MATH 3012 Quiz 2, March 15, 2013, WTT

1. Consider the poset shown below. The ground set is \( X = \{a, b, c, d, e, f, g, h\} \). In the space to the right of the figure, write the reflexive, antisymmetric and transitive relation on \( X \) which defines this poset.

\[
\begin{align*}
P &= \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (g, g), (h, h), \\
&\quad (a, d), (c, d), (f, a), (f, g), (f, c), (f, d), (b, a), (b, c), (b, d), \\
&\quad (e, f), (e, a), (e, g), (e, c), (e, d)\}.
\end{align*}
\]

2. Consider the following poset.

a. Find all points comparable to \( k \): \( \{e, g, f, a, m, q, r, n\} \).

b. Find all points which cover \( k \): \( \{e, f\} \).

c. Find a maximal chain of size 2: \( \{f, h\} \) or \( \{j, h\} \).

d. Using the algorithm taught in class (recursively removing the set of minimal elements), find the height \( h \) of the poset and a partition of \( P \) into \( h \) antichains. Also find a maximum chain. You may indicate the partition by writing directly on the diagram.

The height \( h \) is 6 and \( \{g, e, k, a, m, q\} \) is a maximum chain.

3. Find by inspection the width \( w \) of the following poset and find a partition of the poset into \( w \) chains. Also find a maximum antichain. You may indicate the partition by writing directly on the diagram.

a. The width \( w \) is 4 and \( \{a, b, c, h\} \) is a maximum antichain.

b. This poset is not an interval order. Find four points which form a copy of \( 2 + 2 \): \( \{a, f, g, c\} \), also \( \{a, f, g, h\} \).
4. Shown below is the diagram of an interval order. Use the algorithm taught in class to find an interval representation by computing the down-sets and up-sets in the space provided. Then use the First Fit coloring algorithm to find the width \( w \) and a partition of the poset into \( w \) chains. Also, find a maximum antichain.

\[
D(a) = b, e, h \\
D(b) = \emptyset \\
D(c) = b, e, f, h \\
D(d) = a, b, c, e, f, h \\
D(e) = \emptyset \\
D(f) = e \\
D(g) = e, f \\
D(h) = \emptyset \\
U(a) = d \\
U(b) = a, c, d \\
U(c) = d \\
U(d) = \emptyset \\
U(e) = a, c, d, f, g \\
U(f) = a, c, d, g \\
U(g) = \emptyset \\
U(h) = c, d
\]

The width \( w \) is 3 and \{a, c, g\} is a maximum antichain (there are several others).

5. Let \( 2^{15} \) be the poset consisting of all subsets of \{1, 2, 3, \ldots, 15\}, ordered by inclusion.

a. What is the height of this poset: 16.

b. What is the width of this poset: \( \binom{15}{7} \), which of course is the same as \( \binom{15}{8} \).

c. How many maximal chains does the poset have: 15!.

d. How many maximal chains in this poset pass through the set \{2, 3, 8, 13\}: 4! \cdot 11!.
6. Write the general solution to the homogeneous advancement operator equation:
\[
[A - (7 - 2i)]^3 (A - 1)^4 f = 0.
\]
f(n) = c_1(7 - 2i)^n + c_2 n(7 - 2i)^n + c_3 n^2(7 - 2i)^n + c_4 + c_5 n + c_6 n^2 + c_7 n^3.

7. Find a particular solution to the advancement operator equation:
\[
(A^2 - 3A + 5) f = 4 \cdot 3^n.
\]
We try \( f(n) = c 3^n \). This requires:
\[
4 \cdot 3^n = (A^2 - 3A + 5)c 3^n
\]
\[
= c3^{n+2} - 3c3^{n+1} + 5c3^n
\]
\[
= 9c3^n - 9c3^n + 5c3^n
\]
\[
= 5c3^n
\]
This implies \( 4 = 5c \) so that \( c = 4/5 \) and \( f(n) = 4/5 3^n \) is a solution.

8. Write the inclusion-exclusion formula for \( S(n, m) \), the number of surjections from \( \{1, 2, \ldots, n\} \) to \( \{1, 2, \ldots, m\} \). Then use this formula to calculate \( S(6, 4) \).
\[
S(n, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.
\]
\[
S(6, 4) = \sum_{k=0}^{4} (-1)^k \binom{4}{k} (4-k)^6
\]
\[
= \binom{4}{0} 4^6 - \binom{4}{1} 3^6 + \binom{4}{2} 2^6 - \binom{4}{3} 1^6 + \binom{4}{4} 0^6
\]
\[
= 1 \cdot 4096 - 4 \cdot 729 + 6 \cdot 64 - 4 \cdot 1
\]
\[
= 1560.
\]

9. Write the inclusion formula for the number \( d_n \) of derangements of \( \{1, 2, \ldots, n\} \). Then use this formula to calculate \( d_6 \).
\[
d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!
\]
\[
d_6 = \sum_{k=0}^{6} (-1)^k \binom{6}{k} (6-k)!
\]
\[
= \binom{6}{0} 6! - \binom{6}{1} 5! + \binom{6}{2} 4! - \binom{6}{3} 3! + \binom{6}{4} 2! - \binom{6}{5} 1! + \binom{6}{6} 0!
\]
\[
= 1 \cdot 720 - 6 \cdot 120 + 15 \cdot 24 - 30 \cdot 6 + 15 \cdot 2 - 6 \cdot 1 + 1 \cdot 1
\]
\[
= 265.
\]
10. Note that $1800 = 25 \cdot 9 \cdot 8$. Use this information and the inclusion-exclusion formula to determine $\phi(1800)$, where $\phi$ is the Euler $\phi$-function studied in class.

The prime factors of 1800 are 2, 3 and 5. So

$$\phi(1800) = 1800 \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

$$= 1800 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot 45$$

$$= \frac{2400}{5}$$

$$= 480.$$


F  1. There is a graph on 928 vertices in which no two vertices have the same degree.

F  2. There is a poset with 7403 points having width 65 and height 98.

T  3. There is a poset with 7403 points having width 85 and height 98.

F  4. The permutation $(8, 1, 4, 9, 3, 6, 2, 7, 5)$ is a derangement.

F  5. The number of partitions of an integer $n$ into even parts is the same as the number of partitions of $n$ into parts that are all the same.

Fun!  6. The partitions of a deranged surjection can be effectively computed using inclusion-exclusion and the process will consistently result in a maximum antichain of prime factors.