MATH 3012 Quiz 1, February 10, 2009, WTT

1. Consider the 16-element set consisting of the ten digits \( \{0, 1, 2, \ldots, 9\} \) and the six capital letters \( \{A, B, C, D, E, F\} \).

a. How many strings of length 11 can be formed if repetition of symbols is permitted?

\[ 16^{11} \]

b. How many strings of length 11 can be formed if repetition of symbols is not permitted?

\[ P(16, 11) \]

c. How many strings of length 11 can be formed using exactly three 5's, six A's and two D's?

\[ \binom{11}{3, 6, 2} = \frac{11!}{3! 6! 2!} \]

d. How many strings of length 11 can be formed if exactly three characters are digits and exactly five of the remaining characters are B's?

\[ \binom{11}{3}(8)^3 5^3 \]

2. How many lattice paths from (3, 2) to (23, 17) pass through (9, 6)?

\[ \frac{\binom{9}{6} \cdot \binom{23}{17}}{\binom{4}{17}} \]

3. How many integer valued solutions to the following equations and inequalities:

a. \( x_1 + x_2 + x_3 + x_4 = 59, \) all \( x_i \geq 0. \)

\[ \binom{62}{3} \]

b. \( x_1 + x_2 + x_3 + x_4 = 59, \) all \( x_i > 0. \)

\[ \binom{58}{3} \]

c. \( x_1 + x_2 + x_3 + x_4 < 59, \) all \( x_i \geq 0. \)

\[ \binom{62}{4} \]

d. \( x_1 + x_2 + x_3 + x_4 \leq 59, \) all \( x_i > 0. \)

\[ \binom{59}{4} \]

e. \( x_1 + x_2 + x_3 + x_4 \leq 59, \) all \( x_i > 0, x_2 \geq 7. \)

\[ \binom{53}{4} \]

f. \( x_1 + x_2 + x_3 + x_4 + x_5 \leq 59, \) all \( x_i > 0, x_2 \leq 6. \)

Note: I meant the correct answer to be

\[ \binom{59}{4} - \binom{51}{4} \]

But actually it's

\[ \frac{59!}{5!} - \frac{53!}{5!} \].

Either answer accepted.

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4. Use the Euclidean algorithm to find $d = \text{gcd}(17160, 168)$.

\[
\begin{array}{c|cccc}
 & 102 & 17160 & 24 & 168 \\
\hline
168 & 17160 & 168 & & \\
360 & 24 & & \hline
336 & & & 24 & 0 \\
\end{array}
\]

\[\text{gcd}(17160, 168) = 24\]

5. Use your work in the preceding problem to find integers $a$ and $b$ so that $d = 17160a + 168b$.

\[17160 = 102 \cdot 168 + 24\]
\[168 = 7 \cdot 24\]

\[24 = 1 \cdot 17160 - 102 \cdot 168\]

so

\[a = 1 \quad b = -102\]

6. For a positive integer $n$, let $t_n$ count the number of ways to tile a $2 \times n$ checkerboard with figures of five types:

1. A horizontal strip of height 1 and width 2, i.e. a block of size $1 \times 2$, one row and two columns. Such strips can only be oriented horizontally, and not vertically.

2. An “L” shaped region consisting of three $1 \times 1$ squares. This figure can be oriented in any of the four possible ways (see drawing on the board).

Find a recurrence equation satisfied by $t_n$ and use it to calculate $t_8$.

\[t_1 = 0 \quad \text{when} \quad n \geq 5\]
\[t_2 = 1\]
\[t_3 = 2\]
\[t_4 = 3\]
\[t_5 = 2 + 2t_3 + 2t_2 = 2 + 2 \cdot 2 + 2 = 6\]
\[t_6 = t_4 + 2t_3 + 2t_2 + 2 = 3 + 2 \cdot 2 + 2 + 2 = 11\]
\[t_7 = t_5 + 2t_4 + 2t_3 + 2t_2 + 2 = 6 + 2 \cdot 3 + 2 \cdot 2 + 2 + 2 = 20\]
\[t_8 = t_6 + 2t_5 + 2t_4 + 2t_3 + 2t_2 + 2 = 11 + 2 \cdot 6 + 2 \cdot 3 + 2 \cdot 2 + 2 = 37\]
7. Use the algorithm developed in class to find an Euler circuit in the following graph:

8. Consider the following graph:

   a. Explain why this graph does not have an Euler circuit. It has vertices of odd degree such as 6 and 4.
   b. Provide a listing of the vertices that constitutes a Hamiltonian cycle.
   c. Find a set of vertices that forms a maximal clique but not a maximum clique. Many correct answers, e.g., 5, 1, 2, 3.
   d. What is $\omega(G)$ for this graph? 4
   e. Find a set of vertices which forms a maximum clique in this graph. 5, 3, 4, 8, 9
   f. Show that $\chi(G) = \omega(G)$ for this graph by providing an optimum coloring. You may write directly on the figure.
9. Prove the following identity by Mathematical Induction:

\[ 7 + 11 + 15 + \ldots + 4n + 3 = 2n^2 + 5n \]

Note: We intend that the expression on the left is just the integer 7 when \( n = 1 \). Furthermore, when \( n \geq 2 \), we intend that we are summing up the first \( n \) terms in the sequence which begins with \( s_1 = 7 \) and satisfies \( s_n = s_{n-1} + 4 \).

Proof. When \( n = 1 \), LHS = 7 while RHS = \( 2 \cdot 1^2 + 1 = 7 \) so the formula is valid when \( n = 1 \).

Now assume the formula holds when \( n = k \) where \( k \geq 1 \), i.e., we assume

\[ 7 + 11 + 15 + \ldots + 4k + 3 = 2k^2 + 5k \]

Then

\[ 7 + 11 + 15 + \ldots + 4k + 3 + 4(k + 1) + 3 = \frac{2}{2} + 5k + [4(k+1) + 3] \]

\[ = 2k^2 + 9k + 7 \]

This shows that the formula also holds when \( n = k+1 \). Therefore, by the principle of mathematical induction, it holds for all \( n \geq 1 \).

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