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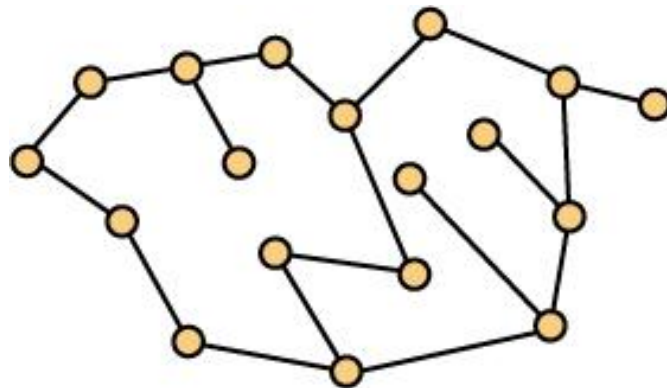
9 - Graph Theory Advanced Topics

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The Girth of a Graph

Definition A graph containing no cycles is called a **forest**. In a forest, every component is a tree. So a tree is a forest. We say that the **girth** of a forest is infinite.

Definition When G is not a forest, we define the **girth** of G as the size of the smallest cycle in G . The graph shown below has girth 8.



Chromatic Number and Girth

Observation The three constructions studied earlier produce triangle-free graphs with large chromatic number produce graphs with small girth. Although the proof is a bit beyond our scope in this course, here is a historically very important result in applications of probability to combinatorics.

Theorem (Erdős, '59) For every pair (g, t) of positive integers with $g, t \geq 3$, there is a graph G with girth g and chromatic number t .

On-Line Coloring - A Two Person Game

Builder constructs a graph one vertex at a time.

Assigner colors the graph in an on-line manner.

Fact Even in the class of forests, Builder can force n colors on a graph with 2^{n-1} vertices.

Explanation Let S_n be the Builder's strategy for forcing n colors. Then S_{n+1} can be viewed as adding one new vertex to the disjoint application of $S_1, S_2, S_3, \dots, S_n$ and then adding one new vertex.

On-Line Coloring for Interval Graphs

Theorem (Kierstead and Trotter, '82) In the class of interval graphs, there is a strategy for Assigner that will enable her to color an interval graph with $3k - 2$ colors provided Builder keeps the maximum clique size at most k . Builder does not need to know the value of k in advance. Furthermore, this bound is best possible, since there is a strategy for Builder that will force assigner to use at least $3k - 2$ colors, regardless of the strategy used in assigning colors.

Game Coloring for Graphs

Definition The game chromatic number of a graph is the least positive integer t for which there is a strategy for Alice that will enable her, working in "cooperation" with Bob, to color the graph using t colors and alternating turns.

Note The issue as to who goes first can be important.

Theorem (Kierstead and Trotter, '94) The game chromatic number of a planar graph is at most 33.

Two Challenging Exercises

Observation The chromatic number of a tree is two if it has an edge. However, the game chromatic number of a tree is at most 4 and this result is best possible. This is a good exercise for a senior level undergraduate course in graph theory.

Follow-Up Note Kierstead and Zhu have been carrying on a running competition for 20 years, and it is now known that the game chromatic number of a planar graph is at most 17 with Zhu in the winning position for now. From below, a lower bound of 7 is known. If you really want to get an A+++, move either bound.

List Colorings of Graphs

Definition The **list chromatic number** of a graph is the smallest integer t so that a proper coloring of the graph can always be found using colors from prescribed lists of size t , one list for each vertex. Note that different vertices can have different lists.

Example When $n = C(2t-1, t)$, the complete bipartite graph $K_{n,n}$ has list chromatic number $t + 1$.

Theorem (Thomassen, 1994) The list chromatic number of a planar graph is at most 5.