MATH 3012 Quiz 1, February 8, 2013, WTT

1. Consider the 62-element alphabet consisting of the ten digits \{0,1,2,\ldots,9\} and the letters \{a,A,b,B,c,C,\ldots,z,Z\} of the English language, including both lower-case and upper case letters, i.e., the letters are case sensitive.

   a. How many strings of length 39 can be formed if repetition of symbols is not permitted?

   \[ P(62, 39) \quad \text{or} \quad 62 \cdot 61 \cdot 60 \cdots 25 \cdot 24 \]

   b. How many strings of length 39 can be formed if repetition of symbols is permitted?

   \[ 62^{39} \]

   c. How many strings of length 39 can be formed using exactly twenty 5's, eight B's and eleven b's?

   \[ \binom{39}{20, 8, 11} \quad \text{or} \quad \frac{39!}{20! \cdot 8! \cdot 11!} \]

   d. How many strings of length 39 can be formed using exactly twenty 5's, eight B's and eleven b's if the eight B's are required to occur consecutively in the string?

   \[ \binom{32}{20, 1, 11} \quad \text{or} \quad \frac{32!}{20! \cdot 1! \cdot 11!} \quad \text{Notes: Treat block of BS as a single character} \]

2. How many lattice paths from (3, 5) to (42, 69) pass through (28, 52)?

   \[ \binom{72}{25} \binom{31}{14} \quad \text{Notes: } 29 - 3 = 25 \quad \text{and} \quad 52 - 5 = 47 \quad \text{and} \quad (42 - 29) + (69 - 52) = 14 + 17 = 31 \]

3. How many integer valued solutions to the following equations and inequalities:

   a. \( x_1 + x_2 + x_3 = 42 \), all \( x_i > 0 \).

   \[ \binom{44}{2} \]

   b. \( x_1 + x_2 + x_3 = 42 \), all \( x_i \geq 0 \).

   \[ \binom{44}{2} \]

   c. \( x_1 + x_2 + x_3 < 42 \), all \( x_i > 0 \).

   \[ \binom{44}{3} \]

   d. \( x_1 + x_2 + x_3 \leq 42 \), all \( x_i \geq 0 \).

   \[ \binom{45}{3} \]

   e. \( x_1 + x_2 + x_3 = 42 \), all \( x_i > 0, x_3 \geq 10 \).

   \[ \binom{32}{2} \]

   Add three artificial's

   Add new variable \( x_4 \geq 0 \) and solve \( x_1 + x_2 + x_3 + x_4 = 42 \)

   weaken \( x_4 \geq 0 \). Now set aside 9 for \( x_3 \), set aside 9 for \( x_3 \).
4. Use the Euclidean algorithm to find \( d = \gcd(231, 504) \).

\[
\begin{array}{c|c|c}
& 2 & 231 \\
231 & 504 & 42 \\
462 & 231 & 21 \\
42 & 0 & 0 \\
\end{array}
\]

\[ d = 21 \quad \text{(last positive remainder)} \]

5. Use your work in the preceding problem to find integers \( a \) and \( b \) so that \( d = 231a + 504b \).

\[
\begin{align*}
504 &= 2 \cdot 231 + 42 \\
231 &= 5 \cdot 42 + 21 \\
1 \cdot 504 - 2 \cdot 231 &= 42 \\
1 \cdot 231 - 5 \cdot 42 &= 21 \\
21 &= 1 \cdot 231 - 5 \cdot 42 \\
&= 1 \cdot 504 - 2 \cdot 231 \\
&= 11 \cdot 231 - 5 \cdot 504 \\
\end{align*}
\]

\[ a = 11 \quad b = -5 \]

6. For a positive integer \( n \), let \( t_n \) count the number of ternary strings of length \( n \) that do not contain 001 as a substring. Note that \( t_1 = 3 \), \( t_2 = 9 \) and \( t_3 = 26 \). Develop a recurrence relation for \( t_n \) and use it to compute \( t_4 \), \( t_5 \) and \( t_6 \).

Consider last digit. If it's a 0 or 2, then in front of it, it is a good sequence. On the other hand, if it's a 1, then in front of it, it is a good sequence except that it cannot end 00. This leads to

\[
t_{n+1} = 2t_n + (t_n - t_{n-2}) = 3t_n - t_{n-2}
\]

So
\[
\begin{align*}
t_4 &= 3t_3 - t_1 = 3 \cdot 26 - 3 = 78 - 3 = 75 \\
t_5 &= 3t_4 - t_2 = 3 \cdot 75 - 9 = 225 - 9 = 216 \\
t_6 &= 3t_5 - t_3 = 3 \cdot 216 - 26 = 648 - 26 = 622
\end{align*}
\]
7. Use the algorithm developed in class, with vertex 1 as root, to find an Euler circuit in the graph $G$ shown below:

\[(1, 2, 7, 3, 1)
(2, 10, 11, 2)
(1, 2, 10, 11, 2, 7, 3, 1)
(7, 4, 5, 7, 8, 5, 9, 4, 6, 9, 7)
(1, 2, 10, 11, 2, 7, 4, 5, 7, 8, 5, 9, 4, 6, 9, 7, 3, 1)\]

8. Consider again the graph $G$ from the preceding problem.

a. Show that there is a path starting at 10 and ending at 8 which visits each of the vertices, exactly once, along the way. You may answer this question either by listing the eleven vertices in a suitable order, or by darkening edges directly on the figure.

\[(10, 11, 2, 1, 3, 7, 9, 6, 4, 5, 8)\] Other answers are possible

b. What is $\omega(G)$?

$\omega(G) = 4$  Note: $\{4, 5, 7, 9\}$ is a maximum clique

c. What is the chromatic number of the complete bipartite graph $K_{11,96}$?

$\chi(K_{11,96}) = 2$  Note: There are two parts. All vertices in the same part are colored the same.

1. \( \left( \frac{10}{3} \right) = 160 \).

2. \( P(10, 7) = 10 \cdot 9 \cdot 8 \cdot 7 \).

3. The answer to question 1, part a, on this test is less than 1,000,000,000.

4. There is a graph \( G \) on 238 vertices with \( \chi(G) = 17 \) and \( \omega(G) = 35 \).

5. All graphs with 1286 vertices and 5973 edges are non-planar.

6. There is a hamiltonian graph on 684 vertices in which every vertex has degree 10.

7. Every connected graph on 684 vertices in which every vertex has degree 10 has an Euler circuit.

8. Every graph with 21 vertices and 231 edges is hamiltonian.

9. Every graph with 21 vertices in which every vertex has degree at least 11 is hamiltonian.

10. If \( G \) is an interval graph, then \( \chi(G) = \omega(G) \).

11. There is a planar graph on 458 vertices which is a homeomorphic copy of the complete bipartite graph \( K_{2,3} \).

12. When \( n \geq 3 \), the shift graph \( S_n \) has \( \binom{n}{2} \) edges and \( \binom{n}{3} \) vertices.

13. When \( n \geq 3 \), the maximum clique size of the shift graph \( S_n \) is given by \( \omega(S_n) = 2 \).

14. When \( n \geq 3 \), the chromatic number of the shift graph \( S_n \) is given by \( \chi(S_n) = 3n - 6 \).

15. Euler’s formula asserts that if \( V \), \( E \) and \( F \) count the number of vertices, edges and faces in a drawing (without crossings) of a connected planar graph, then \( V - E + F = 2 \).

16. The number of lattice paths from \( (0, 0) \) to \( (n, n) \) which do not pass through a point above the diagonal is the Catalan number \( \binom{2n}{n} / (n + 1) \).

17. Any modern computer can quickly add two 300 digit numbers.

18. Any modern computer can quickly multiply two 300 digit numbers.

19. Any modern computer can quickly test whether a 300 digit number is prime.

20. Any modern computer can accept a file of 1000 positive integers, each at most 2000, and quickly determine whether 947 is one of the integers in the file.

21. Any modern computer can accept a file of 1000 positive integers, each at most 2000 and quickly determine whether there are two integers \( m \) and \( n \) in the file so that \( m + n = 947 \).

22. Merge Sort proceeds by: (1) splitting a sequence of length \( n \) into a planar shift graph and a connected multilinear coefficient; (2) extracting an Euler circuit; (3) forming the sum \( \sum_{i=1}^{n} 3i - 6 \); (4) producing a combinatorial proof of \( 2 = 1 + 1 \); (5) repeating the previous part by induction; and finally (6) showing that interval graphs are homeomorphs of Catalan numbers.

Just testing your sense of humor!!

Of course, this last question isn't graded.