Complete Solutions for MATH 3012 Quiz 2, October 25, 2011, WTT

Note. The answers given here are more complete than is expected on an actual exam. It is intended that the more comprehensive solutions presented here will be valuable to students in studying for the final exam. In a few places, the wording of a problem is changed slightly to reflect the modifed layout. A table providing point values for the problems is given at the very end.

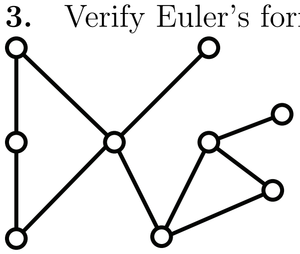
1. Dave claims there is a triangle-free graph with 38 vertices and 412 edges. Yolanda says he is wrong. Explain who is right.

> The intent of this question is to determine whether you know Turán's theorem, which states that the maximum number of edges in a triangle-free graph on n vertices is $\lfloor n^2/4 \rfloor$. This value is the number of edges in a complete bipartite graph in which the two part sizes are as balanced as possible. In this case, since $n = 38$, the two parts would both have 19 vertices, so the maximum number of edges, provided there were no triangles, is $19^2 = 361$. Since the graph Dave refers to has 412 edges, it must have a triangle. So Yolanda is right.

2. Zori claims that there is planar graph with 1012 vertices and 3672 edges. Carlos says she is wrong. Explain who is right.

> The intent here is to determine whether you remember that a planar graph on n vertices has at most $3n - 6$ edges when $n \geq 3$. We showed this in class by starting with an arbitrary planar graph and showing that if any face is not a triangle, then we can always add an edge and preserve planarity. When all faces (including the external face) are triangles, then Euler's formula will show that the number of edges is exactly $3n - 6$. In the case at hand, Carlos is right. The graph has 1012 vertices and 3672 edges. Since $3672 > 3(1012) - 6$, it can't be planar.

3. Verify Euler's formula for this planar graph.



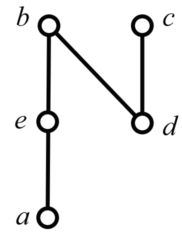
We count V, the number of vertices; E, the number of edges; and F, the number of faces, and see that $V = 9$, $E = 10$ and $F = 3$. So,

$$V - E + F = 9 - 10 + 3 = 2.$$

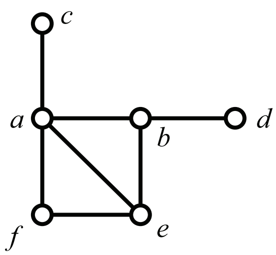
4. How many symmetric binary relations are there on $\{1, 2, \ldots, n\}$? Of these how many are reflexive?

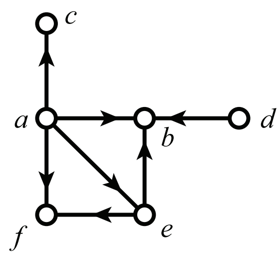
> A binary relation R on a set X is symmetric when $(x, y) \in R$ if and only if $(y, x) \in R$, for every $x, y \in X$. There are n pairs of the form (x, x), where $x \in \{1, 2, \ldots, n\}$. For each of these pairs, we decide whether to put (x, x) in the relation or not. Similarly there are $\binom{n}{2}$ 2-element subsets of the form $\{x, y\}$ with $x, y \in \{1, 2, \ldots, n\}$ and $x \neq y$. For each such pair, we must either put both (x, y) and (y, x) in the relation or put neither in the relation. So the total number of symmetric relations is $2^n 2^{\binom{n}{2}}$. For the second part, a binary relation R is reflexive if $(x, x) \in R$ for every $x \in X$. Now, we lose the issue of choice. All pairs (x, x) with $x \in \{1, 2, \ldots, n\}$ *must* belong to R, and we have a choice only for the distinct 2-element subsets. So the total number of relations which are both symmetric and reflexive is $2^{\binom{n}{2}}$.

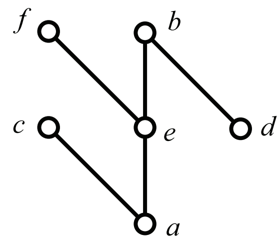
5. **a.** Let $X = \{a, b, c, d, e\}$ and let $P = \{(a, a), (b, b), (c, c), (d, d), (e, e), (d, b), (d, c), (a, e), (a, b), (e, b)\}$. Draw a diagram for the poset (X, P).



6. Show that the following graph is a comparability graph by transitively orienting the edges. In the space to the right, draw the diagram of the associated poset.

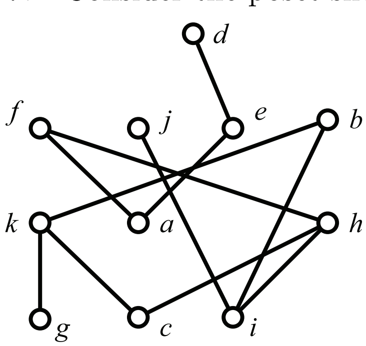


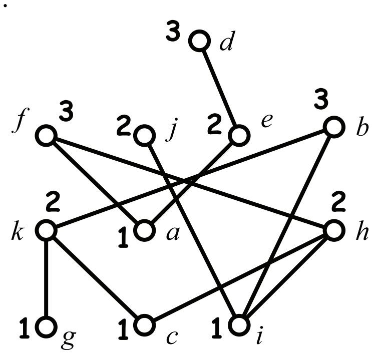




The figure on the left is the original graph given on the test. In the center is a transitive orientation. I started with the edge ac which I oriented as (a, c), i.e., a directed edge from a to c. This edge together with ab forms a "Vee", since bc is *not* an edge in the graph. This forces the directed edge (a, b). The edge (a, b) forces (d, b), which then forces (e, b). In turn, (e, b) forces (e, f). Now (a, c) forces (a, e) and (a, b) forces (a, f). It is easy to see that the orientation is essentially unique, i.e., the only option is to reverse the directions on *all* the edges.
The figure on the right is a diagram for the resulting poset. Of course, if the arrows are reversed in the middle figure, then the diagram on the right is turned upside down.

7. Consider the poset shown on the left.





a. Find all points comparable to a.

A point u is *comparable* to a when either $u < a$ or $u > a$. In this poset, a is a minimal point, so no other point is less than a. However, d, e and f are all bigger than a. Answer: $\{d, e, f\}$.

b. Find all points which cover a.

A point u *covers* a when $u > a$ and there is no point v between them. Here e and f cover a. Note that $d > a$ but d does not cover a since e is between them. Answer: $\{e, f\}$.

c. Find a maximal chain of size 2.

A chain C is *maximal* when there is no point u which is not in C for which $C \cup \{x\}$ is also a chain. In this poset $\{a, f\}$, $i, j\}$ and $\{b, i\}$ are 2-element maximal chains.

d. Find a maximal antichain of size 3.

In this poset, $\{a, i, k\}$ is a 3-element maximal antichain.

e. Find the set of all minimal elements.

An element u is *minimal* when there is no v with $v < u$ in the poset. Here the set of minimal elements is $\{a, c, g, i\}$.

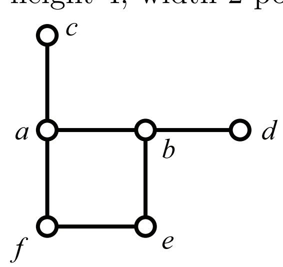
f. Using the algorithm taught in class (recursively removing the set of minimal elements), find the height h of the poset and a partition of P into h antichains. Also find a maximum chain. You may indicate the partition by writing directly on the diagam.

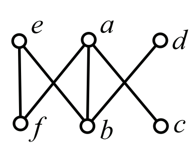
The antichain partition is drawn to the right of the original poset, i.e., for each $i = 1, 2, 3$, the antichain A_i consists of the points labelled with a boldface i.

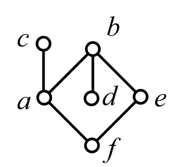
Find the height h of this poset and find a maximum chain.

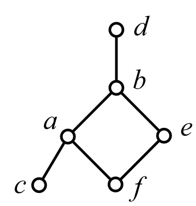
To answer this question, you take an element from the highest antichain, and then perform back-tracking. Here you might start with f which comes from A_3. Why is f not in A_2? Because f is over something in A_2. Here we see h is in A_2 and $f > h$. Why is h not in A_1? Because it is over something in A_1. Now we see that c is in A_1 and $h > c$. So the height is 3 and $\{f, h, c\}$ is a maximum chain. Note that there are several maximum chains in this poset.

8. In the space to the right, draw the diagrams of three different posets all having the following graph as their cover graphs (a) a height 2, width 3 poset; (b) a height 3, width 3 poset; and (c) a height 4, width 2 poset.





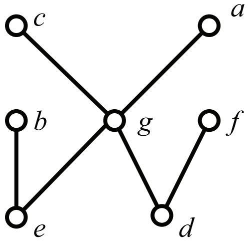




The answer for part (a) is unique up to duality, i.e., the poset shown can be turned upside down. Also, there are many ways to draw the diagram for the same poset, so correct answers can appear to be somewhat different but they are really drawings of the same poset.

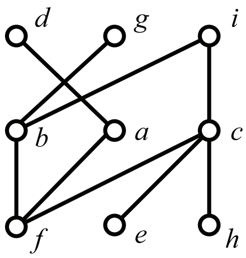
On the other hand, there are multiple correct answers for both part (b) and for part (c). This reflects the fact that it is difficult to determine whether a graph is a cover graph. Also, it is difficult to count the number of distinct posets with the same cover graph.

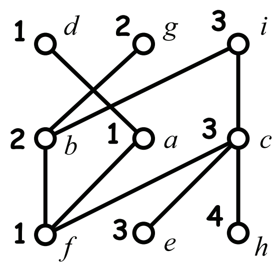
9. By inspection (not by algorithm), find four points in the following poset that form a $\mathbf{2+2}$.



The four points in $\{b, d, e, f\}$ form two 2-element chains with both points in one chain incomparable with both points in the other.

10. Find by inspection the width w of the poset shown below on the left and find a partition of the poset into w chains. Also find a maximum antichain. You may indicate the partition by writing directly on the diagram.



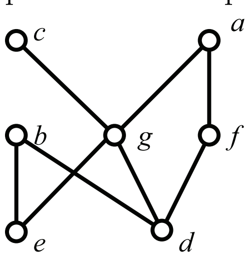


The chain partition is indicated by the figure on the right.

Find the width w of this poset and find a maximum antichain.

By inspection, the width is 4 and $\{b, d, e, h\}$ is a 4-element antichain. There are other 4-element antichains in this poset. One such is $\{a, b, e, h\}$.
Note that except for when P is an interval order, we do not have an algorithm for finding the width w of a poset P and a partition of P into w chains. From Dilworth's theorem, we know that such a partition exists. We just don't know how to find it. This issue will be addressed right at the end of our course.

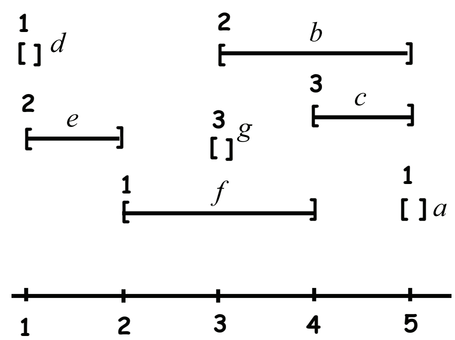
11. Shown below is the diagram of an interval order. Use the algorithm taught in class to find an interval representation. Then use the First Fit coloring algorithm to find the width w and a partition of the poset into w chains. Also, find a maximum antichain.



We list below the down sets and the up sets. There are 5 distinct down sets and 5 distinct up sets (these numbers will always be the same. Can you explain why?), and we label the down sets from little to big and the up sets from big to little. This labelling is shown in bold face type.

$D(a) = \{d, e, f, g\}$ **5** $\quad$ $U(a) = \emptyset$ **5**
$D(b) = \{d, e\}$ **3** $\quad$ $U(b) = \emptyset$ **5**
$D(c) = \{d, e, g\}$ **4** $\quad$ $U(c) = \emptyset$ **5**
$D(d) = \emptyset$ **1** $\quad$ $U(d) = \{a, b, c, f, g\}$ **1**
$D(e) = \emptyset$ **1** $\quad$ $U(e) = \{a, b, c, g\}$ **2**
$D(f) = \{d\}$ **2** $\quad$ $U(f) = \{a\}$ **4**
$D(g) = \{d, e\}$ **3** $\quad$ $U(g) = \{a, c\}$ **3**

We show below the interval representation which this labelling produces. Note that this representation is unique and it uses the fewest number of endpoints.



Find the width w of this poset and find a maximum antichain.

In contrast to the situation in Problem 10, we have a clean algorithm for finding the width w, a maximum antichain and a partition of the poset into w chains. This is accomplished by applying the first fit coloring algorith in the order of left end points. This is illustrated in the figure by the coloring with the bold face colors placed near the left end points. Here I have broken ties by proceeding from top to bottom in the picture.

Furthermore, we see that $w = 3$ and $\{b, f, g\}$ is a maximum antichain. To see how this is done, let k be the largest color assigned. When a vertex x is assigned color k, then for each $i = 1, 2, \dots, k-1$, the vertex x must be adjacent to a vertex x_i colored i which has already been colored. This means that the interval for x_i contains the left endpoint of the interval for x, so $\{x, x_1, x_2, \dots, x_{k-1}\}$ is an antichain of size k.

12. Let $\mathbf{2}^{13}$ be the poset consisting of all subsets of $\{1, 2, 3, \dots, 13\}$, ordered by inclusion.

a. What is the height of this poset?

We note that all maximal chains are maximum and all have 14 points. The reason is that a maximal chain starts with $(0, 0, 0, \dots, 0$ as its least element (13 zeroes in this string). Then as we proceed up the chain, at each step a zero is swapped out for a one. This is repeated until we get to the very top element $(1, 1, 1, \dots, 1)$.

b. What is the width of this poset?

The intent of this question is to see if you remember Sperner's theorem which states that the width of the lattice of all subsets of $\{1, 2, 3, \dots, n\}$ is the middle binomial coefficient $C(n, \lfloor n/2 \rfloor)$.

So in this case, the width is $C(13, 6)$ which of course can also be written as $\binom{13}{6}$.

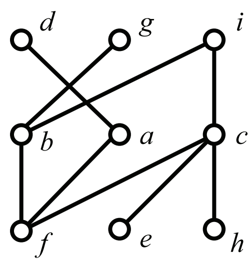
c. How many maximal chains does the poset have?

We have already commented on the structure of maximal chains. Using this structure, we see that there are $n!$ maximal chains. This follows from the fact that at step 1, we can choose any of the n zeroes in $(0, 0, 0, \ldots, 0)$ to toggle to a one. For the second step, we choose one of the remainging $n - 1$ zeroes to flip. Then $n - 2$, etc.

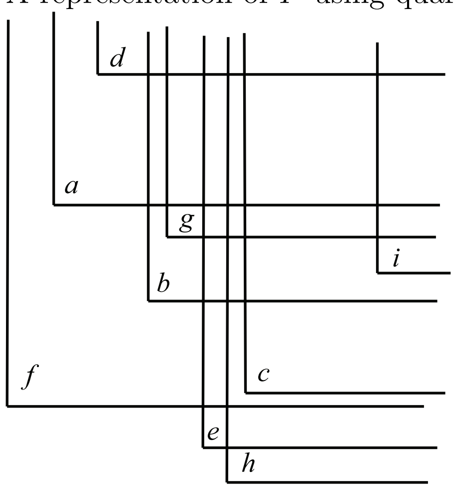
d. How many maximal chains in this poset pass through the set $\{2, 5, 8, 10\}$?

To answer this question, we apply the reasoning of the proceeding answer in both directions, so the answer is 4!9!.

13. Extra Credit. Show that the poset shown below has dimension 2 by finding two linear extensions whose intersection is the partial order.



A representation of P using quarterplanes is shown below.



From this representation, we see that we may take:

$$L_1 = [f, a, d, b, g, d, h, c, i] \quad \text{and} \quad L_2 = [h, e, f, c, b, i, g, a, d].$$

Point Totals

1. Seven points.
2. Seven points.
3. Seven points.
4. Six points.
5. Six points.
6. Eight points.
7. Sixteen points. $16 = 2 + 2 + 2 + 2 + 2 + 6$.
8. Six points. $6 = 2 + 2 + 2$.
9. Six points.
10. Eight points.
11. Fifteen points.
12. Eight points. $8 = 2 + 2 + 2 + 2$.

The total is 100 and the Extra Credit problem could earn up to +10 points.