path P.

The Exchange Principle (4)



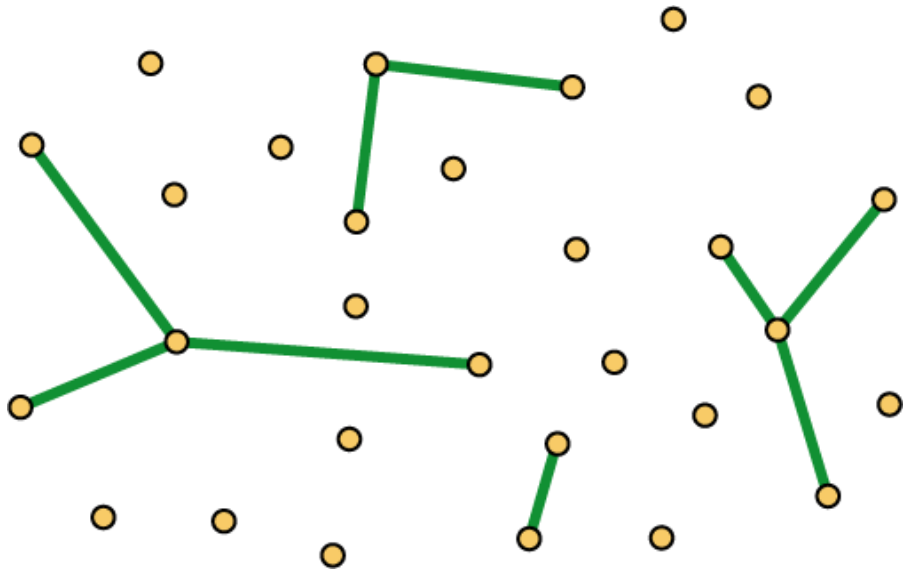
Important Fact Then $T' = T - f + e$ is again a tree.

A New Kind of Vector Space

Mathematical Framework Let G be a connected graph, and let E be the edge set of G. We consider subsets of E and call a subset S independent if it contains no cycles. Note that the maximal independent sets are the spanning trees of G.

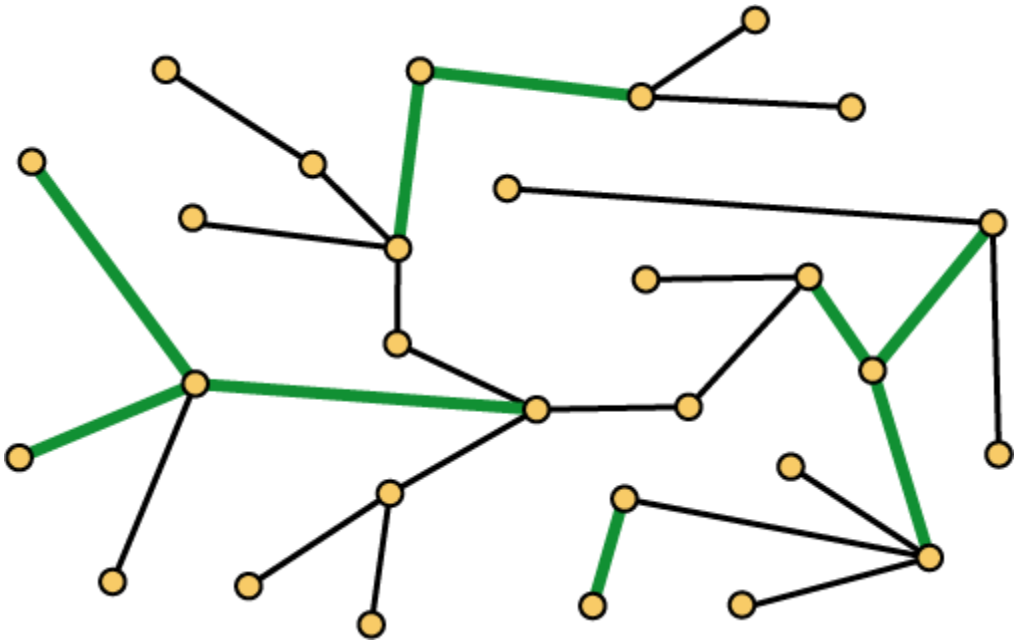
Note In combinatorial mathematics, a family of independent sets satisfying an exchange property is called a matroid, and these structures are studied extensively, just like graphs and posets.

Constrained Spanning Trees



Modified Problem Find the minimum weight spanning tree with some choices made by management. These choices may or may not be good ones??!!

Constrained Spanning Trees

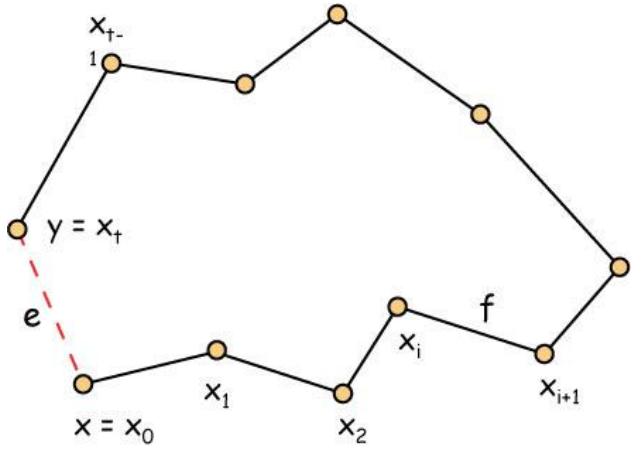


Remark Neither of our two algorithms has a provision for "starting with a handicap.

Fundamental Lemma

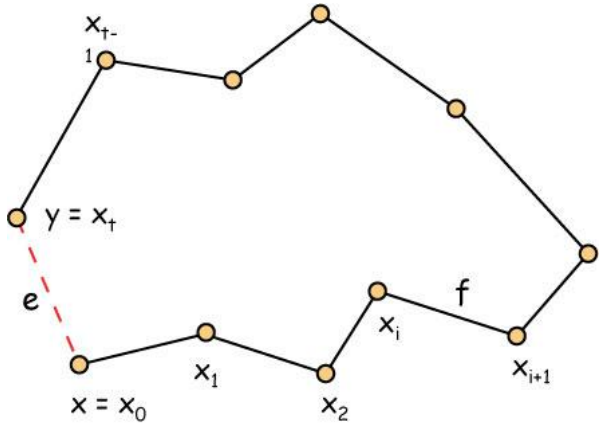
Lemma Let G be a graph with non-negative weights on the edges, let F be a spanning forest of G and let C be a component of F. Also, among all edges with one endpoint in C and the other not in C, let edge e be one of minimum weight. Then among all the spanning trees of G that contain F, there is one of minimum weight that contains the edge e.

Applying the Exchange Principle



Proof Suppose the lemma is false and let T be a spanning tree of minimum weight among all that contain the spanning forest F. Then we know that e is not an edge in T. Let e = xy with x a vertex in the component C. there are none of minimum weight containing the edge e. Then consider the unique path $x = x_0$, $x_1, \ldots, x_t = y$ in T.

Applying the Exchange Principle (2)



Proof (continued) Let i be the least integer so that x_i is in the component C (together with x) while x_{i+1} is not in C. Then let $f = x_i x_{i+1}$. Since S is optimal, and $S - f + e$ is a spanning tree, we know that $w(f) \leq w(e)$. But by the rule used in the selection of e, we know $w(e) \leq w(f)$. So $w(e) = w(f)$. Thus $T - f + e$ has the same weight as T and contains e. The contradiction completes the proof.

The Correctness of Kruskal's algorithm

Proof We proceed by induction on the number of edges specified by management, except now we consider management as both benevolent and enlightened. At step i when i edges have already been selected, management considers the set of admissible edges (those avoiding cycles) and takes one of minimum weight. In turn, manager declares the component C to contain one of the end points of C.

Note Additional details provided in class.

The Correctness of Prim's algorithm

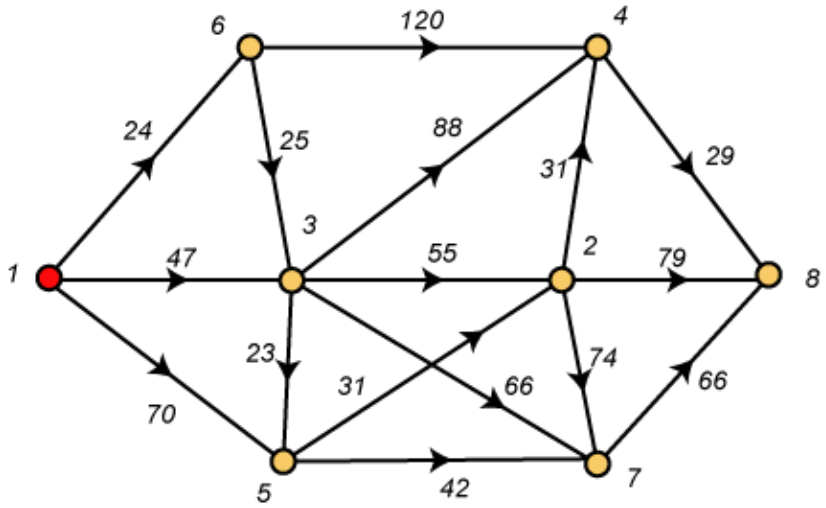
Proof We proceed by induction on the number of edges specified by management, except now we consider management as both benevolent and enlightened. Now we simply take the component C as the set of edges determined by all preceding choices, with the initial case being C as just the root vertex of the graph.

Note Additional details provided in class.

Data Structure and Computational Issues

1. Implementing Kruskal's Algorithm seems to require sorting the edges by weight as a preliminary step. Are there alternatives?
2. Implementing Prim's algorithm seems to require keeping track of the edge of minimum weight with one endpoint in a component, but the components are changing. How does one do this?

Finding Shortest Paths



Problem Given a connected digraph G with non-negative weights on the edges and a root vertex r, find for each vertex x, a directed path $P(x)$ from r to x so that the sum of the weights on the edges in $P(x)$ is as small as possible.

Dijkstra's Algorithm

1. At each step, and each vertex x, keep track of a "distance" $d(x)$ and a directed path $P(x)$ from root to vertex x of length $d(x)$.
2. Scan first from the root and take initial paths $P(r,x) = (r, x)$ with $d(x) = w(rx)$ when rx is an edge, and $d(x) = \infty$ when rx is not an edge.
3. Vertices are either "permanent" or "temporary". At first, the root r is the only permanent vertex. For each permanent vertex x, the current value of $d(x)$ is the length of a shortest path from r to x.

Dijkstra's Algorithm - The Inductive Step

1. Of all temporary vertices, choose one, say x, whose "distance" to the root is minimum. Mark it permanent.
2. For each temporary vertex y distinct from x, set

   $$d(y) = \min\{d(y), d(x)+w(xy)\}$$

3. If this assignment lowers the value of $d(y)$, set $P(y)$ to be the path obtained by appending y to the end of $P(x)$

Initialization

Remark Initially, the paths are just the edges (if they exist) joining each vertex to the root. If there is no edge, take the length of the edge to be infinite. The sequence of permanent vertices is the trivial sequence (1).

1. d(1) = 0; P(1) = (1)
2. d(2) = ∞; P(2) = (1,2)
3. d(3) = 47; P(3) = (1,3)
4. d(4) = ∞; P(4) = (1,4)
5. d(5) = 70; P(5) = (1,5)
6. d(6) = 24; P(6) = (1,6)
7. d(7) = ∞; P(7) = (1,7)
8. d(8) = ∞; P(8) = (1,8)

Step 1

Among the temporary vertices, choose one closest to the root. This is vertex 6. Make vertex 6 permanent. Scan edges from 6 and see edge (6,3) of weight 25 and edge (6,4) of weight 120. This results in improvements for vertex 4 but not for 3.

1. d(1) = 0; P(1) = (1)
2. d(2) = ∞; P(2) = (1,2)
3. d(3) = 47; P(3) = (1,3)
4. d(4) = ∞; P(4) = (1,4) New d(4) = 144; P(4) = (1,6,4)
5. d(5) = 70; P(5) = (1,5)
6. d(6) = 24; P(6) = (1,6)
7. d(7) = ∞; P(7) = (1,7)
8. d(8) = ∞; P(8) = (1,8)

Step 2

Among the temporary vertices, choose one closest to the root. This is vertex 3. Make vertex 3 permanent. Scan edges from 3 and see edge (3,2) of weight 55, edge (3,4) of weight 88, edge (3,5) of weight 23 and edge (3,7) of weight 66. These edges result in improvements for vertices 2, 4 and 7 but not 5 (where there is a tie).

1. d(1) = 0; P(1) = (1)
2. d(2) = ∞; P(2) = (1,2) New d(2) = 102; P(2) = (1,3,2)
3. d(3) = 47; P(3) = (1,3)
4. d(4) = 144; P(4) = (1,6,4) New d(4) = 135; P(4) = (1,3,4)
5. d(5) = 70; P(5) = (1,5)
6. d(6) = 24; P(6) = (1,6)
7. d(7) = ∞; P(7) = (1,7) New d(7) = 113; P(7) = (1,3,7)
8. d(8) = ∞; P(8) = (1,8)

Step 3

Among the temporary vertices, choose one closest to the root. This is vertex 5. Make vertex 5 permanent. Scan edges from 5 and see edge (5,2) of weight 32 and edge (5,7) of weight 42. These edges result in improvements for vertices 2 and 7.

1. d(1) = 0; P(1) = (1)
2. d(2) = 102; P(2) = (1,3,2) New d(2) = 101; P(2) = (1,5,2)
3. d(3) = 47; P(3) = (1,3)
4. d(4) = 135; P(4) = (1,3,4)
5. d(5) = 70; P(5) = (1,5)
6. d(6) = 24; P(6) = (1,6)
7. d(7) = 113; P(7) = (1,3,7) New d(7) = 112; P(7) = (1,5,7)
8. d(8) = ∞; P(8) = (1,8)

Step 4

Among the temporary vertices, choose one closest to the root. This is vertex 2. Make vertex 2 permanent. Scan edges from 2 and see edge (2,4) of weight 31, edge (2,7) of weight 74 and edge (2,8) of weight 79. These edges result in improvements for vertices 4 and 8, but not vertex 7.

1. d(1) = 0; P(1) = (1)
2. d(2) = 101; P(2) = (1,5,2)
3. d(3) = 47; P(3) = (1,3)
4. d(4) = 135; P(4) = (1,3,4) New d(4) = 132; P(4) = (1,5,2,4)
5. d(5) = 70; P(5) = (1,5)
6. d(6) = 24; P(6) = (1,6)
7. d(7) = 112; P(7) = (1,5,7)
8. d(8) = ∞; P(8) = (1,8) New d(8) = 180; P(8) = (1,5,2,8)

Step 5

Among the temporary vertices, choose one closest to the root. This is vertex 7. Make vertex 7 permanent. Scan edges from 7 and see edge (7,8) of weight 66. This edge results in an improvement for vertex 8.

1. d(1) = 0; P(1) = (1)
2. d(2) = 101; P(2) = (1,5,2)
3. d(3) = 47; P(3) = (1,3)
4. d(4) = 132; P(4) = (1,5,2,4)
5. d(5) = 70; P(5) = (1,5)
6. d(6) = 24; P(6) = (1,6)
7. d(7) = 112; P(7) = (1,5,7)
8. d(8) = 180; P(8) = (1,5,2,8) New d(8) = 178; P(8) = (1,5,7,8)

Step 6

Among the temporary vertices, choose one closest to the root. This is vertex 4. Make vertex 4 permanent. Scan edges from 4 and see edge (4,8) of weight 29. This edge results in an improvement for vertex 8.

1. d(1) = 0; P(1) = (1)
2. d(2) = 101; P(2) = (1,5,2)
3. d(3) = 47; P(3) = (1,3)
4. d(4) = 132; P(4) = (1,5,2,4)
5. d(5) = 70; P(5) = (1,5)
6. d(6) = 24; P(6) = (1,6)
7. d(7) = 112; P(7) = (1,5,7)
8. d(8) = 178; P(8) = (1,5,7,8) New d(8) = 161; P(8) = (1,5,2,4,8)

Step 7

The last temporary vertex becomes permanent. There are no edges to scan. DONE!!

1. d(1) = 0; P(1) = (1)
2. d(2) = 101; P(2) = (1,5,2)
3. d(3) = 47; P(3) = (1,3)
4. d(4) = 132; P(4) = (1,5,2,4)
5. d(5) = 70; P(5) = (1,5)
6. d(6) = 24; P(6) = (1,6)
7. d(7) = 112; P(7) = (1,5,7)
8. d(8) = 161; P(8) = (1,5,2,4,8)

The Correctness of the Algorithm (1)

Proof A very important first observation is that Dijkstra's Algorithm determines a permutation

$$\sigma = (x_1, x_2, x_3, x_4, \ldots, x_n)$$

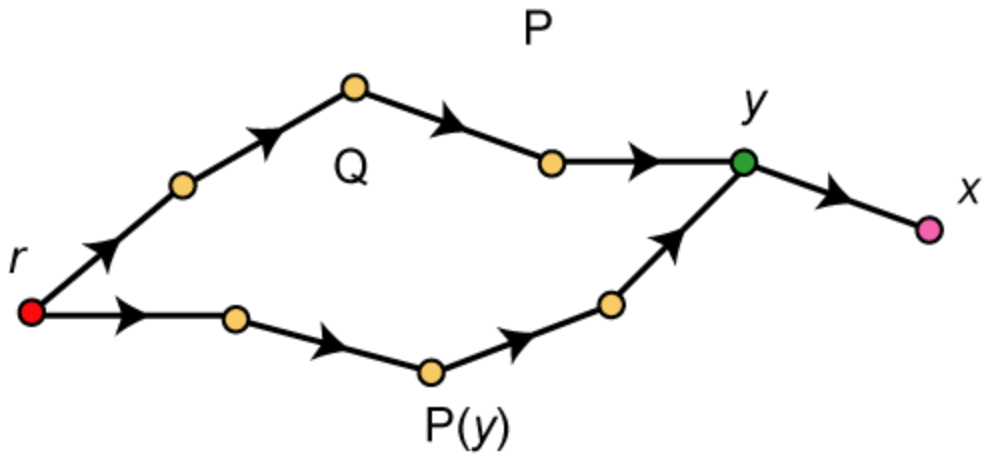
of the vertex set of the digraph according to the order in which the vertices are marked permanent. Of course, x_1 is the root r. For each vertex x_i, the algorithm has determined a path $P(x_i)$ from r to x_i having length $d(x_i)$. At this stage, it is not clear that $d(x_i)$ is really the shortest distance from r to x_i. However, we do know that these values are increasing, i.e.,

$$d(x_1) \leq d(x_2) \leq d(x_3) \leq d(x_4) \leq \ldots \leq d(x_n)$$

The Correctness of the Algorithm (2)

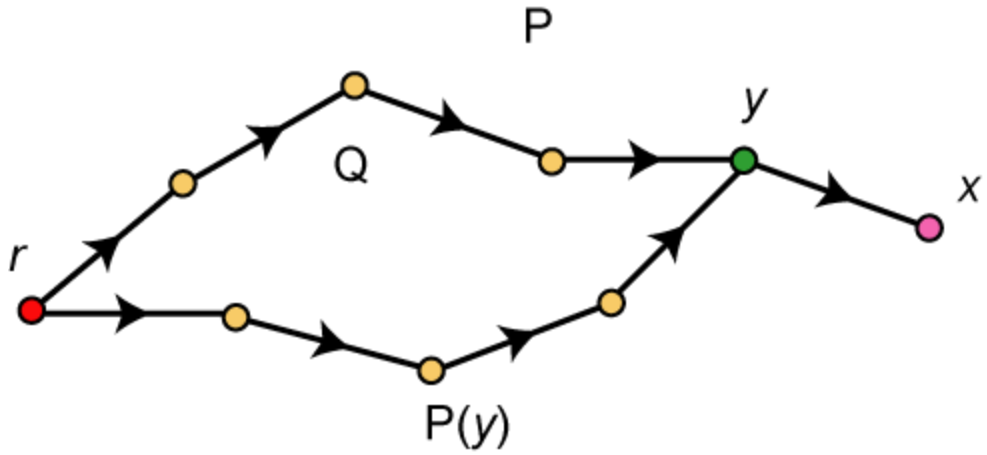
1. We show that for each vertex x, the length d(x) of the path P(x) is the shortest distance from r to x. The argument proceeds by induction on the minimum number k of edges in a shortest path from r to x. Note that the claim holds for $k = 1$, since we scan the edge (r,x) at Step 1.
2. Now assume that for some positive integer k, Dijkstra's Algorithm find a shortest path from r to x whenever the minimum number of edges in such a path is at most k. Then let x be a vertex for which the minimum number of edges in a shortest path from r to x is $k+1$. Let P be such a path and let y b the point immediately before x on P.

The Correctness of the Algorithm (3)



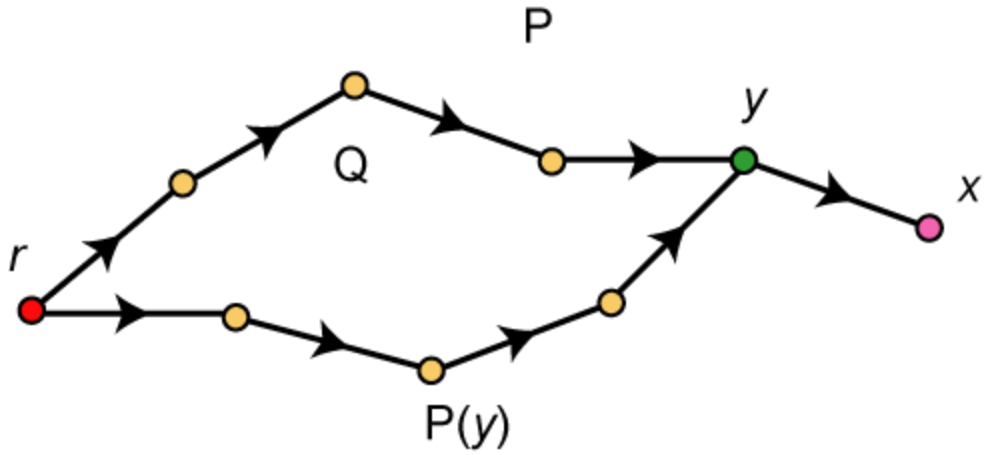
Let Q be the initial segment of P beginning at *r* and ending at *y*. Then Q is a shortest path from *r* to *y*, so the minimum number of edges in a shortest path from *r* to *y* is at most *k*. Therefore Dijkstra's algorithm finds a shortest path P(y) from *r* to *y*. Note that P(y) need not be the same as Q. However, Q and P(y) both length d(y).

The Correctness of the Algorithm (4)



The length of path P is $d(y) + w(x,y) \geq d(y)$ since all weights are non-negative. If x is marked permanent before y, then the algorithm has already found a path $P(x)$ from r to x of length $d(x) \leq d(y)$. This implies that $d(x) = d(y)$ and $w(x,y) = 0$. So the algorithm has found a shortest length path from r to x, albeit one where the last edge is "free", i.e., has weight zero.

The Correctness of the Algorithm (5)



Now suppose that y is marked permanent before x. When we scan from y, we will see the edge (x,y) having weight w(x,y). Therefore, $d(x) \leq d(y) + w(x,y)$, i.e., the algorithm will find a shortest path from r to x. This observation completes the proof.

Data Structure and Computational Issues

1. Dijkstra's Algorithm has modest space requirements since we only maintain information about the candidate optimal path.
2. If the graph has n vertices, each iteration marks a new vertex as permanent, so there are only n iterations. Also, each scan involves $O(n)$ calculations. So the running time is $O(n^2)$. In fact, the running time is essentially the same as the time it takes to read the data.