MATH 3012 Quiz 3, April 19, 2007, WTT

1. Write the general solution of the advancement operator equation:

\[ (A + 2)^3 A^2 \delta = 0. \]

\[ f(n) = c_1 (-2)^n + c_2 n (-2)^n + c_3 n^2 (-2)^n + c_4 8^n + c_5 n 3^n \]

2. Find a particular solution to the advancement operator equation:

\[ (A^2 + 2A - 15) f(n) = 27(4)^n. \]

Try \( f(n) = c \cdot 4^n. \)

\[
\begin{aligned}
   &c \cdot 4^n + 2c \cdot 4^{n+1} - 15c \cdot 4^n = 27 \cdot 4^n \\
   &16c \cdot 4^n + 8c \cdot 4^n - 15c \cdot 4^n = 27 \cdot 4^n
\end{aligned}
\]

\[
\begin{cases}
   c = 3 \\
   9c = 27
\end{cases}
\]

\[ f(n) = 3(4)^n \]

3. Find the unique solution to the advancement operator equation:

\[ (A^2 + 2A - 15) f(n) = 27(4)^n \text{ with } f(0) = 18 \text{ and } f(1) = 1. \]

\[ A^2 + 2A - 15 = (A + 5)(A - 3) \]

4. Write the Inclusion-Exclusion formula for \( d_n \), the number of derangements of \( \{1, 2, \ldots, n\} \):

\[ d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! \]

5. Use the formula in the preceding question to find the value of \( d_4 \).

\[ d_4 = \binom{4}{0} 4! - \binom{4}{1} 3! + \binom{4}{2} 2! - \binom{4}{3} 1! + \binom{4}{4} 0! \]

\[ = 24 - 24 + 6.2 - 4.1 + 1 = 9 \]
6. Verify your answer to the previous question by listing all the derangements on \{1, 2, 3, 4\}.

\[
\begin{array}{cccc}
2 & 4 & 3 & \\
2 & 3 & 4 & 1 \\
2 & 4 & 1 & 3 \\
3 & 1 & 4 & 2 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3 \\
4 & 3 & 1 & 2 \\
4 & 3 & 2 & 1 \\
\end{array}
\]

7. A data file `digraph.data.txt` has been read for a digraph whose vertex set is \{6\}. The weights on the directed edges are shown in the matrix below. In the space to the right, apply Dijkstra's algorithm to find the distance from vertex 1 to all other vertices in the graph. Also, for each \(x\), find a shortest path from 1 to \(x\).

\[
\begin{array}{cccccc}
W & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 54 & 58 & 17 & 22 & 97 \\
2 & 60 & 0 & 28 & 9 & 19 & 8 \\
3 & 46 & 24 & 0 & 19 & 9 & 12 \\
4 & 16 & 36 & 40 & 0 & 8 & 73 \\
5 & 23 & 29 & 30 & 3 & 0 & 47 \\
6 & 19 & 8 & 82 & 16 & 28 & 0 \\
\end{array}
\]
8. The data file for a graph with vertex set \{1, 2, \ldots, 7\} is shown below. In the space to the right, list in order the edges that would be found in carrying out Kruskal's algorithm (avoid cycles) and Prim's algorithm (build tree). Vertex 1 is the root.

<table>
<thead>
<tr>
<th>graph1.txt</th>
<th>Kruskal</th>
<th>Prim</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2 7 24</td>
<td>6 1 35</td>
</tr>
<tr>
<td>6 1 35</td>
<td>6 1 35</td>
<td>2 1 35</td>
</tr>
<tr>
<td>2 1 38</td>
<td>2 1 38</td>
<td>2 7 24</td>
</tr>
<tr>
<td>7 6 39</td>
<td>1 4 45</td>
<td>2 7 24</td>
</tr>
<tr>
<td>1 4 45</td>
<td>1 4 45</td>
<td>1 4 45</td>
</tr>
<tr>
<td>6 4 47</td>
<td>3 1 53</td>
<td>3 1 53</td>
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<td>3 1 53</td>
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<td>3 1 53</td>
</tr>
<tr>
<td>7 3 54</td>
<td>5 2 58</td>
<td>5 2 58</td>
</tr>
<tr>
<td>4 7 56</td>
<td>4 7 56</td>
<td></td>
</tr>
<tr>
<td>5 2 58</td>
<td>5 2 58</td>
<td></td>
</tr>
<tr>
<td>4 5 60</td>
<td>4 5 60</td>
<td></td>
</tr>
</tbody>
</table>

9. Let $R(n, m)$ denote the least positive integer $t$ so that every graph on $t$ vertices contains a complete subgraph of size $n$ or an independent set of size $m$. Bob has a new computer and prides himself on being a good programmer. One day, he boasts to Alice that with the assistance of his new computer, he has succeeded in verifying that $R(100, 150) \leq 2^{300}$. Alice is not impressed. Can you explain why?

We know $R(n, m) \leq \left( \frac{n + m - 2}{n - 1} \right)^{50}$

$R(100, 150) \leq \left( \frac{248}{99} \right) \leq 2^{48} \leq 2^{300}$
a. What is the current value of the flow? \[ 12 + 8 + 9 = 29 \]
b. What is the capacity of the cut \( V = \{ S, A, C, E \} \cup \{ B, D, F, T \} \)? \[ 24 + 15 + 13 + 14 = 66 \]
c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

\[
\begin{align*}
S & (\ast, +, \infty) \\
C & (S, +, 5) \\
E & (S, +, 11) \\
B & (C, +, 5) \\
D & (E, +, 11) \\
A & (D, -, 5) \\
F & (A, +, 3) \\
T & (F, +, 2)
\end{align*}
\]
d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram.

Augmenting path \( (S, E, D, A, F, T) \)
e. What is the value of the new flow? \[ 24 + 2 = 31 \]

Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled. Find a cut whose capacity is equal to the value of the flow.

\[
\begin{align*}
S & (\ast, +, \infty) \\
C & (S, +, 5) \\
E & (S, +, 9) \\
B & (C, +, 5) \\
D & (E, +, 9) \\
A & (D, -, 3) \\
F & (A, +, 3) \\
T & \text{capacity} = 16 + 15 \\
\text{Unlabeled} & \geq 31
\end{align*}
\]