MATH 3012 Quiz 2, October 12, 2004, WTT

1. Note that $67375 = 5^3 \times 7^2 \times 11$. Compute $\phi(67375)$.

\[
\phi(67375) = 67375 \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)
= 5^3 \cdot 7^2 \cdot 11 \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11}
= 5^2 \cdot 7 \cdot 4 \cdot 6 \cdot 10
= 42000
\]

2. (a) Write all the partitions of the integer 8;

\begin{align*}
8 & = 8 \text{ distinct parts} \\
& = 7 + 1 \text{ distinct parts, odd parts} \\
& = 6 + 2 \text{ distinct parts} \\
& = 6 + 1 + 1 \\
& = 5 + 3 \text{ distinct parts, odd parts} \\
& = 5 + 2 + 1 \text{ distinct parts} \\
& = 5 + 1 + 1 + 1 \text{ odd parts} \\
& = 4 + 4 \\
& = 4 + 3 + 1 \text{ distinct parts} \\
& = 4 + 2 + 2 \\
& = 4 + 2 + 1 + 1 \\
& = 4 + 1 + 1 + 1 + 1 \\
& = 3 + 3 + 2 \\
& = 3 + 3 + 1 + 1 \text{ odd parts} \\
& = 3 + 2 + 2 + 1 \\
& = 3 + 2 + 1 + 1 + 1 \\
& = 3 + 1 + 1 + 1 + 1 + 1 \text{ odd parts} \\
& = 2 + 2 + 2 + 2 \\
& = 2 + 2 + 2 + 1 + 1 \\
& = 2 + 2 + 1 + 1 + 1 + 1 \\
& = 2 + 1 + 1 + 1 + 1 + 1 + 1 \\
& = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \text{ odd parts}
\end{align*}
(b) Of the partitions listed in part (a), how many use distinct parts?

There are 6 partitions of the integer 8 into distinct parts.

(c) Of the partitions listed in part (a), how many use odd parts?

There are 6 partitions of the integer 8 into odd parts. More generally, for every integer \( n \), the number of partitions of \( n \) into odd parts equals the number of partitions of \( n \) into distinct parts.

3. Write the inclusion/exclusion formula for the number of onto functions from \( \{1, 2, \ldots, m\} \) to \( \{1, 2, \ldots, n\} \).

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)^m
\]

4. Write the inclusion/exclusion formula for the number of derangements of \( \{1, 2, \ldots, n\} \).

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)!
\]

5. Let \( A \) denote the advancement operator, i.e., \( Af(n) = f(n + 1) \). Find the general solution of the following equation:

\[
(2A^2 + 7A - 15)f(n) = 0
\]

Note that we can factor the quadratic \( 2A^2 + 7A - 15 \) as \( (2A - 3)(A + 5) \) so the roots are \( \frac{3}{2} \) and \(-5\). Therefore the general solution is \( c_1 \left( \frac{3}{2} \right)^n + c_2 (-5)^n \).

6. For the equation in the preceding problem, find the particular solution given \( f(0) = 6 \) and \( f(1) = -4 \).

Substituting \( n = 0 \) and \( n = 1 \) in the formula for the general solution, we obtain the following two equations for \( c_1 \) and \( c_2 \):

\[
\begin{align*}
c_1 + c_2 &= 6 \\
\frac{3}{2} c_1 - 5c_2 &= -4
\end{align*}
\]

The solution to this system is \( c_1 = 4 \) and \( c_2 = 2 \). So the answer is then \( 4 \left( \frac{3}{2} \right)^n + 2 (-5)^n \).

7. Find the general solution of the following equation:

\[
(A - 1)^2 (A - 3)^4 (A - 4 + i)^3 f(n) = 0
\]

\[
f(n) = c_1 + c_2 n + c_3 3^n + c_4 n 3^n + c_5 n^2 3^n + c_6 n^3 3^n + c_7 (4 - i)^n + c_8 n (4 - i)^n + c_9 n^2 (4 - i)^n
\]
8. Let $r_n$ denote the number of regions in the plane determined by $n$ circles—provided each pair of circles intersects in exactly two points. (a) Write a recurrence equation for $r_n$.

Label the $n$ circles as $C_1, C_2, \ldots, C_n$. Circle $C_n$ intersects each other circle in exactly two points, so there are $2(n-1)$ points of intersection on $C_n$. These points divide circle $C_n$ into $2(n-1)$ arcs, and each of these arcs divides an “old” region into two “new” ones. So the recursion is

$$r_n = r_{n-1} + 2(n-1)$$

(b) Solve the recurrence equation in part (a).

The general solution to the homogeneous equation $r_n = r_{n-1}$ is $f(n) = c$. We look for a particular solution to the non-homogeneous equation of the form $f(n) = An + Bn^2$. Substituting, we obtain:

$$An + Bn^2 = A(n-1) + B(n-1)^2 + 2(n-1)$$

$$= An - A + Bn^2 - 2Bn + B + 2n - 2$$

Equating coefficients, we obtain the two equations:

$$2 - 2B = 0$$
$$-A + B = 2$$

Thus $B = 1$ and $A = -1$. So the solution is $f(n) = n^2 - n + c$. Substituting $n = 1$ and noting that $r_1 = 2$, we obtain $2 = f(1) = 1^2 - 1 + c = c$. It follows that the final answer is $f(n) = n^2 - n + 2$.

9. (Extra Credit) Explain how the principle of inclusion/exclusion is used to derive the formula in Problem 3 for the number of onto functions.

Consider the set $X$ of all functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. For each $j = 1, 2, \ldots, n$, we say that a function $f \in X$ satisfies property $P_j$ if $j$ is NOT in the range of $f$. Now let $S$ be a set of $i$ properties. Then the number of functions from $X$ which satisfy the properties in $S$ is $(n-i)^m$. By the principle of inclusion/exclusion, the number of onto functions is then:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$