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A Forbidden Subposet Characterization of an Order – Dimension Inequality

by

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ABSTRACT

Dushnik and Miller defined the dimension of a partially ordered set X, Dim X, as the minimum number of linear extensions of X whose intersection is the partial ordering on X. The concept of dimension for a partially ordered set has applications to preference structures and the theory of measurement. Hiraguchi proved that Dim $X \leq [|X|/2]$ when $|X| \geq 4$. Bogart, Trotter, and Kimble gave a forbidden subposet characterization of Hiraguchi's inequality by constructing for each $n \geq 2$ the minimum collection $\mathscr{B}_n$ of posets such that if $[|X|/2] = n \geq 2$, then Dim $X < n$ unless X contains one of the posets in $\mathscr{B}_n$. Recently Trotter gave a simple proof of Hiraguchi's inequality based on the following theorem. If A is an antichain of X and $|X-A| = n \geq 2$, then Dim $X \leq n$. In this paper we give a forbidden subposet characterization of this last inequality.

1. Introduction. Dushnik and Miller [3] defined the dimension of a partially ordered set (poset) X, denoted Dim X, as the smallest positive integer t for which there exist t linear extensions $L_1, L_2, \cdots, L_t$ of X whose intersection is the partial ordering on X, i.e., $x < y$ in X if and only if $x < y$ in L_i for all $i \leq t$.

A collection $L_1, L_2, \cdots, L_n$ of linear orders on a set X can be interpreted as a record of the preferences among the elements of X by n different observers. The partial order obtained by taking the set theoretic intersection of these linear orders then reflects precisely those preferences on which all n observers agree. The dimension of a poset can thus be interpreted as a measurement of the complexity of the partial order since it measures the minimum number of observers required to produce the given partial order as a statement of the unanimous opinion of the observers. We refer the reader to [1], [7], and [8] for additional preliminary material in the dimension theory of posets.

If x and y are distinct points in a poset X, $x \nless y$, and $y \nless x$, then x and y are said to be incomparable, and we write $x I y$. A subset A of X is called an antichain if each pair of distinct points from A is an incomparable pair. The following theorem is proved in [8].

THEOREM 1. *Let A be an antichain of a poset X with $|X-A| = n \geq 2$. Then Dim $X \leq n$.*

In this paper we provide a "forbidden subposet" characterization of the inequality in Theorem 1. We will construct, for each $n \geq 2$, the minimum collection $\mathscr{C}_n$ of posets such that if A is an antichain of a poset X and $|X - A| = n \geq 2$, then Dim $X < n$ unless X contains a subposet isomorphic to one of the posets in $\mathscr{C}_n$.

Since a poset has dimension 1 if and only if it is a chain, the determination of $\mathscr{C}_2$ is trivial. It is the singleton collection containing a two element antichain. The case $n = 3$ involves some pathology and will be deferred temporarily. However for $n \geq 4$, there is sufficient regularity for us to tackle these values simultaneously.

2. The Forbidden Subposets for n $\geq$ 4. We first construct a poset $X(n)$ containing as subposets all the posets in the list $\mathscr{C}_n$ of forbidden subposets in the characterization of Theorem 1. For each $n \geq 4$, the height of $X(n)$ is one and there are n minimal elements $b_1, b_2, \cdots, b_n$. For each $i \leq n$, there is a maximal element a_i covering all b's except b_i. For each i,j with $1 \leq i < j \leq n$, there is a maximal element a_{ij} covering all b's except b_i and b_j. For each $i \leq n$, there is a maximal element a^i which is incomparable with all b's except b_i. Finally there is a maximal element a_o which covers all b's.

We are now ready to extract from $X(n)$ the posets belonging to $\mathscr{C}_n$. Let $B = \{b_1, b_2, \cdots, b_n\}$ and let k be an integer with $0 \leq k \leq n$. We define a subposet $X(n,k) \subset X(n)$ as follows. The point set of $X(n,n)$ is $B \cup \{a_i: 1 \leq i \leq n\}$, and for each k with $0 \leq k < n$, the point set of $X(n,k)$ is $B \cup \{a_i: 1 \leq i \leq k\} \cup \{a_{ij}: 1 \leq i \leq j \leq n, j > k\} \cup \{a^j: k < j \leq n\} \cup \{a_o\}$

The poset $X(n,n)$ is isomorphic to the poset consisting of all $(n-1)$-element and one-element subsets of an n-element set ordered by inclusion. Hiraguchi [4] discussed this poset and noted that it was n-dimensional. This poset is also discussed in [7], where it is denoted S_n^0 and called a crown. It provides a forbidden subposet characterization of Hiraguchi's inequality [4] Dim $X \leq [|X|/2]$ for $|X| \geq 4$.

THEOREM 2. (*Bogart and Trotter* [2] *and Kimble* [5]): *Let X be a poset with* $|X| \geq 8$; *then Dim* $X < [|X|/2]$ *unless X contains a subposet isomorphic to* S_n^0.

Theorem 1 implies that Dim $X(n, k) \leq n$ since in every case the set of maximal elements of $X(n, k)$ is an antichain whose removal from $X(n, k)$ leaves n minimal elements. We now show Dim $X(n, k) = n$. As noted before, Dim $X(n, n) = n$.

THEOREM 3. *Dim* $X(n, k) = n$ *for every k with* $0 \leq k < n$.

Proof. Suppose Dim $X(n, k) = t < n$ and let $L_1, L_2 \cdots, L_t$ generate $X(n, k)$. Then let $D = \{b \in B$: There exists $m \leq t$ such that b is the largest element of B in $L_m\}$. If $1 \leq i \leq k$, then there must exist some $m \leq t$ with $b_i > a_i$ in L_m and thus b_i is the largest element of B in L_m. Now if $1 \leq i < j \leq n$ and neither b_i nor b_j is in $D, j > k$, $a_{ij} \in X(n, k)$, $a_{ij} \, I \, b_i$ in $X(n, k)$, and $a_{ij} > b$ in $X(n, k)$ for every $b \in D$. But this implies that $a_{ij} > b_i$ in L_m for every $m \leq t$. The contradiction shows that $|D| = t = n-1$.

Now let b_i be the unique element of $B - D$ and let b_j be the largest element of B in L_m for some $m \leq n-1$. It is easy to see that there must exist a maximal element $a \in X(n, k)$ which covers all minimal elements except b_i and b_j. Since $a \, I \, b_i$, there must exist an integer $p \leq n-1$ with $b_i > a$ in L_p. It follows that $p = m$ and that b_i is the second largest element of B in L_m. Since m was arbitrary, we conclude that b_i is the second largest element of B in L_m for every $m \leq n-1$.

Since $i > k$ and $k < n$, we know that a_o and a^i are elements of $X(n, k)$. If $m \leq n-1$ and b_j is the largest element of B in L_m, then $a^i I b_j$ in $X(n, k)$ and thus there exists an integer $p \leq n-1$ with $b_j > a^i$ in L_p. It follows that $p = m$ and thus $b_j > a^i > b_i$ in L_m. Since m was arbitrary and a_o covers all elements of B, we conclude that $a_o > b_j > a^i > b_i$ in L_m for every $m \leq n-1$, i.e., $a_o > a^i$ in L_m for every $m \leq n-1$. The contradiction completes the proof. □

Since a poset X and its dual $\hat{X}$ have the same dimension, the collection $\mathscr{C}_n = \{X(n, k) : 0 \leq k \leq n\} \cup \{\hat{X}(n, k) : 0 \leq k \leq n\}$ consist of n-dimensional posets each containing an antichain and n additional points. Since $X(n, n)$ is self dual, we note that $|\mathscr{C}_n| = 2n-1$.

3. The Characterization Theorem for n ≥ 4. The following theorem is due to R. Kimble [5].

THEOREM 4. *If A is an antichain of a poset X and $|X-A| = n \geq 4$, then Dim $X < n$ unless one of the sets $\{x \in X : x > a$ for some $a \in A\}$ and $\{x \in X : x < a$ for some $a \in A\}$ is empty and the other is an n-element antichain of X.*

The following lemma follows immediately from Theorems 1, 4 and the well known result [4] that if X is the free sum $X = X_1 + X_2 + \cdots + X_n$ of 2 or more components, then Dim X = Max $\{2, \text{Dim } X_1, \text{Dim } X_2, \cdots, \text{Dim } X_n\}$.

LEMMA. *If A is an antichain of X and $|X-A| = n \geq 4$, then Dim $X < n$ unless A and $X-A$ are maximal antichains in X, one is the set of maximal elements of X, and the other is the set of minimal elements.*

If X is a set, $X = A_1 \cup A_2 \cup \cdots \cup A_m$ is a partition of X, and each A_i has a partial order P_i defined on it, we denote by Q: $[P_1(A_1), P_2(A_2), \cdots, P_m(A_m)]$ the partial order Q on X defined by $x > y$ in Q if and only if $x > y$ in some P_i or $x \in A_i$, $y \in A_j$, and $i < j$. If some A_i is an antichain under P_i, we will replace $P_i(A_i)$ by A_i in this notation. In this case A_i will also be an antichain under Q.

Let $\mathscr{C}_n$ be the collection of posets defined in Section 2. We now state and prove the following characterization theorem.

THEOREM 5. *Let A be an antichain of a poset X with $|X-A| = n \geq 4$. Then Dim $X < n$ unless X contains a subposet isomorphic to a poset in $\mathscr{C}_n$.*

Proof. Let A be an antichain of a poset X with $|X-A| = n \geq 4$ and suppose that X does not contain a subposet isomorphic to a poset in $\mathscr{C}_n$. We show that Dim $X < n$. In view of our remarks on duality, and the above lemma, we may assume that A is the set of maximal elements of X, $B = X - A = \{b_1, b_2, \cdots, b_n\}$ is the set of minimal elements, and both A and B are maximal antichains of X.

Furthermore we may assume (see lemmas 1 and 2 in [8]) that holdings are not duplicated, i.e., there do not exist distinct elements of A which cover exactly the same set of elements from B nor do there exist distinct elements from B which are covered by exactly the same set of elements from A.

Among the elements of A, we will be primarily concerned with those which cover 1, $n-2$, $n-1$, or all n of the minimal elements. We will give such maximal elements the labels assigned to the corresponding maximal elements of $X(n)$ as defined in Section 2.

For each $a \in A$, let $I(a) = \{b \in B : b\, I\, a \text{ in } X\}$; also let $I(b) = \{a \in A : a\, I\, b\}$ for each $b \in B$. Now define a partial order P on A by $a \leq a'$ in P if and only if $I(a') \subset$

$I(a)$ and let L be a linear extension of P. Then let M be an arbitrary linear order on B.

Let $D = \{b_i \in B\colon a_i \in A\}$. If $|D| = k \geq 1$, we may assume without loss of generality that $D = \{b_1, b_2, b_3, \cdots, b_k\}$. Since X does not contain a subposet isomorphic to $X(n, n)$, we also assume that $k < n$. We now proceed to prove that Dim $X < n$.

First let j be any integer with $k < j \leq n$. We show that Dim $X < n$ unless a^j, $a_o \in X$. To accomplish this, define for each $i \leq n$ with $i \neq j$ a linear order L_i on X by L_i: $[L(A - I(b_i)), b_i, L(I(b_i) - I(b_j)), b_j, L(I(b_j) \cap I(b_j)), M(B - \{b_i, b_j\})]$. It is straightforward to verify that this construction produces $n-1$ linear extensions which generate X unless $a_o, a^j \in X$ in which case $a_o > a^j$ in each of these extensions. Therefore we will assume that $a_o \in A, a^j \in A$ for every j with $k < j \leq n$.

Since X does not contain a subposet isomorphic to a poset in $\mathscr{C}_n$, we conclude that there exist integers i, j with $k < j \leq n$ and $1 \leq i < j$ such that $a_{ij} \notin A$. Choose an integer $m \leq n-1$ with $m \neq i, m \neq j$. For each p with $1 \leq p \leq n-1, p \neq i$, let L_p be the linear order on X defined by L_p: $[L(A - I(b_p)), L(I(b_p) - I(b_j)), b_j$ $L(I(b_p) \cap I(b_j)), M(B - \{b_p, b_j\})]$. Then let L_i be the linear order on X defined by L_i: $[L(A - I(b_i)), L(I(b_i) - I(b_m)), b_m, L(I(b_i) \cap I(b_m) - I(b_j)), b_j, L(I(b_i) \cap I(b_m) \cap I(b_j)),$ $M(B - \{b_i, b_j, b_m\})]$. Finally let L_m: $[L(A - (I(b_m) - \{a^j\})), b_m, L(I(b_m) - \{a^j\} - I(b_j)),$ $b_j, L(I(b_m) \cap I(b_j)), M(B - \{b_j, b_m\})]$. It is straightforward to verify that these $n-1$ linear extensions generate X unless $a_{ij} \in X$, and in this case $a_{ij} > b_j$ in each of these extensions.

We have now proved that Dim $X < n$ unless X contains a subposet isomorphic to $X(n, k)$ and the proof of our theorem is complete.

4. The case $\mathbf{n} = 3$. The following result is due to R. Kimble [5].

THEOREM 6. *Let A be an antichain of a poset X with $|X - A| = 3$. Then Dim $X < 3$ unless one of the following conditions holds.*

(1) *One of the sets $\{x \in X : x > a$ for some $a \in A\}$ and $\{x \in X : x < a$ for some $a \in A\}$ is empty and the other is a three-element antichain.*

(2) *All points in $\{x \in X : x > a$ for some $a \in A\}$ are greater in X than all points in $\{x \in X : x < a$ for some $a \in A\}$. Furthermore one of the sets is a singleton and the other is a two element antichain.*

Using Theorem 6 and the lemmas on holdings mentioned in Section 3, it is straightforward to verify the following theorem.

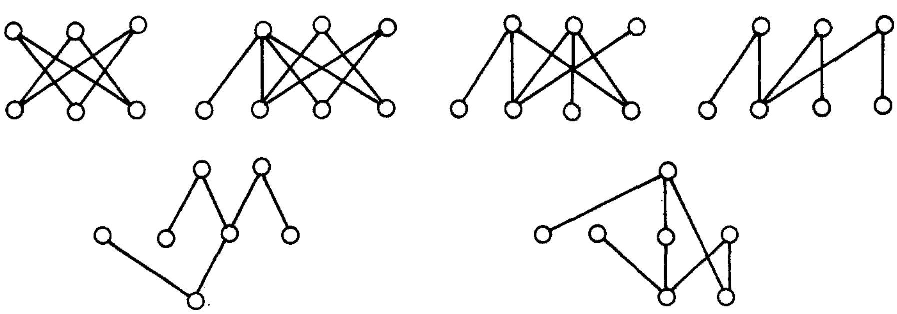

Figure 1

THEOREM 7. *Let A be an antichain of a poset X with $|X - A| = 3$. Then Dim $X < 3$ unless X contains a subposet isomorphic to one of the posets of Figure 1 or its dual.*

We do not include the details of the arguments necessary to prove Theorem 7 in this paper. However we comment that the posets in $\mathscr{C}_3$ are among the 24 irreducible posets of dimension 3 on six and seven points [9]. We also comment that the task of testing a poset X for dimension two is simplified considerably by the technique of choosing a point p_x in the plane for each $x \in X$ such that $x > y$ in X if and only if p_x is above and to the right of p_y. Of course we are using here the alternate definition of Dim X due to Ore [6]; specifically Dim X is the smallest positive integer n for which X is isomorphic to a subposet of R^n, where R^n has the usual product ordering. The selection of points is further simplified by drawing horizontal and vertical rays emanating in the positive direction from each point p_x. For example consider the poset in Figure 2; the diagram in Figure 3 shows that it has dimension two.

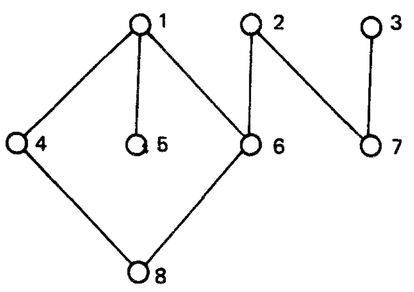

Figure 2

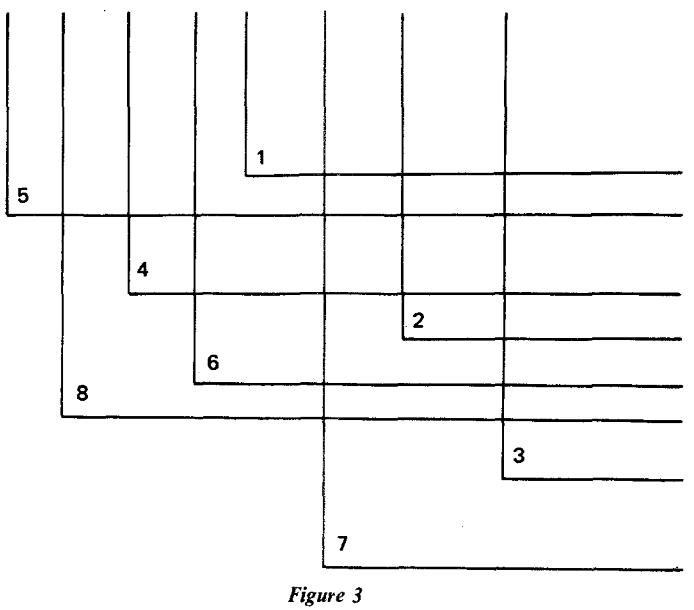

Figure 3

5. An Open Problem. A forbidden poset characterization of Hiraguchi's inequality Dim $X \leq [|X|/2]$ has recently been completed by *R.* Kimble [5]; the result for $|X| \geq 8$ was stated in Section 2. The principal value of the inequality given in Theorem 1 is that it gives, when combined with the well known inequality Dim $X \leq$ width (X), a simple proof of Hiraguchi's inequality. Hiraguchi proved this last inequality but was apparently unaware of Theorem 1. It would be interesting to provide a forbidden subposet characterization of the inequality Dim $X \leq$ width (X). This characterization is likely to be somewhat more difficult than the result given in this paper. However, some examples which must be included in any list of forbidden subposets may found in [9].

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