**Reminder**   Test 1, Thursday September 17, 2015. Taken here in MRDC 2404. Final listing of material for test will be made via email after class on Thursday, September 10.

**Homework Due Date**   Tuesday, September 15, 2015. Papers will be returned with tests - with a target of Tuesday, September 22, 2015. Scores posted on T-Square.
Hamiltonian Paths and Cycles

**Definition** When $G$ is a graph on $n \geq 3$ vertices, a cycle $C = (x_1, x_2, \ldots, x_n)$ in $G$ is called a **Hamiltonian cycle**, i.e., the cycle $C$ visits each vertex in $G$ exactly one time and returns to where it started.

**Definition** When $G$ is a graph on $n \geq 3$ vertices, a path $P = (x_1, x_2, \ldots, x_n)$ in $G$ is called a **Hamiltonian path**, i.e., the path $P$ visits each vertex in $G$ exactly one time. In contrast to the first definition, we no longer require that the last vertex on the path be adjacent to the first.
Hamiltonian Paths

**Question** Does the graph shown below have a Hamiltonian path?
Answer: Yes!

(12, 9, 17, 14, 3, 1, 15, 5, 10, 13, 16, 8, 2, 11, 7, 6, 4)
Hamiltonian Cycles

**Question** Does the graph shown below have a Hamiltonian cycle?
Answer: Yes!!

(1, 3, 14, 17, 9, 12, 7, 11, 2, 4, 6, 10, 13, 8, 16, 5, 15)
Certificates for “Yes” Answer

Remark  Given a graph $G$, a “yes” answer to the question: Does $G$ have a Hamiltonian path? can be validated by providing a certificate in the form of a permutation of the vertex set of $G$. An impartial referee (computer) can quickly check the essential details. Is every vertex listed exactly once? Are consecutive vertices adjacent in the graph?

Remark  An analogous statement applies for Hamiltonian cycles.
**Question**
Does this graph have a Hamiltonian path?

**Answer**
Yes!!

**Certificate**
(6, 3, 1, 4, 5, 2)

**Note**
The correctness of the answer can be verified quickly by an impartial referee (computer).

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**Graph_data.txt**

```
6
5 2
1 4
6 5
3 6
1 3
4 5
4 6
6 1
```
Hamiltonian Cycles (3)

**Question** Does this graph have a Hamiltonian cycle?

**Answer** Yes!!

**Certificate** (6, 3, 1, 4, 5, 2)

**Note** The correctness of the answer can be verified quickly by an impartial referee (computer).
Certificates for “No” Answer

Remark  Given a graph $G$, there does not seem to be a way to provide a certificate to validate a “no” answer to the question: Does $G$ have a Hamiltonian cycle?"  To be more precise, there does not seem to be a way to provide an impartial referee (computer) with information which can be effectively checked and will satisfy the referee that your answer is correct.
Question  Does this graph have a Hamiltonian cycle?

\[\text{Graph\_data.txt}\]

\begin{verbatim}
6
5 2
1 4
6 5
3 6
1 3
4 5
4 6
6 1
\end{verbatim}

Answer  No!!

Certificate  Vertex 2 has degree 1. If a graph has a Hamiltonian cycle, every vertex has degree at least 2.

Note  The correctness of the answer can be verified quickly by an impartial referee (computer).
Remark  Given a graph $G$, there does not seem to be a way to provide a certificate to validate a “no” answer to the question: Does $G$ have a Hamiltonian cycle? To be more precise, there does not seem to be a way to provide an impartial referee (computer) with information which can be effectively checked and will satisfy the referee that your answer is correct, at least not in general. This does not preclude there being a justification for a “no” answer in some cases.
A Very Informal Perspective  The class $\mathbf{P}$ consists of all “yes-no” questions for which the answer can be determined using an algorithm which is provably correct and has a running time which is polynomial in the input size.

Examples

1. Given a list of $n$ numbers, is 2388643 in the list?
2. Given a list of $n$ numbers, can you find three distinct numbers $a$, $b$ and $c$ in the list so that $a + b = c$?
3. Given a graph $G$, does it have an Euler circuit?
A Very Informal Perspective  The class $\text{NP}$ consists of all “yes-no” questions for which there is a certificate for a “yes” answer whose correctness can be verified with an algorithm whose running time is polynomial in the input size. Any question in $\text{P}$ is also in $\text{NP}$.

Examples

1. Given a list of $n$ numbers, is there a fair division?
2. Given a graph $G$, is there a clique whose size is at least $n/2$?
3. Given a graph $G$, does it have a Hamiltonian cycle?
Observation  As we have already noted, any problem which is in $P$ is also in $NP$, but no one knows whether the converse statement is true or not. The current reward for settling this question:

\[ P = NP ? \]

Stands at $1,000,000$ USD.

http://www.claymath.org/millennium-problems
Remark  Given a graph $G$, a “no” answer to the question: Does $G$ have an Euler circuit?“ can be validated by providing a certificate. Now this certificate is one of the following. Either the graph is not connected, so the referee is told of two specific vertices for which the graph does not contain a path between them. On the other hand, if the graph is connected, then the referee is told that there is vertex of odd degree.
Definition  A bipartite graph is a triple \((A, B, E)\) where \(A\) and \(B\) are disjoint finite sets and \(E\) is a collection of 2-element sets, each of which contains one element of \(A\) and one element of \(B\). In the bipartite graph shown below, \(A = \{a, b, c, d, e, f, g\}\) and \(B = \{1, 2, 3, 4, 5\}\)
**Caution** In a discussion of unlabelled bipartite graphs, care has to be exercised regarding which elements belong to $A$ and which belong to $B$. The potential for confusion is minor when the graph is connected.
Caution  But there are real problems when the graph is disconnected. For example, consider the red, blue and green points in the graph shown below. Which side are they on?
**Complete Bipartite Graphs**

**Definition**  For $m, n \geq 1$, the complete bipartite graph $K_{m, n}$ has $m + n$ vertices, with $m$ on one side and $n$ on the other. There are $mn$ edges in $K_{m, n}$, i.e., each vertex on one side is adjacent to every vertex on the other. Here is a drawing of $K_{7, 5}$. 
Observation  If a bipartite graph $G = (A, B, E)$ has a Hamiltonian cycle, then it is connected and $|A| = |B|$.
**Observation**  In particular, the complete bipartite graph $K_{n, n+1}$ does not have a Hamiltonian cycle, even though every vertex is adjacent to (nearly) half the other vertices.
Dirac’s Theorem

**Theorem** If $G$ is a graph on $n$ vertices and every vertex in $G$ has at least $n/2$ neighbors, then $G$ has a Hamiltonian cycle.

**Note** The complete bipartite graph $K_{n, n+1}$ has $2n + 1$ vertices but the vertices in the larger part have only $n$ neighbors and $n < (2n + 1)/2$. 
An Algorithm to Find a Hamiltonian Cycle

Initialization: Build Long Path

Note We may assume that all the neighbors of the end (red) vertices are on the path; otherwise we get a longer path. This implies $t > 1 + n/2$. 
A Two-Phase Algorithm

Phase 1 - Turn long path into cycle of same size

Note Using the pigeon-hole principle, there are consecutive vertices \( i \) and \( i+1 \) on the path with \( \{1, i+1\} \) and \( \{i, t\} \) as edges in \( G \).
Chromatic Number

**Definition** A \( t \)-coloring of a graph \( G \) is an assignment of integers (colors) from \( \{1, 2, \ldots, t\} \) to the vertices of \( G \) so that adjacent vertices are assigned distinct colors. We show a 7-coloring of the graph below.
Optimization Problems  Given a graph $G$, what is the least $t$ so that $G$ has a $t$-coloring? This integer is called the chromatic number of $G$ and is denoted $\chi(G)$. The coloring below is the same graph but now we illustrate a 5-coloring, so $\chi(G) \leq 5$. 
Chromatic Number (3)

**Optimization Problems**  The coloring below is the same graph but now we illustrate a 4-coloring, so $\chi(G) \leq 4$. 
**Maximum Clique Size**

**Definition**  Given a graph $G$, the maximum clique size of $G$, denoted $\omega(G)$, is the largest integer $k$ for which $G$ contains a clique (complete subgraph) of size $k$.

**Trivial Lower Bound**  $\chi(G) \geq \omega(G)$ so in this case, we know $\chi(G) = \omega(G) = 4$. 

![Diagram of a graph with colored vertices]
Observation  When $n \geq 2$, the odd cycle $C_{2n+1}$ satisfies $\chi(C_{2n+1}) = 3$ and $\omega(C_{2n+1}) = 2$ so the inequality

$$\chi(G) \geq \omega(G)$$

need not be tight. In our next lecture, we will investigate this inequality in greater detail.
Computing Chromatic Number

**Computational Complexity Detail**  Given a graph $G$ and an integer $t$, the yes-no question: “Is $\chi(G) \leq t$?” belongs to the class $\text{NP}$.

**Explanation**  It is obvious that a “yes” answer has a certificate that can be checked very efficiently. The certificate is just the assignment of colors to vertices.
Chromatic Number - A Special Case

**Computational Complexity Detail**  Given a graph $G$ and an integer $t$, the yes-no question: “Is $\chi(G) \leq 2$?” belongs to the class $P$.

**Basic Idea**  It is easy to see that $\chi(G) \geq 3$ when $G$ contains an odd cycle. The algorithm we present will show that $\chi(G) \leq 2$ if and only if $G$ does not contain an odd cycle. CS students will recognize that the algorithm uses “breadth-first” search. We will revisit this concept in greater detail later in the course.
Algorithm Choose an arbitrary vertex \( x \) and color it 1. Then find all uncolored vertices that are neighbors of colored vertices and color them with 2. Pause to check if you have an edge among the vertices colored 2. If yes, there is a triangle, so \( \chi(G) \geq 3 \) and the answer is “no”. If no, find all uncolored neighbors of colored neighbors and color them 1. Pause to see if there are any edges among the vertices just colored. If yes, there is a 5-cycle in \( G \) and the answer is “no”. If yes, continue, alternating colors 1 and 2. Either the graph will be eventually 2-colored or we will find an odd cycle.
Applying the Algorithm (1)
Applying the Algorithm (2)
Applying the Algorithm (3)
Applying the Algorithm (4)

**Observation**  After several more steps, the algorithm halts with a 2-coloring of $G$. 

![Graph Diagram]
Observation  Here’s an example (using a different graph) of how the algorithm will detect an odd cycle.
Another Way to Earn a Million Bucks!!

**Computational Complexity Question**  Given a graph \( G \) and an integer \( t \), the yes-no question: “Is \( \chi(G) \leq 3? \)” belongs to the class \( \text{NP} \). Does it also belong to \( \text{P} \)?

**Remark**  As was stated explicitly in our lectures, I am not encouraging Math 3012 students to ponder on this question, as the greatest minds in the world have spent enormous amounts of time on it without success. However, it does represent just how challenging the delightful world of discrete mathematics can be.