September 8, 2015

Math 3012 – Applied Combinatorics
Lecture 7

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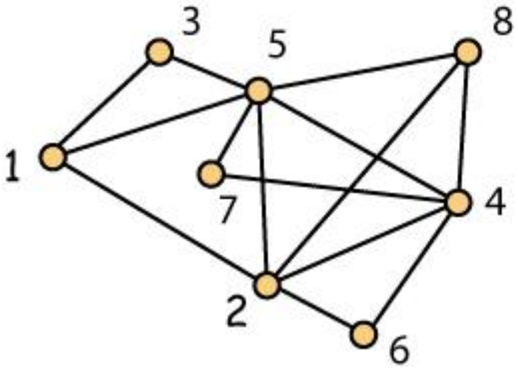
Test 1 and Homework Due Date

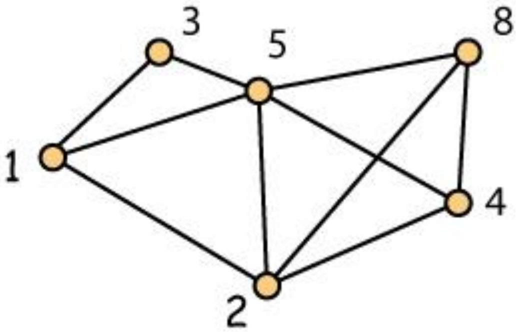
Reminder Test 1, Thursday September 17, 2015. Taken here in MRDC 2404. Final listing of material for test will be made via email after class on Thursday, September 10.

Homework Due Date Tuesday, September 15, 2015. Papers will be returned with tests – with a target of Tuesday, September 22, 2015. Scores posted on T-Square.

Induced Subgraphs

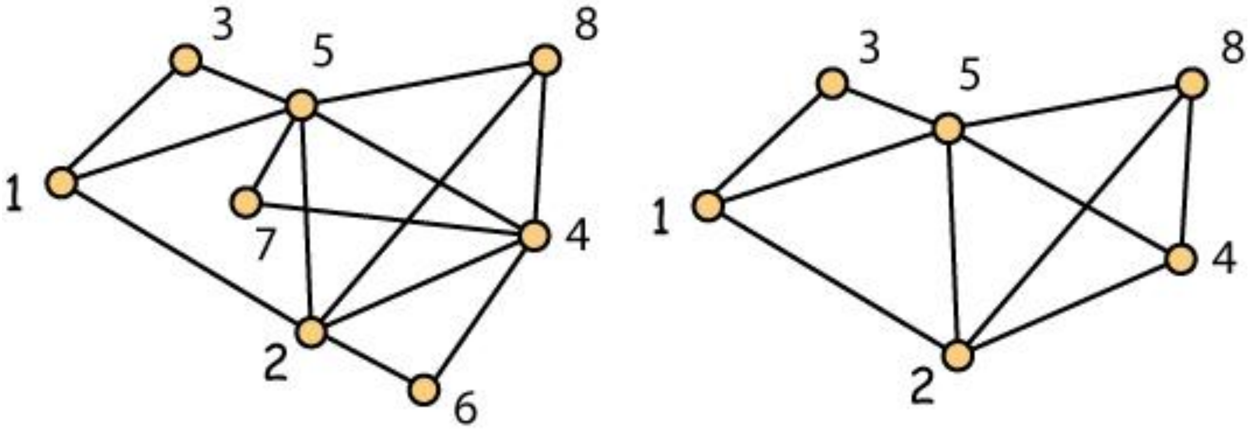
Definition A graph $H = (V', E')$ is an induced subgraph of a graph $G = (V, E)$ if $V' \subseteq V$ and xy is an edge in H whenever x and y are distinct vertices in V' and xy is an edge in G. In the drawing below, the graph on the right is an induced subgraph of the graph on the left.





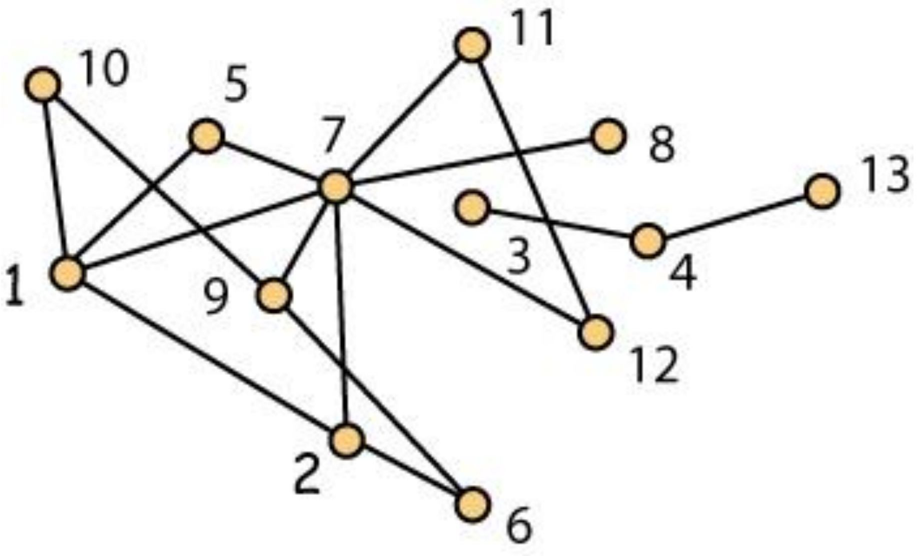
Induced Subgraphs (2)

Remark When $G = (V, E)$ is a graph, an induced subgraph of G is determined entirely by its vertex set, so for example, the induced subgraph on the right can be denoted as $G - \{6, 7\}$.



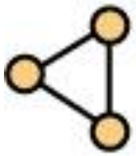
Cut Vertices

Definition A vertex x in a graph G is called a cut vertex of G if the induced subgraph $G - x$ has more components than G. In the graph shown below, 4 and 7 are cut vertices.

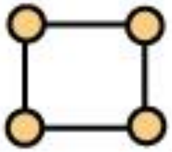


Special Classes of Graphs

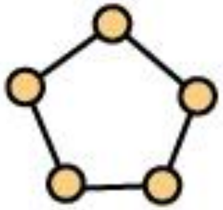
Definition For $n \geq 3$, C_n denotes a cycle on n vertices. Here are C_3, C_4, C_5 and C_6.



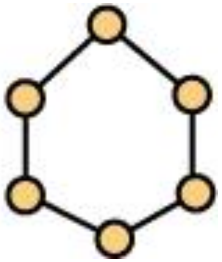
C_3



C_4



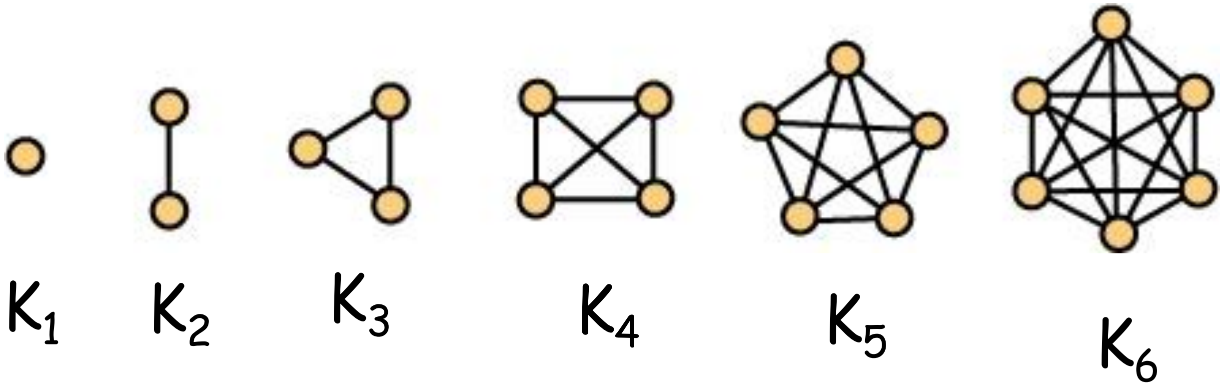
C_5



C_6

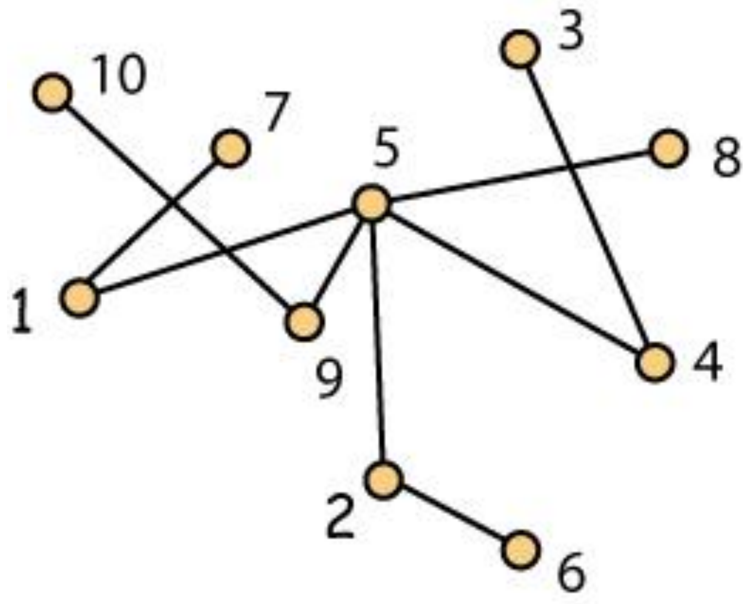
Special Classes of Graphs (2)

Definition For $n \geq 1$, K_n denotes a complete graph (also called a clique) on n vertices. Here are K_1, K_2, K_3, K_4, K_5 and K_6.



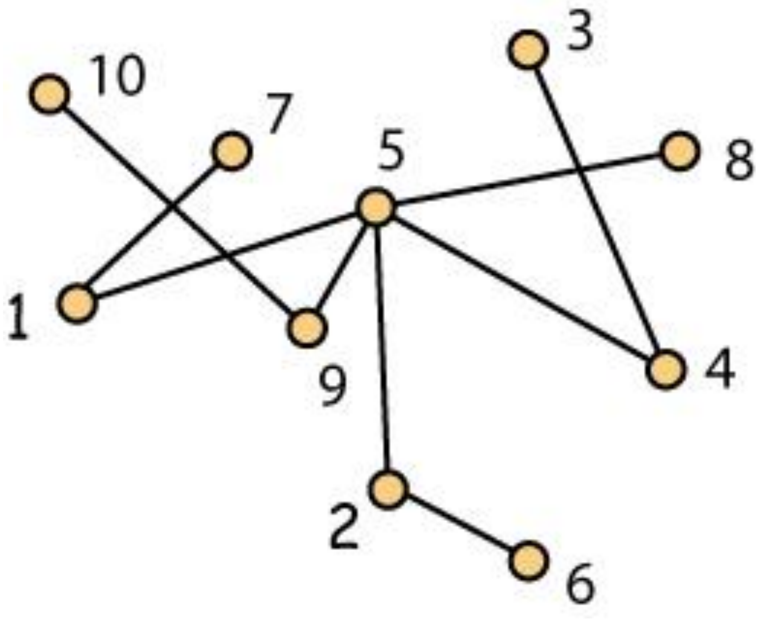
Special Classes of Graphs (3)

Definition A graph G on n vertices is a tree if G is connected and contains no cycles.



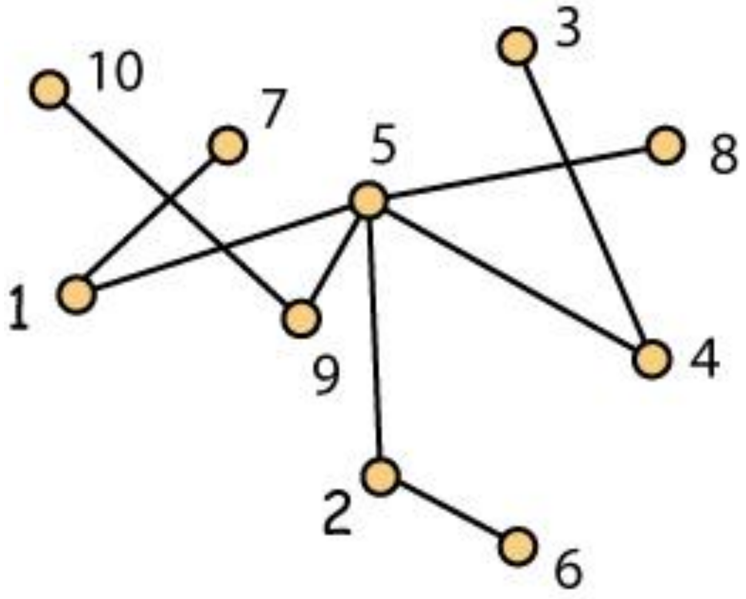
Properties of Trees

Definition When T is a tree, a vertex of degree 1 is called a leaf. This tree has five leaves: 3, 6, 7, 8, 10. The other vertices are cut vertices.



Properties of Trees (2a)

Theorem When T is a tree on n vertices and $n \geq 2$, then T has at least two leaves.



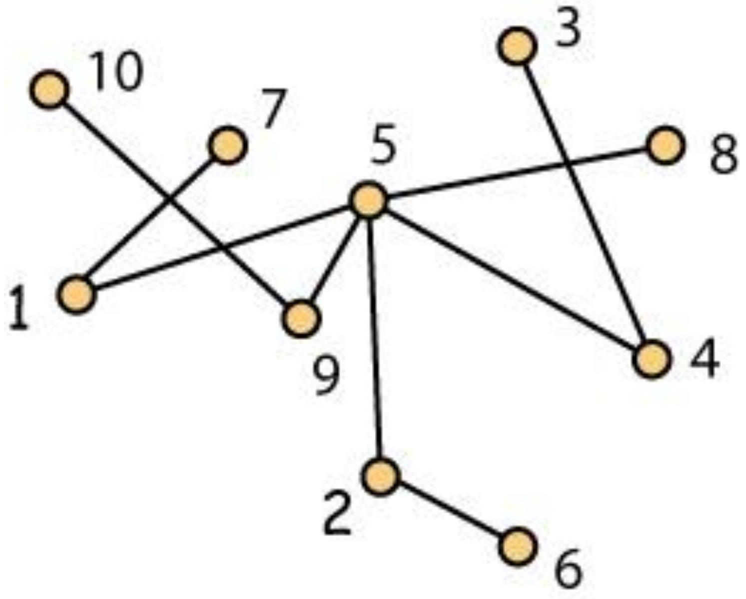
Properties of Trees (2b)

Theorem When T is a tree on n vertices and $n \geq 2$, then T has at least two leaves.

Proof Induction on n. Obviously true when $n = 2$. Assume valid when T is a tree with at least 2 and at most k vertices for some $k \geq 2$. Then let T be a tree with $k + 1$ vertices. Then $k + 1 \geq 3$, so if T does not have 3 leaves, it has a cut vertex x. It follows that if C is a component of $G - x$, then $C + x$ is a tree and has at least 2 leaves. One of these is distinct from x and is therefore a leaf in T.

Properties of Trees (3a)

Theorem When T is a tree on n vertices, T has n – 1 edges.



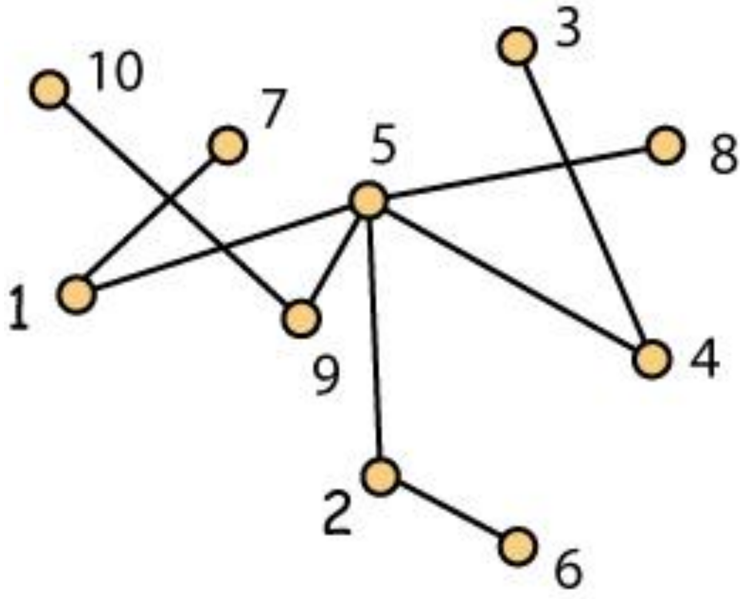
Properties of Trees (3b)

Theorem When T is a tree on n vertices, T has $n - 1$ edges.

Proof Induction on n. True when $n = 1$. Now assume valid when $n = k$ for some integer $k \geq 1$. Then let T be a tree on $k + 1$ vertices. Choose a leaf x (there are at least two from which to choose). Then $\deg(x) = 1$ while the tree $T - x$ has k vertices and $k - 1$ edges. Therefore T has $(k - 1) + 1 = k$ edges.

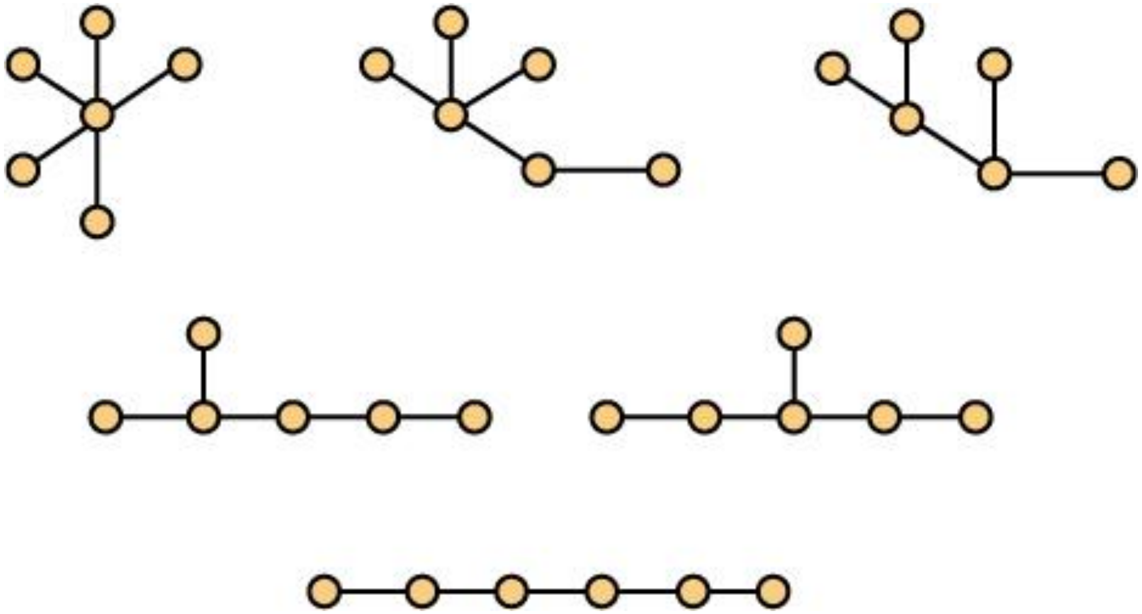
Paths and Trees

Theorem When T is a tree, T is a path unless it has more than two leaves.



Counting Trees

Exercise Explain why there are 6 unlabelled trees on 6 vertices. They are shown below.



The Unlabelled Trees on 6 Vertices

Exercise Show that when $1 \leq n \leq 6$, the number of trees with vertex set $\{1, 2, \ldots, n\}$ is n^{n-2}. Actually, we did the work when $1 \leq n \leq 5$ in class, so all you really have to do is the case $n = 6$.

Remark Later in the course, we will show that this is true for all $n \geq 1$.

Trails and Circuits in Graphs

Definition A sequence $(x_1, x_2, x_3, \ldots, x_t)$ of vertices is called a trail (also a walk) in a graph G if for every $i = 1, 2, \ldots, t-1$, $x_i x_{i+1}$ is an edge of G.

Note We do not require that the vertices in the sequence be distinct. This is what differentiates a trail from a path.

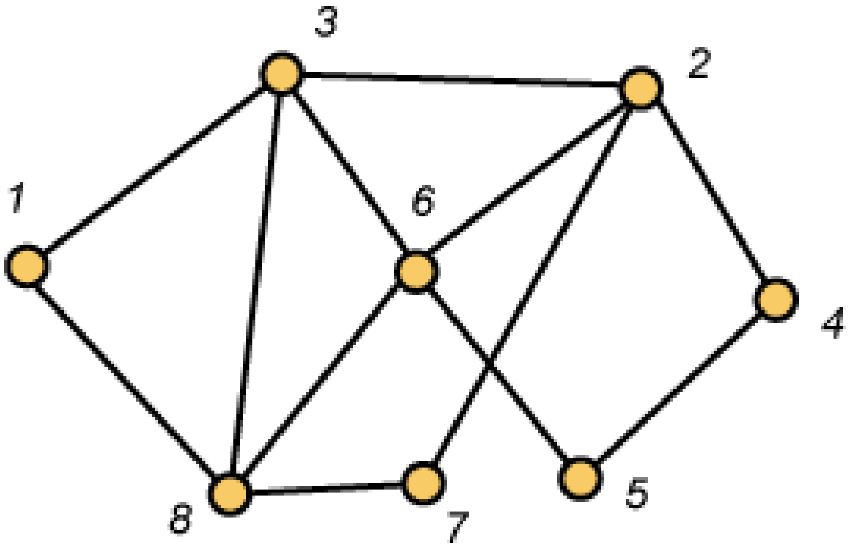
Definition A trail $(x_1, x_2, x_3, \ldots, x_t)$ *is called a* *circuit* *if* $x_t x_1$ *is also an edge in* G.

Euler Trails and Circuits

Definition A trail $(x_1, x_2, x_3, \ldots, x_{t)}$ in a graph G is called an Euler trail in G if for every edge e of G, there is a unique i with $1 \leq i < t$ so that $e = x_i x_{i+1}$.

Definition A circuit $(x_1, x_2, x_3, \ldots, x_t)$ *in a graph* G *is called an* *Euler circuit* *if for every edge* e *in* G, *there is a unique* i *with* $1 \leq i \leq t$ so that $e = x_i x_{i+1}$. Note that in this definition, we intend that $x_t x_{t+1} = x_t x_1$.

Euler Circuits in Graphs



Here is an Euler circuit for this graph:

(1,8,3,6,8,7,2,4,5,6,2,3,1)

Euler's Theorem

Theorem A non-trivial connected graph G has an Euler circuit if and only if every vertex has even degree.

Theorem A non-trivial connected graph has an Euler trail if and only if there are exactly two vertices of odd degree.

Algorithm for Euler Circuits

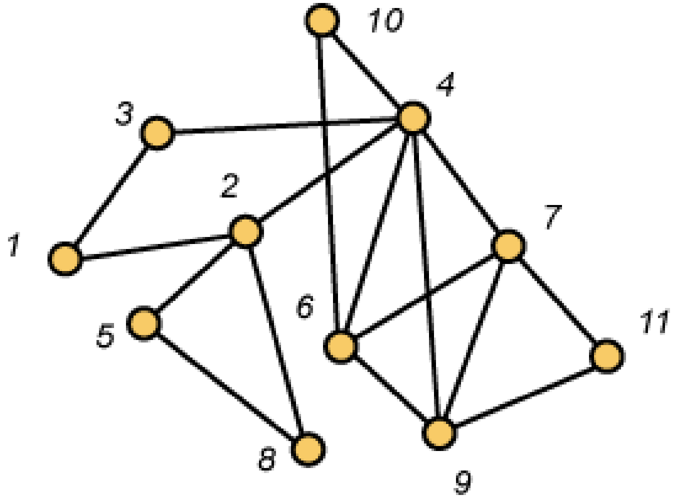
1. Choose a root vertex r and start with the trivial partial circuit (r).

2. Given a partial circuit $(r = x_0, x_1, \dots, x_t = r)$ that traverses some but not all of the edges of G containing r, remove these edges from G. Let i be the least integer for which x_i is incident with one of the remaining edges. Form a greedy partial circuit among the remaining edges of the form $(x_i = y_0, y_1, \dots, y_s = x_i)$.

3. Expand the original circuit by setting

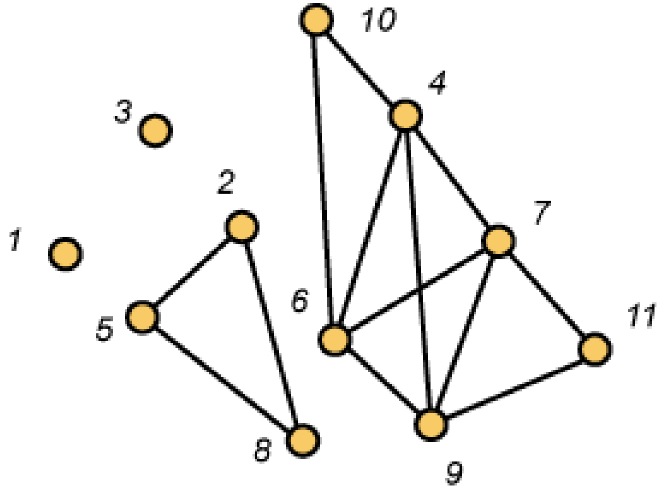
   $r = (x_0, x_1, \dots, x_{i-1}, x_i = y_0, y_1, \dots, y_s = x_i, x_{i+1}, \dots, x_t = r)$

An Example



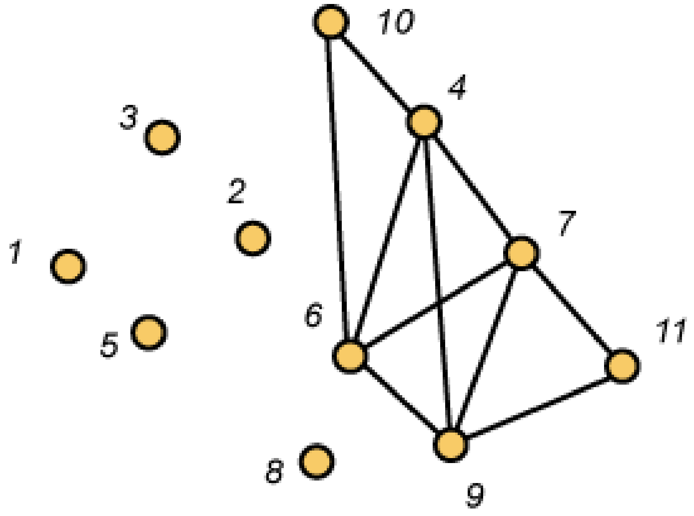
Start with the trivial circuit (1). Then the greedy algorithm yields the partial circuit (1,2,4,3,1).

Remove Edges and Continue



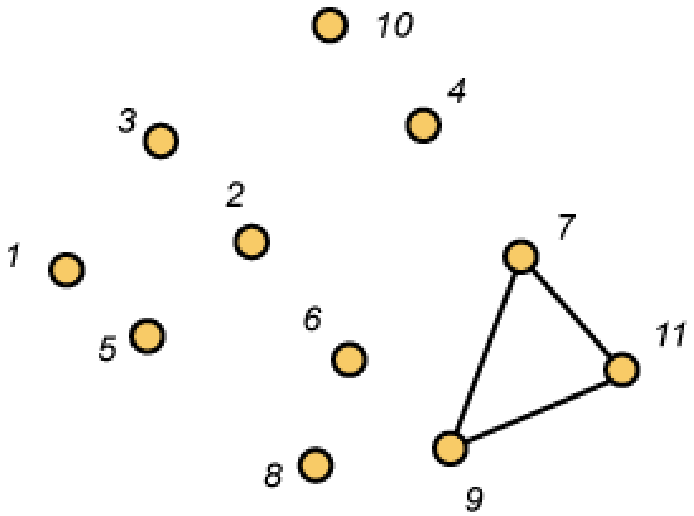
Start with the partial circuit (1,2,4,3,1).
First vertex incident with an edge remaining is 2. A greedy approach yields (2,5,8,2).
Expanding, we get the new partial circuit (1,2,5,8,2,4,3,1)

Remove Edges and Continue



Start with the partial circuit (1,2,5,8,2,4,3,1). First vertex incident with an edge remaining is 4. A greedy approach yields (4,6,7,4,9,6,10,4). Expanding, we get the new partial circuit (1,2,5,8,2,4,6,7,4,9,6,10,4,3,1)

Remove Edges and Continue



Start with the partial circuit (1,2,5,8,2,4,6,7,4,9,6,10,4,3,1) First vertex incident with an edge remaining is 7. A greedy approach yields (7,9,11,7). Expanding, we get the new partial circuit (1,2,5,8,2,4,6,7,9,11,7,4,9,6,10,4,3,1). This exhausts the edges and we have an euler circuit.

Interpreting Halting Conditions

Remark Suppose any loop halts with a starting vertex x and a terminating vertex y which is distinct from x. The conclusion is that y has odd degree. If we are searching for an Euler circuit, there isn't one. End of story. But if we are willing to accept an Euler trail, start over with y as root.

Remark If we halt with another odd pair, then there's not even an Euler trail.

Remark If we halt, there are unvisited edges and there's no place to start the next loop, then the graph has two non-trivial components.

Data Structure Issues

Remark When we read the data for the graph, we must build for each vertex x a structure that keeps track of the neighbors of x. As the algorithm progresses, we must keep track of the neighbors of x for which we have already walked on the edge xy. So either we have to "flag" edges already visited or have a convenient way to delete them from the neighborhood.

Remark The Greedy Algorithm taught in class tries to capture the spirit of these complexities, but in fact an actual implementation might follow a quite different track.

Hamiltonian Paths and Cycles

Definition When G is a graph on $n \geq 3$ vertices, a cycle $C = (x_1, x_2, \ldots, x_n)$ in G is called a Hamiltonian cycle, i.e, the cycle C visits each vertex in G exactly one time and returns to where it started.

Definition When G is a graph on $n \geq 3$ vertices, a path $P = (x_1, x_2, \ldots, x_n)$ in G is called a Hamiltonian path, i.e, the path P visits each vertex in G exactly one time. In contrast to the first definition, we no longer require that the last vertex on the path be adjacent to the first.

Hamiltonian Paths and Cycles (2)

Remark In contrast to the situation with Euler circuits and Euler trails, there does not appear to be an efficient algorithm to determine whether a graph has a Hamiltonian cycle (or a Hamiltonian path). For the moment, take my word on that but as the course progresses, this will make more and more sense to you.

Hamiltonian Paths and Cycles (2)

Remark In contrast to the situation with Euler circuits and Euler trails, there does not appear to be an efficient algorithm to determine whether a graph has a Hamiltonian cycle (or a Hamiltonian path). For the moment, take my word on that but as the course progresses, this will make more and more sense to you.