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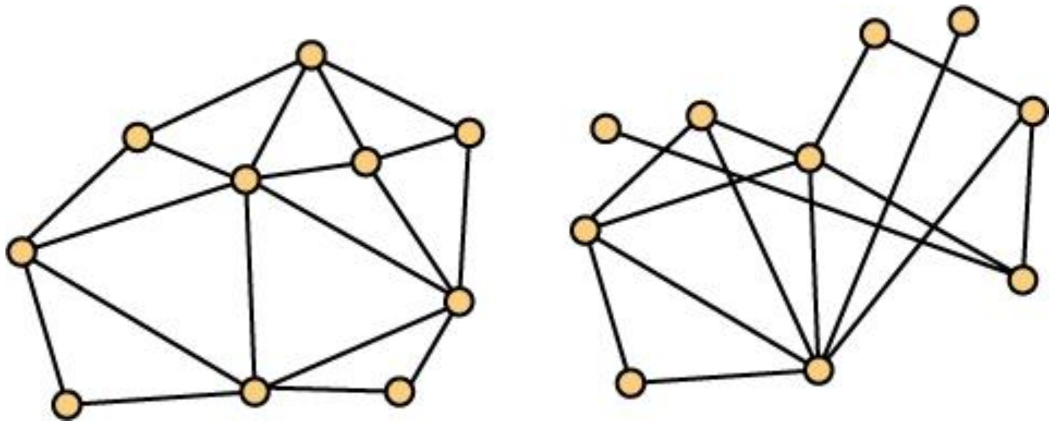
8 – Planar Graphs

William T. Trotter
trotter@math.gatech.edu

Planar Graphs

Definition A graph G is planar if it can be drawn in the plane with no edge crossings.

Exercise The two graphs shown below are both planar. Explain why.



Problems for Planar Graphs

Question Do planar graphs have any interesting properties? What can be said about maximum clique size and chromatic number? Do plane drawings exhibit any interesting properties that we should note?

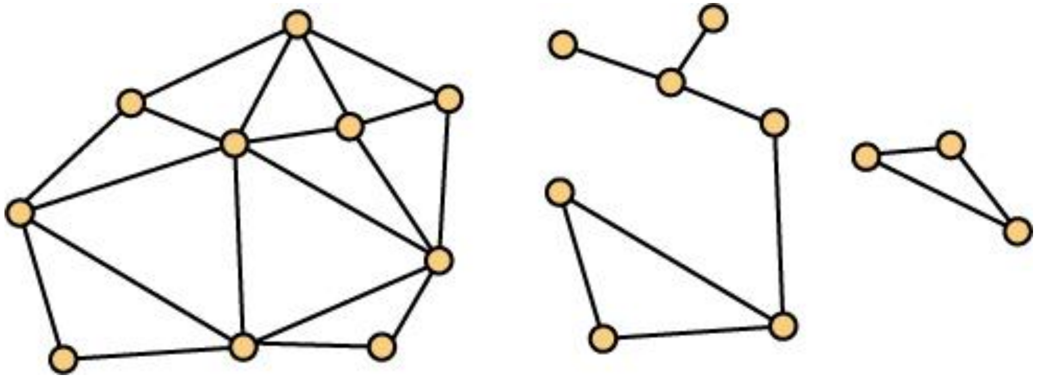
Question Given the data for a graph G, can you efficiently test whether G is planar? When the answer is yes, can you provide a "nice" plane drawing (a drawing with no crossings, straight line segments for edges and vertices positioned at points from a small grid). If no, can you provide a certificate to justify this negative response.

Euler's Formula

Theorem If n, q and f denote respectively, the number of vertices, edges and faces in a plane drawing of a planar graph with t components, then

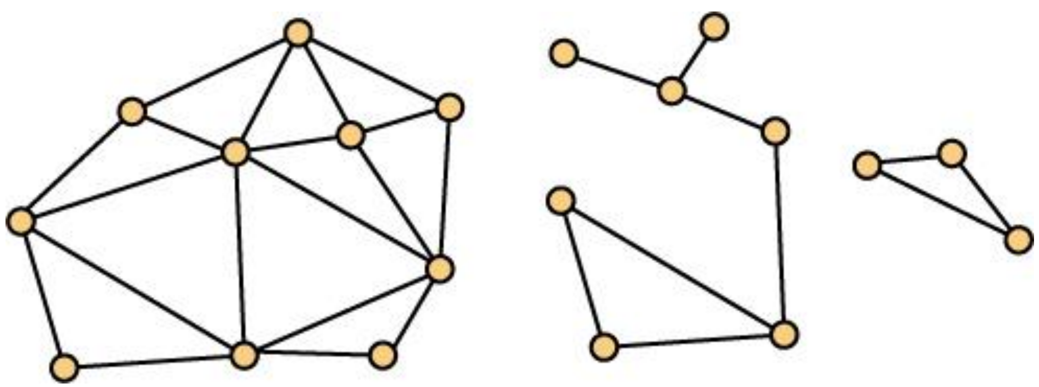
$$n - q + f = 1 + t$$

Exercise For the plane drawing shown below, $t = 3$. Determine n, q and f and show that the formula holds.



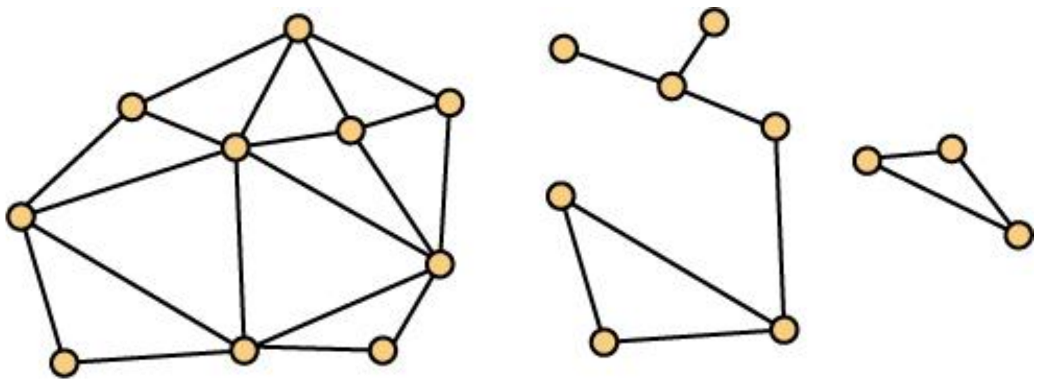
Bridges in Graphs

Definition An edge e in a graph G with t components is called a bridge when the removal of e leaves a subgraph with $t + 1$ components. In the graph shown below, $t = 3$ and there are four bridges.



2-Connected Graphs

Definition A connected graph G is said to be 2-connected when it has no bridges. In the graph shown below, there are three components. The component on the left is a 2-connected graph. So is the component on the right. But the component in the middle is not 2-connected.



Proof of Euler's Formula

Proof Fix the value of n. Then proceed by induction on q. Base case is $q = 0$. In this case, $f = 1$ and $t = n$. Check!

Inductive Step Suppose it holds when $q = k$ for some $k \geq 0$. Then suppose $q = k + 1$ edges.

Case 1 Suppose G has an edge e which is a bridge.

Case 2 G has no bridges.

Maximum Number of Edges

Theorem If G is a planar graph with $n \geq 3$ vertices and q edges, then $q \leq 3n - 6$.

Proof Fix the value of n and consider a plane drawing of a planar graph G on n vertices having the maximum number of edges. Clearly, G is connected and has no bridges so that every edge of G belongs to exactly two faces.

For each $m \geq 3$, let f_m be the number of faces whose boundary is a cycle of size m.

Maximum Number of Edges (2)

$2q = 3f_3 + 4f_4 + 5f_5 + \ldots$

$\geq 3f_3 + 3f_4 + 3f_5 + \ldots$

$= 3(f_3 + f_4 + f_5 + \ldots)$

$= 3f$ So

$3f \leq 2q$.

Multiply Euler by 3 to obtain $3n - 3q + 3f = 6$.

This implies

$6 = 3n - 3q + 3f \leq 3n - 3q + 2q = 3n - q$, so

$q \leq 3n - 6$

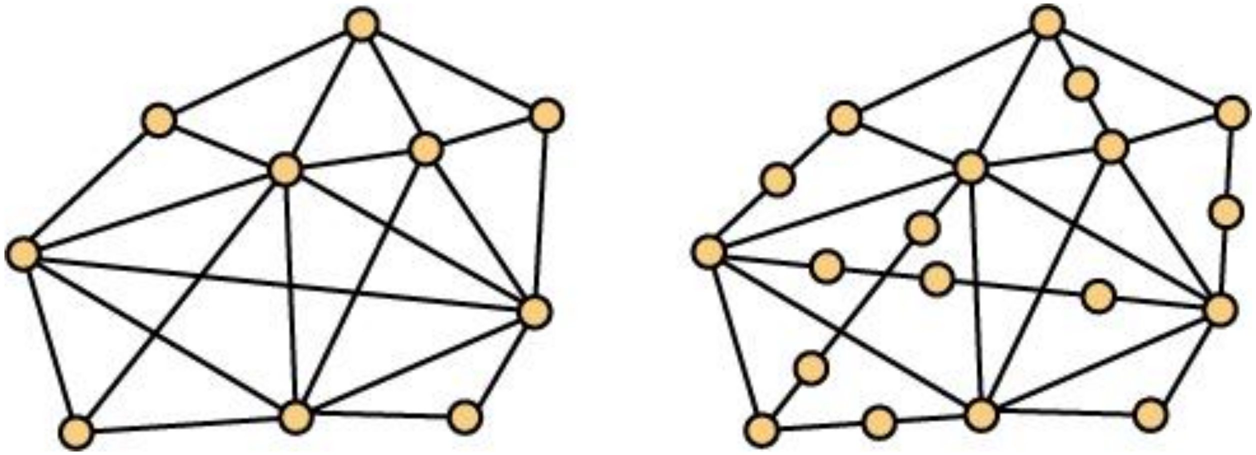
Using Euler to Determine Non-Planarity

Theorem The complete graph K_5 is non-planar.

Proof The complete graph K_5 has $n = 5$ vertices and $q = 10 = C(5, 2)$ edges. Since $10 > 3 \cdot 5 - 6 = 15 - 6 = 9$, K_5 cannot be planar.

Homeomorphs of a Graph

Definition A graph H is a homeomorph of a graph G if H is obtained by "inserting" one or more vertices on some of the edges of G. The graph on the right is a homeomorph of the graph on the left.



Homeomorphs and Planarity

Observation If a graph G is planar, then any subgraph of G is planar.

Observation If a graph H is a homeomorph of a graph G, then H is planar if and only if G is planar.

Consequence A graph is non-planar if it contains a homeomorph of the complete graph K_5 as a subgraph.

Triangle-Free Planar Graphs

Theorem If G is a triangle-free planar graph with $n \geq 3$ vertices and q edges, then $q \leq 2n - 4$.

Proof Trivially, the theorem holds when $n = 3$, so we may assume $n \geq 4$. Now consider a plane drawing of a triangle-free planar graph G on n vertices having the maximum number of edges. Clearly, G is 2-connected.

For each $m \geq 4$, let f_m be the number of faces whose boundary is a cycle of size m. Note that since G is triangle-free, $f_3 = 0$.

Triangle-Free, Maximum Number of Edges

$2q = 4\,f4 + 5\,f5 + 6f6 \ldots$

$\geq 4\,f4 + 4\,f5 + 4\,f6 + \ldots$

$= 4\,(f4 + f5 + f6 + \ldots)$

$= 4f$ So

$4f \leq 2q$ and $2f \leq q$

Multiply Euler by 2 to obtain $2n - 2q + 2f = 4$.

This implies

$4 = 2n - 2q + 2f \leq 2n - 2q + q = 2n - q$, so

$q \leq 2n - 4$.

Using Euler to Determine Non-Planarity (2)

Theorem The complete bipartite graph $K_{3,3}$ is non-planar.

Proof The complete bipartite graph $K_{3,3}$ is triangle-free and has $n = 6$ vertices and $q = 9$ edges. Since $9 > 2 \cdot 6 - 4 = 12 - 4 = 8$, $K_{3,3}$ cannot be planar.

Kuratowski's Theorem

Theorem (Kuratowski, '30) *A graph is non-planar if and only if it contains a homeomorph of the complete graph K_5 or a homeomorph of the complete bipartite graph $K_{3,3}$ as a subgraph.*

Remark Highly efficient algorithms for planarity testing are known, and they have running time which is linear in the input size. This issue remains an active research topic. These algorithms use advanced data structures and are beyond the level of our course.

Maximum Clique Size and Chromatic Number

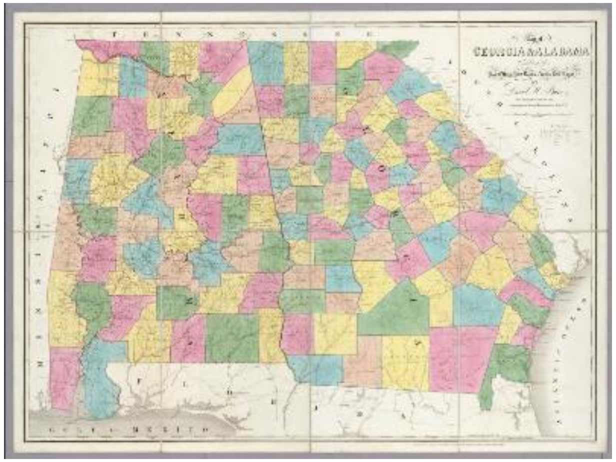
Fact Since the complete graph K_5 is non-planar, if G is a planar graph, then it has maximum clique size at most 4.

Note The following result, known as the "four color theorem" has a history spanning more than 100 years.

Theorem If G is a planar graph, then the chromatic number of G is at most 4, i.e., G can be 4-colored.

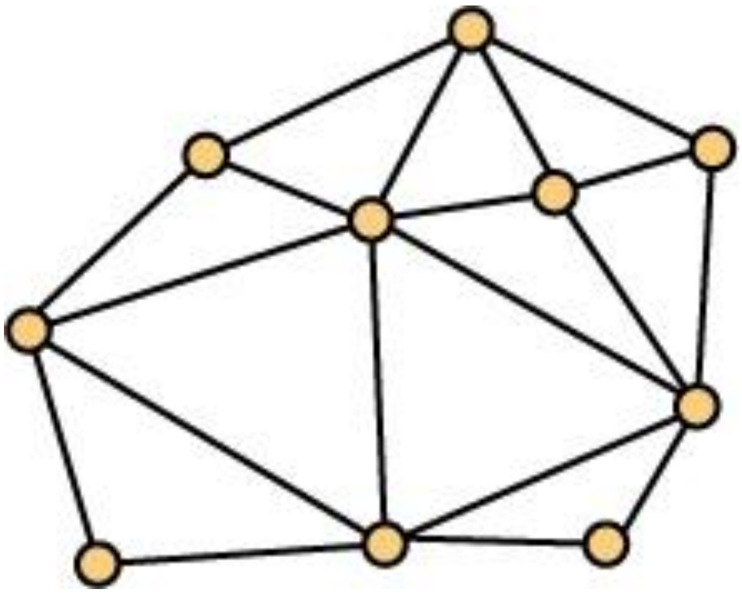
Coloring Planar Maps

Historical Note The problem of coloring a planar map so that states/countries/regions sharing a common boundary have different colors is a problem with a several hundred year history. The map shown below is the state of Georgia and is due to David H. Burr (1803 – 1875). Note that it has been 5-colored, so Mr. Burr could have done better!!!



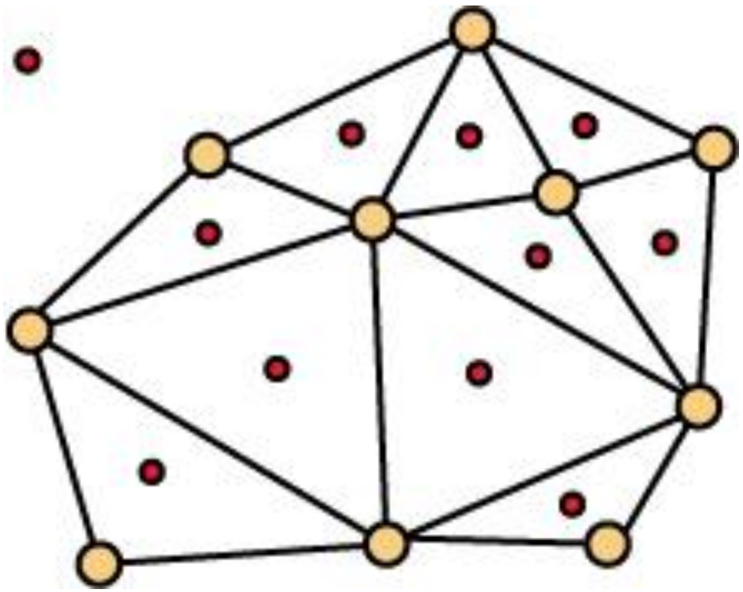
Planar Graphs and Planar Maps (1)

Observation A planar graph has a dual graph which is also planar. In the dual graph, the concepts of faces and vertices are interchanged.



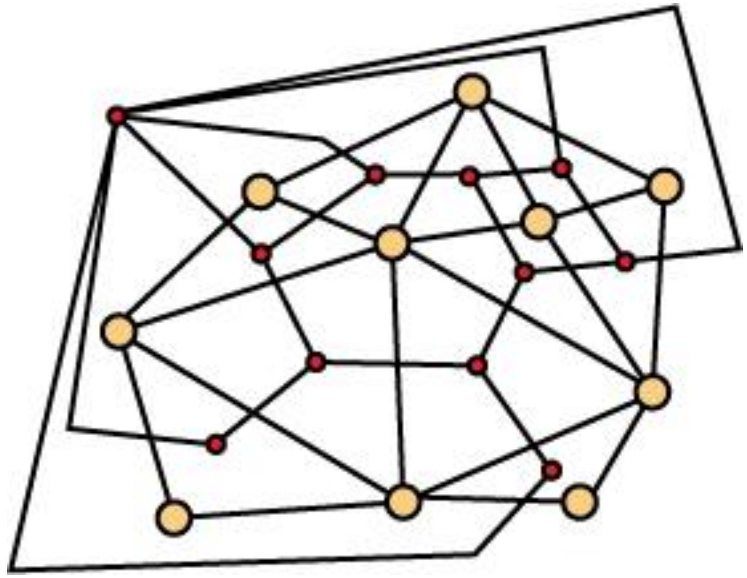
Planar Graphs and Planar Maps (2)

Observation Insert a “capital city” in each region. Don’t forget the infinite region.



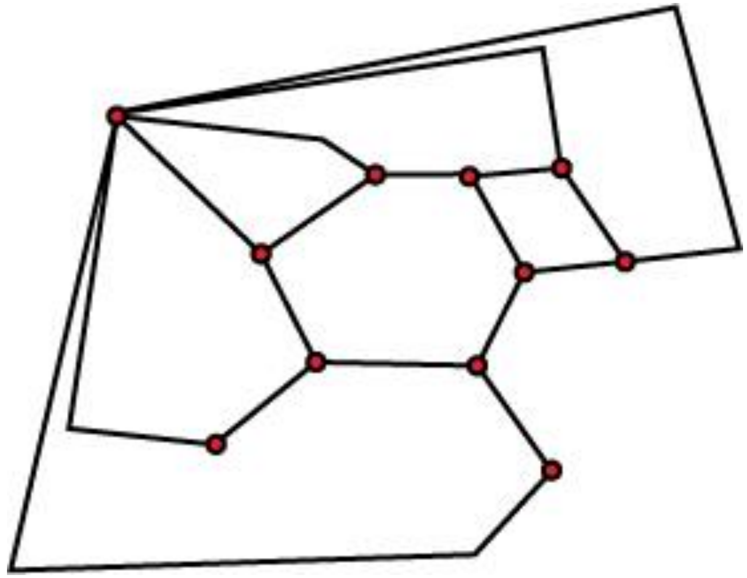
Planar Graphs and Planar Maps (3)

Observation Join two capitals with an edge when their respective regions share a boundary edge.



Planar Graphs and Planar Maps (4)

Observation Remove the original graph to obtain the dual graph.



The Four Color Theorem

Theorem (Appel and Haken, 1977) If G is a planar graph, then G can be 4-colored.

Historical Note The proof remains a controversial issue … for two reasons. First, it used extensive computing to verify certain claims. Second, some researchers are not convinced that the paper and pencil reductions to the computational stage are complete and correct.

The Four Color Theorem (2)

Follow-up Note Robertson, Sanders, Seymour and Thomas, 1996, have given a definitive, albeit still computer based proof, of the Four Color Theorem. The correctness of their programs has been verified independently by multiple sources. You can find a number of fascinating stories about this problem by doing a web search … but a word of caution that the final chapter in this story has yet to be written!! Nevertheless, the basic approach of Appel and Haken has been validated … and RSST were clear in their work about this fact.