$14 \times 7 = 98$

KEY

Student Name and ID Number

MATH 3012 Quiz 3, April 11, 2006, WTT

1. Write the general solution of the advancement operator equation:
$(A-1)^4(A-3)^5(A+4)^2(A-4)f = 0.$

$$f(n) = c_1 + c_2 n + c_3 n^2 + c_4 n^3 + c_5 3^n + c_6 n 3^n + c_7 n^2 3^n + c_8 n^3 3^n + c_9 n^4 3^n + c_{10}(-4)^n + c_{11} n(-4)^n + c_{12} 4^n$$

2. Find a particular solution to the advancement operator equation: $(A^2 - 6A + 8)f(n) = 9(5)^n$.

Try $f(n) = c(5)^n$

$(A^2 - 6A + 8)f(n) = c5^{n+2} - 6c5^{n+1} + 8c5^n = (25c - 30c + 8c)5^n = 3c5^n$

$\therefore 3c = 9 \quad c = 3$ and $f(n) = 3(5)^n$ is a particular solution.

3. Find the unique solution to the advancement operator equation: $(A^2 - 6A + 8)f(n) = 9(5)^n$ with $f(0) = 6$ and $f(1) = 17$. $(A^2 - 6A + 8) = (A-2)(A-4)$ So

General solution is $f(n) = c_1 2^n + c_2 4^n + 3(5)^n$

$f(0) = 6 = c_1 + c_2 + 3 \qquad c_1 + c_2 = 3 \qquad 2c_1 + 2c_2 = 6$

$f(1) = 17 = 2c_1 + 4c_2 + 15 \qquad 2c_1 + 4c_2 = 2 \qquad 2c_2 = -4 \qquad c_2 = -2$

$c_1 = 5$

$\therefore$ Unique solution is $f(n) = 5(2)^n - 2(4)^n + 3(5)^n$

4. Let t_n count the number of ternary strings of length n which do not have a substring of three consecutive positions that form $(0,0,1)$. So $f(1) = 3$, $f(2) = 9$ and $f(3) = 26$. Find an advancement operator equation satisfied by t_n.

$\boxed{0}$ ok $n+1$ | $\boxed{1}$ Maybe bad $n+1$ | $\boxed{2}$ ok $n+1$

Middle case is bad $\Leftrightarrow$ $\boxed{0\,0\,1}$ $n-2$, $n-1$, n, $n+1$; t_{n-2}

$t_{n+1} = 2t_n + (t_n - t_{n-2}) = 3t_n - t_{n-2}$ $\qquad t_{n+1} - 3t_n + t_{n-2} = 0$

$\therefore (A^3 - 3A^2 + 1)f = 0$

5. Let X be a set and let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a family of properties. For each subset $S \subseteq \{1, 2, \ldots, m\}$, let $N(S)$ denote the number of elements of X which satisfy property P_i whenever $i \in S$. Write the Inclusion-Exclusion formula for the number of elements of X which satisfy none of the properties in $\mathcal{P}$:

$$\sum_{S \subseteq \{1,2,\ldots,m\}} (-1)^{|S|} N(S)$$

6. Write the Inclusion-Exclusion formula for d_n, the number of derangements of $\{1, 2, \ldots, n\}$:

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!$$

7. Use the formula in the preceding question to find the value of d_6.

$$d_6 = \binom{6}{0}6! - \binom{6}{1}5! + \binom{6}{2}4! - \binom{6}{3}3! + \binom{6}{4}2! - \binom{6}{5}1! + \binom{6}{6}0!$$

$$= 1 \cdot \cancel{720} - 6 \cdot \cancel{120} + 15 \cdot 24 - 20 \cdot 6 + 15 \cdot 2 - 6 \cdot 1 + 1 \cdot 1$$

$$= 360 - 120 + 30 - 6 + 1 = 265$$

8. The Euler ϕ-function $\phi(n)$ counts the number of integers from $\{1, 2, \ldots, n\}$ that are relatively prime to n. Write the Inclusion-Exclusion formula for $\varphi(n)$.

$$\varphi(n) = n \prod_{i=1}^{m} \frac{p_i - 1}{p_i}$$ where $p_1, p_2, \ldots, p_m$ are the distinct prime factors of n

9. Using the formula from the preceding problem, find $\phi(n)$ when $n = 7 \times 11^2 \times 101$.

$$\varphi(7 \times 11^2 \times 101) = \cancel{7} \cdot 11^{\cancel{2}} \cdot \cancel{101} \, \frac{6 \cdot 10 \cdot 100}{\cancel{7} \cdot \cancel{11} \cdot \cancel{101}}$$
$$= 66 \cdot 1000 = 66{,}000$$

10. For positive integers n and m, let $S(n, m)$ count the number of surjections from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$. Write the Inclusion-Exclusion formula for $S(n, m)$:

$$S(n, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

11. Use the formula from the preceding problem to find the value of $S(7, 4)$.

$$S(7,4) = \binom{4}{0} 4^7 - \binom{4}{1} 3^7 + \binom{4}{2} 2^7 - \binom{4}{1} 1^7$$
$$= 4^7 - 4 \cdot 3^7 + 6 \cdot 2^7 - 4$$

12. Let $R(n, m)$ denote the least positive integer t so that every graph on t vertices contains a complete subgraph of size n or and independent set of size m. Bob claims that with the assistance of his new computer and a clever search program he has written, he has determined that $R(12, 3) = 81$. Alice says that either Bob's reasoning or his computer program is defective as $R(12, 3)$ cannot be 81. Explain why Alice is right.

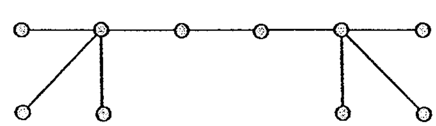
We know that $R(n,m) \leq \binom{n+m-2}{n-1} = \binom{n+m-2}{m-1}$

so $R(12,3) \leq \binom{12+3-2}{3-1} = \binom{13}{2} = \frac{13 \cdot 12}{2} = 13 \cdot 6 = 78$

13. What is the formula for the number of labeled trees with vertex set $\{1, 2, \ldots, n\}$?

$$n^{n-2}$$

14. How many ways are there to assign labels from the set $\{1, 2, \ldots, 10\}$ to the unlabeled tree shown below.



$$\frac{10!}{3!\,3!\,2!}$$