Applications of the Probabilistic Method to Partially Ordered Sets

William T. Trotter

Department of Mathematics, Arizona State University, Tempe, Arizona 85287, U.S.A.

This paper is dedicated to Paul Erdős with appreciation for his impact on mathematics and the lives of mathematicians all over the world.

Summary. There are two central themes to research involving applications of probabilistic methods to partially ordered sets. The first of these can be described as the study of random partially ordered sets. One central theme to this research is to define appropriate definitions of a random poset, and G. Brightwell's excellent survey article [1] provides a summary of work in this direction. A second theme involves the application of random methods to more general classes of posets. After this brief introductory section, we present four examples of this theme. The first example is quite elementary and involves fibers and co-fibers, concepts which generalize the notions of chains and antichains. The principal result here is an application of random methods to provide a non-trivial upper bound on the minimum size of fibers.

Our second example is more substantial. It involves the dimension of subposets of the subset lattice, an instance in which much of the classic techniques and results pioneered by Paul Erdős play major roles. The third example involves an application of the Lovász Local Lemma and leads naturally to the the investigation of the dimension of a random poset of height two.

Our last example involves fractional dimension for posets—an area where there are many attractive open problems. This topic leads to natural questions involving Ramsey theory for probability spaces.

1. Introduction

Probabilistic methods have been used extensively throughout combinatorial mathematics, so it is no great surprise to see that researchers have applied these techniques with great success to finite partially ordered sets. One central theme to this research is to define appropriate definitions of a random poset, and G. Brightwell's excellent survey article [1] provides a summary of work in this direction.

A second theme involves the application of random methods to more general classes of posets. After this brief introductory section, we present four examples of this theme. The first example is quite elementary and involves fibers and co-fibers, concepts which generalize the notions of chains and antichains. The principal result here is an application of random methods to provide a non-trivial upper bound on the minimum size of fibers.

Our second example is more substantial. It involves the dimension of subposets of the subset lattice, an instance in which much of the classic techniques and results pioneered by Paul Erdős play major roles. The third example involves an application of the Lovász Local Lemma and leads naturally to the the investigation of the dimension of a random poset of height two.

Our last example involves fractional dimension for posets—an area where there are many attractive open problems. This topic leads to natural questions involving Ramsey theory for probability spaces.

The remainder of this section is notation necessary for the remainder of this paper. A partially ordered set (or poset) $P = (X, \leq)$ is a set $X$ and a reflexive, antisymmetric relation $\leq$, which we call $X$ the ground set of the poset $P$. The notations $x \leq y$ in $P$, $y \geq x$ in $P$, $y \succ x$ in $P$, and $x \npreceq y$ in $P$ are throughout the discussion. We write $x \equiv y$ in $P$ if $x \leq y$ and $y \leq x$ in $P$. When $x, y \in X$, $(x, y)$ is incomparable and write $x \nparallel y$ in $P$.

Although we are concerned almost exclusively with posets with finite ground sets, we find it useful to denote $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ to denote respectively the integers, rational numbers, and natural numbers, and $\mathbb{R}$ to denote the real numbers. We use $a < b$ to mean $a = b$ and $b$ is not an element of $a$. A subset $A \subseteq X$ is called an antichain if no two elements of $A$ are comparable. We also use $P + Q$ to denote the direct product of posets $P$ and $Q$.

In the remainder of this article, we consider posets with the basic concepts for partially ordered sets, elements, chains, chains, sums, products, and Hasse diagrams. For additional background, we refer the reader to volume 3 of this series. For further details, we refer the reader to the author's monograph [2].

2. Fibers and Co-Fibers

The classic theorem of Dilworth [4] states that every partially ordered set (or poset) $P = (X, \leq)$ can be partitioned into $n$ chains. All theorems state that comparability graphs have devoted considerable energy to studying antichains. Here is one such example.

Let $P = (X, \leq)$ be a poset. Let $n \in \mathbb{N}$ if it intersects every non-trivial maximal chain in $P$ for $P$ taken over all $n$-element posets. Let $\text{cof}(n)$ denote the maximum number of maximal elements which are not maximal in any other chain whose maximum is not maximal in any other chain. Then $\text{cof}(n) \leq [n/2]$. On the other hand, let $\text{fib}(n)$ denote the size of a height 2 poset with $[n/2]$ minimal elements and $\text{cof}(n)$ maximal elements. So $\text{cof}(n) = \lfloor n/2 \rfloor$.

Dually, a subset $B \subseteq X$ is called an antichain. Let $\text{fib}(P)$ denote the least value $n$ such that $\text{fib}(n)$ is the maximum value of $\text{fib}(P)$.

Trivially, $\text{fib}(n) \geq \lfloor n/2 \rfloor$, and $\text{fib}(n)$ is the maximum value of $\text{fib}(P)$. In [6], Duffus, Sands, Sauer, and Trotter prove that if $P$ is an $n$-element poset, then there exists a

---

1991 Mathematics Subject Classification. 06A07, 05C35.

Key words and phrases. Partially ordered set, poset, graph, random methods, dimension, fractional dimension, chromatic number

Research supported in part by the Office of Naval Research.
The remainder of this section is a very brief condensation of key ideas and notation necessary for the remaining five sections. In this article, we consider a partially ordered set (or poset) \( P = (X, P) \) as a discrete structure consisting of a set \( X \) and a reflexive, antisymmetric and transitive binary relation \( P \) on \( X \). We call \( X \) the ground set of the poset \( P \), and we refer to \( P \) as a partial order on \( X \). The notations \( x \leq y \) in \( P \), \( y \geq x \) in \( P \) and \((x, y) \in P \) are used interchangeably, and the reference to the partial order \( P \) is often dropped when its definition is fixed throughout the discussion. We write \( x < y \) in \( P \) and \( y > x \) in \( P \) when \( x \leq y \) in \( P \) and \( x \neq y \). When \( x, y \in X \), \((x, y) \notin P \) and \((y, x) \notin P \), we say \( x \) and \( y \) are incomparable and write \( x \nparallel y \) in \( P \).

Although we are concerned almost exclusively with finite posets, i.e., those posets with finite ground sets, we find it convenient to use the familiar notation \( \mathbb{R} \), \( \mathbb{Q} \), \( \mathbb{Z} \) and \( \mathbb{N} \) to denote respectively the reals, rationals, integers and positive integers equipped with the usual orders. Note that these four infinite posets are total orders; in each case, any two distinct points are comparable. Total orders are also called linear orders, or chains. We use \( n \) to denote an \( n \)-element chain with the points labelled as \( 0 < 1 < \ldots < n-1 \).

A subset \( A \subseteq X \) is called an antichain if no two distinct points in \( A \) are comparable. We also use \( P + Q \) to denote the disjoint sum of \( P \) and \( Q \).

In the remainder of this article, we will assume that the reader is familiar with the basic concepts for partially ordered sets, including maximal and minimal elements, chains and antichains, sums and cartesian products, comparability graphs and Hasse diagrams. For additional background information on posets, the reader is referred to the author’s monograph [23], the survey article [14] on dimension by Kelly and Trotter and the author’s survey articles [21], [22], [25] and [26]. Another good source of background information on posets is Brightwell’s general survey article [2].

2. Fibers and Co-Fibers

The classic theorem of Dilworth [4] asserts that a poset \( P = (X, P) \) of width \( n \) can be partitioned into \( n \) chains. Also, a poset of height \( h \) can be partitioned into \( h \) antichains. For graph theorists, these results can be translated into the simple statement that comparability graphs are perfect. Against this backdrop, researchers have devoted considerable energy to generalizations of the concepts of chains and antichains. Here is one such example.

Let \( P = (X, P) \) be a poset. Loebl and Rival [18] called a subset \( A \subseteq X \) a co-fiber if it intersects every non-trivial maximal chain in \( P \). Let \( \text{cof}(P) \) denote the least \( m \) so that \( P \) has a co-fiber of cardinality \( m \). Then let \( \text{cof}(n) \) denote the maximum value of \( \text{cof}(P) \) taken over all \( n \)-element posets. In any poset, the set \( A_1 \) consisting of all maximal elements which are not minimal elements and the set \( A_2 \) of all minimal elements which are not maximal elements. As \( A_1 \cap A_2 = \emptyset \), it follows that \( \text{cof}(n) \leq \lfloor n/2 \rfloor \). On the other hand, the fact that \( \text{cof}(n) \geq \lfloor n/2 \rfloor \) is evidenced by a height 2 poset with \( \lfloor n/2 \rfloor \) minimal elements each of which is less than all \( \lfloor n/2 \rfloor \) maximal elements. So \( \text{cof}(n) = \lfloor n/2 \rfloor \) (this argument appears in [18]).

Dually, a subset \( B \subseteq X \) is called a fiber if it intersects every non-trivial maximal antichain. Let \( \text{fib}(P) \) denote the least \( m \) so that \( P \) has a fiber of cardinality \( m \). Then let \( \text{fib}(n) \) denote the maximum value of \( \text{fib}(P) \) taken over all \( n \)-element posets. Trivially, \( \text{fib}(n) \geq \lfloor n/2 \rfloor \), and Loebl and Rival asked whether equality holds.

In [6], Duffus, Sands, Sauer and Woodrow showed that if \( P = (X, P) \) is an \( n \)-element poset, then there exists a set \( F \subseteq X \) which intersects every 2-element
maximal antichain so that $|F| \leq \lfloor n/2 \rfloor$. However, B. Sands then constructed a 17-point poset in which the smallest fiber contains 9 points. This construction was generalized by R. Maltby [19] who proved that for every $\epsilon > 0$, there exist a $n_0$ so that for all $n > n_0$ there exists an $n$-element poset in which the smallest fiber has at least $(8/15 - \epsilon)n$ points.

From above, there is no elementary way to see that there exists a constant $\alpha > 0$ so that $\text{fib}(n) < (1 - \alpha)n$. However, this is an instance where random methods provided real insights into the truth. In the remainder of this paper, we use the notation $[n]$ to denote the $n$-element set $\{1, 2, \ldots, n\}$ (No order is implied on $[n]$, except for the natural order on positive integers).

**Theorem 2.1.** Let $P = (X, P)$ be a poset with $|X| = n$. Then $X$ contains a fiber of cardinality at most $4n/5$. Consequently, $\text{fib}(n) \leq 4n/5$.

**Proof.** Let $C \subseteq X$ be a maximum chain. Then $X - C$ is a fiber. So we may assume that $|C| < n/5$. Label the points of $C$ as $x_1 < x_2 < \ldots < x_t$, where $t = |C| < n/5$. Next, we define two different partitions of $X - C$. First, for each $i \in [t]$, set $U_i = \{x \in X - C : i \text{ is the least integer for which } x \parallel x_i \}$. Then set $D_i = \{x \in X - C : i \text{ is the largest integer for which } x \parallel x_i \}$.

Then for each subset $S \subseteq [t - 1]$, define

$$B(S) = C \cup (\bigcup_{i \in S} D_i) \cup (\bigcup_{i \in [t] \setminus S} U_i).$$

**Claim 1.** For every subset $S \subseteq [t - 1], B(S)$ is a fiber.

**Proof.** Let $S \subseteq [t - 1]$ and let $A$ be a non-trivial maximal antichain. We show that $A \cap B(S) \neq \emptyset$. This intersection is nonempty if $A \cap C \neq \emptyset$, so we may assume that $A \cap C = \emptyset$. Now the fact that $C$ is a maximal chain implies that every point of $C$ is comparable with one or more points of $A$. However, no point of $C$ can be greater than one point of $A$ and less than another point of $A$. Also, $x_1$ can only be less than points in $A$, and $x_t$ can be greater than points in $A$. It follows that $t \geq 2$ and that there is an integer $i \in [t - 1]$ and points $a', a'' \in A$ for which $x_i < a' \in P$ and $a_t > a'' \in P$. Clearly, $a' \in D_i$ and $a'' \in U_i$. If $i \in S$, then $D_i \subseteq B(S)$, and if $i \notin S$, then $U_i \subseteq B(S)$. In either case, we conclude that $A \cap B(S) \neq \emptyset$.

**Claim 2.** The expected cardinality of $B(S)$ with all subsets $S \subseteq [t - 1]$ equally likely is $t + 3(n - t)/4$.

**Proof.** Note that $C \subseteq B(S)$, for all $S$. For each element $x \in X - C$, let $i$ and $j$ be the unique integers for which $x \in D_i$ and $x \in U_j$. Then $j \neq i + 1$. It follows that the probability that $x$ belongs to $B(S)$ is exactly $3/4$.

To complete the proof of the theorem, we note that there is some $S \subseteq [t - 1]$ for which the fiber $B(S)$ has at most $t + 3(n - t)/4$ points. However, $t < n/5$ implies that $t + 3(n - t)/4 < 4n/5$.

The preceding theorem remains an interesting (although admittedly elementary) illustration of applying random methods to general partially ordered sets. Characteristically, it shows that an $n$-point poset has a fiber containing at most $4n/5$ points without actually producing the fiber. Furthermore, this is also an instance in which the constant provided by random methods can be improved by another approach.

The following result is due to Duffus, Kierstead and Trotter [5].

**Theorem 2.2.** (Duffus, Kierstead and Trotter) Let $\mathcal{H}$ be the hypergraph of non-trivial maximal fibers of $P$ when $P$ is a poset. Then $\dim(\mathcal{H}) \geq 3$. Since any two of the $3$-fibers of $\mathcal{H}$ is a fiber, the following interesting result provides an upper bound on the dimension of $\mathcal{H}$.

**Theorem 2.3.** (Lascu) Let $P = (X, P)$ has a fiber of cardinality at most $4n/5$.

I am still tempted to assert that

3. Dimension Theory

When $P = (X, P)$ is a poset, a line $L$ and a fiber $F$ when $x < y \in L$ for all $x, y \in F$. A subset $\mathcal{F}$ of $P$ is called a realization of $P$ when $P(L, F)$ only if $x < y \in L$. For every $L \in \mathcal{F}$, we call this the dimension of $L$ and is denoted $\text{dim}(\mathcal{F})$.

It is useful to have a simple test of whether $P$ is a trivial or a non-trivial fiber. This is the definition. Let $\text{inc}(P) = \text{inc}(X, P)$. Then a family $\mathcal{I}$ of linear extensions $I : (x, y) \in \text{inc}(X, P)$, there exist distinct $L$ and $y > x \in L'$.

Here is a more useful test. Call $P$ a linear extension of $P$. Let $P$ be a linear extension of $P$. Let $P$ be a linear extension of $P$. Let $P$ be a linear extension of $P$. Let $P$ be a linear extension of $P$. Let $P$ be a linear extension of $P$.

The set of all critical pairs $P$, where $x, y \in X$ are $x \leq y \leq x$. For every linear extension $P$, there is some $L \in X$ such that $\text{inc}(P, L)$ if $x < y$. The minimum number of linear extensions.

For every $\mathcal{F}$ of $\mathcal{F}$, let $\mathcal{F}_n$ denote the set of all $\mathcal{F}_n$ of $\mathcal{F}_n$.

Note that $\text{dim}(\mathcal{F}_n)$ is at most $n$, since $\mathcal{F}_n$ is extended to reverse the $n$-dimensional $\mathcal{F}_n$.

4. The Dimension of Subposets

For integers $k, r, n$ with $1 \leq k \leq n$, $\text{dim}(\mathcal{F}_n)$ consisting all $k$-element and all $r$-element by inclusion. By simplicity, we use $d_n$.
Theorem 2.2. (Duffus, Kierstead and Trotter) Let $P = (X, P)$ be a poset and let $H$ be the hypergraph of non-trivial maximal antichains of $P$. Then the chromatic number of $H$ is at most 3.

Theorem 2.2 shows that $\text{fib}(n) \leq 2n/3$, since whenever $X = B_1 \cup B_2 \cup B_3$ is a 3-coloring of the hypergraph $H$ of non-trivial maximal antichains, then the union of any two of $\{B_1, B_2, B_3\}$ is a fiber. Quite recently, Lonc [17] has obtained the following intersesting result providing a better upper bound for posets with small width.

Theorem 2.3. (Lonc) Let $P = (X, P)$ be a poset of width 3 and let $|X| = n$. Then $P$ has a fiber of cardinality at most $11n/18$.

I am still tempted to assert that $\lim_{n \to \infty} \text{fib}(n)/n = 2/3$.

3. Dimension Theory

When $P = (X, P)$ is a poset, a linear order $L$ on $X$ is called a linear extension of $P$ when $x \leq y$ in $L$ for all $x, y \in X$ with $x < y$ in $P$. A set $\mathcal{R}$ of linear extensions of $P$ is called a realizer of $P$ when $P = \cap \mathcal{R}$, i.e., for all $x, y \in X$, $x < y$ in $P$ if and only if $x < y$ in $L$, for every $L \in \mathcal{R}$. The minimum cardinality of a realizer of $P$ is called the dimension of $P$ and is denoted $\dim(P)$.

It is useful to have a simple test to determine whether a family of linear extensions of $P$ is actually a realizer. The first such test is just a reformulation of the definition. Let $\text{inc}(P) = \text{inc}(X, P)$ denote the set of all incomparable pairs in $P$.

Then a family $\mathcal{R}$ of linear extensions of $P$ is a realizer of $P$ if and only if for every $(x, y) \in \text{inc}(X, P)$, there exist distinct linear extensions $L, L' \in \mathcal{R}$ so that $x > y$ in $L$ and $y > x$ in $L'$.

Here is a more useful test. Call a pair $(x, y) \in X \times X$ a critical pair if:

1. $x \parallel y$ in $P$;
2. $z < x$ in $P$ implies $z < y$ in $P$, for all $z \in X$; and
3. $w > y$ in $P$ implies $w > x$ in $P$, for all $w \in X$.

The set of all critical pairs of $P$ is denoted $\text{crit}(P)$ or $\text{crit}(X, P)$. Then it is easy to see that the family $\mathcal{R}$ of linear extensions of $P$ is a realizer of $P$ if and only if for every critical pair $(x, y)$, there is some $L \in \mathcal{R}$ with $x > y$ in $L$. We say that a linear order $L$ on $X$ reverses $(x, y)$ if $x > y$ in $L$. So the dimension of a poset is just the minimum number of linear extensions required to reverse all critical pairs.

For each $n \geq 3$, let $S_n$ denote the height 2 poset with $n$ minimal elements $a_1, a_2, \ldots, a_n$, $n$ maximal elements $b_1, b_2, \ldots, b_n$, and $a_i < b_j$ for $i, j \in [n]$ and $i \neq j$. The poset $S_n$ is called the standard example of an $n$-dimensional poset. Note that $\dim(S_n)$ is at most $n$, since $\text{crit}(S_n) = \{(a_i, b_i) : i \in [n]\}$ and $n$ linear extensions suffice to reverse the $n$ critical pairs in $\text{crit}(S_n)$. On the other hand, $\dim(S_n) \geq n$, since no linear extension can reverse more than one critical pair.

4. The Dimension of Subposets of the Subset Lattice

For integers $k$, $r$ and $n$ with $1 \leq k < r < n$, let $P(k, r; n)$ denote the poset consisting of all $k$-element and all $r$-element subsets of $\{1, 2, \ldots, n\}$ partially ordered by inclusion. For simplicity, we use $\dim(k, r; n)$ to denote the dimension of $P(k, r; n)$.
Historically, most researchers have concentrated on the case \( k = 1 \). In a classic 1950 paper in dimension theory, Dushnik [7] gave an exact formula for \( \dim(1, r; n) \), when \( r \geq 2 \sqrt{n} \).

**Theorem 4.1.** (Dushnik) Let \( n, r \) and \( j \) be positive integers with \( n \geq 4 \) and \( 2 \leq r < n - 1 \). If \( j \) is the unique integer with \( 2 \leq j \leq \sqrt{n} \) for which \[
\frac{n - 2 j + j^2}{j} \leq k < \frac{n - 2(j - 1) + (j - 1)^2}{j - 1},
\]
then \( \dim(1, r; n) = n - j + 1 \). \( \square \)

No general formula for \( \dim(1, r; n) \) is known when \( r \) is relatively small in comparison to \( n \), although some surprisingly tight estimates have been found. Here is a very brief overview of this work, beginning with an elementary reformulation of the problem. When \( L \) is a linear order on \( X \), \( S \subset X \) and \( x \in X - S \), we say \( x > S \) in \( L \) when \( x > s \) in \( L \), for every \( s \in S \).

**Proposition 4.2.** \( \dim(1, r; n) \) is the least \( t \) so that there exist \( t \) linear orders \( L_1, L_2, \ldots, L_t \) of \([n]\) so that for every \( r \)-element subset \( S \subset [n] \) and every \( x \in [n] - S \), there is some \( i \in [t] \) for which \( x > S \) in \( L_i \).

Spencer [20] used this proposition to estimate \( \dim(1, 2; n) \). First, he noted that by the Erdős/Szekeres theorem, if \( n > 2^t \) and \( R \) is any set of \( t \) linear orders on \([n]\), then there exists a 3-element set \( \{x, y, z\} \subset [n] \) so that for all \( L \in R \), either \( x < y < z \) in \( L \) or \( x > y > z \) in \( L \). Thus \( \dim(1, 2; n) > t \) when \( n > 2^t \). On the other hand, if \( n \leq 2^t \), then there exists a family \( R \) of \( t \) linear orders on \([n]\) so that for every 3-element subset \( S \subset [n] \) and every \( x \in S \), there exists some \( L \in R \) so that either \( x < S - \{x\} \) in \( L \) or \( x > S - \{x\} \) in \( L \). Then let \( S \) be the family of 2t linear orders on \( X \) determined by adding to \( R \) the duals of the linear orders in \( R \). Clearly, the 2t linear orders in \( S \) satisfy the requirements of Proposition 4.2 when \( r = 2 \), and we conclude:

**Theorem 4.3.** (Spencer) For all \( n \geq 4 \),
\[
\log \log n < \dim(1, 2; n) \leq 2 \log \log n.
\]

Spencer [20] then proceeded to determine a more accurate upper bound for \( \dim(1, 2; n) \) using a technique applicable to larger values of \( r \). Let \( t \) be a positive integer, and let \( F \) be a family of subsets of \([t]\). Then let \( r \) be an integer with \( 1 \leq r \leq t \). We say \( F \) is \( r \)-scrambling if \( |F| \geq r \) and for every sequence \( (A_1, A_2, \ldots, A_r) \) of \( r \) distinct sets from \( F \) and for every subset \( B \subset [r] \), there is an element \( \alpha \in [t] \) so that \( \alpha \in A_\beta \) if and only if \( \beta \in B \). We let \( M(r, t) \) denote the maximum size of a \( r \)-scrambling family of subsets of \([t]\). Spencer then applied the Erdős/Ko/Rado theorem to provide a precise answer for the size of \( M(2, t) \).

**Theorem 4.4.** (Spencer) \( M(2, t) = \binom{t-1}{\lceil \log_2 t \rceil} \), for all \( t \geq 4 \). \( \square \)

As a consequence, Spencer observed that
\[
\log \log n < \dim(1, 2; n) \leq \log \log n + \left( \frac{1}{2} + o(1) \right) \log \log n.
\]

Almost 20 years later, Füredi, Hajnal, Rödl and Trotter [13] were able to show that the upper bound in this inequality is tight, i.e.,
\[
\dim(1, 2; n) = \log \log n + \left( \frac{1}{2} + o(1) \right) \log \log n.
\]

For larger values of \( r \), Spencer used a different approach. In particular, he used the following theorem.

**Theorem 4.5.** (Spencer) For every \( t \),
\[
M(t, t) > e^t.
\]

**Proof.** Let \( p \) be a positive integer and \( t \) a positive integer whose elements are subsets of \([t]\). Because of such sequences which fail to be \( r \)-scrambling:
\[
\binom{p}{r} 2^{p(t-1)}
\]

So at least one of these sequences is \( \binom{p}{r} 2^{p(t-1)}(2^t - 1)^{q(p-r)t} < 2^{st} \). Clearly, \( e^{2t} \) is a constant larger than 1.

Here's the concept of how the concept of \( \dim(q, r; n) \) is defined:

**Proposition 4.2.** \( \dim(q, r; n) \) is the least \( t \) so that there exist \( t \) linear orders \( L_1, L_2, \ldots, L_t \) of \([n]\) so that for every \( r \)-element subset \( S \subset [n] \) and every \( x \in [n] - S \), there is some \( i \in [t] \) for which \( x > S \) in \( L_i \).

**Theorem 4.6.** (Spencer) If \( p = M(q, r; n) \),
\[
M(q, r; n) > M(q, r; n).
\]

**Proof.** Let \( F \) be an \( r \)-scrambling family of \( n \)-sets, and let \( n = 2^t \). For each \( \alpha \in [t] \), define a linear order \( \alpha \) on \([n] \) by \( x \) and \( y \) be distinct integers from \([n] \) so that \( x > y \) in \( L_\alpha \) if either
\[
1. \alpha \in A_u \text{ and } u \in Q_x - Q_y \text{ or } 2. \alpha \notin A_u \text{ and } u \in Q_y - Q_x.
\]

It is not immediately clear why it is easy to check that this is so. Nevertheless, it is easy to check that this is so. We must show that for every \( \alpha \in [t] \), let \( u_\alpha = \min((Q_x - Q_y) \cup (Q_y - Q_x)) \). Since \( F \) is a \( r \)-scrambling family of subsets of \([n] \), \( \alpha \in A_u \) if and only if \( u \in Q_x \). It is easy to verify that this is so. By putting just a bit of attention to the problem, we actually yield the following upper bounds.

**Theorem 4.7.** (Spencer) For all \( r \geq 4 \),
\[
M(q, r; n) < M(q, r; n).
\]

Of course, this bound is only meaningful for \( r \geq 4 \), but in this range, it is surprising to find such recent results due to Kierstead.

**Theorem 4.8.** (Kierstead) If \( 2 \leq r < n \),
\[
\frac{(r + 2 - \log \log n)}{32 \log (r + 2 - \log \log n)}
\]

We will return to the issue of estimating...
defined on the case \( k = 1 \). In a classic 1950
exact formula for \( \dim(1, r; n) \), when

positive integers with \( n \geq 4 \) and \( 2\sqrt{n} - \sqrt{2} \leq j \leq \sqrt{n} \) for which

\[
(j - 1) + (j - 1)^2/j - 1
\]

when \( r \) is relatively small in comparison
rates have been found. Here is a very
an elementary reformulation of the
\( X \) and \( x \in X - S \), we say \( x > S \) in

so that there exist \( t \) linear orders
subset \( S \subseteq [n] \) and every \( x \in [n] - S \).

\( \dim(1, 2; n) \). First, he noted that
and \( R \) is any set of \( t \) linear orders on \([n]\) so that for all \( L \in R \), either

\( \dim(1, 2; n) > t \) when \( n > 2^{2t} \). On the

by \( \dim(1, 2; n) \leq 2\lg \lg n \).

a more accurate upper bound for
larger values of \( r \). Let \( t \) be a positive in-
then let \( r \) be an integer with \( 1 \leq r \leq t \).

every sequence \( (A_1, A_2, \ldots, A_t) \) of \( r \)
\([n]\), there is an element \( \alpha \in [t] \) so
\( M(r, t) \) denote the maximum size of \( \alpha \) then applied the Erdős/Ko/Rado
size of \( M(2, t) \).

for all \( t \geq 4 \).

\[
\dim(1, 2; n) = \lg \lg n + \left(\frac{1}{2} + o(1)\right) \lg \lg \lg n.
\]

For larger values of \( r \), Spencer used random methods to produce the following
bound.

**Theorem 4.5.** (Spencer) For every \( r \geq 2 \), there exists a constant \( c = c_r > 1 \) so that \( M(r, t) > c_t \).

**Proof.** Let \( p \) be a positive integer and consider the set of all sequences of length \( p \) whose elements are subsets of \([t]\). There are \( 2^{pt} \) such sequences. The number of such sequences which fail to be \( r \)-scrambling is easily seen to be at most

\[
\binom{p}{r} 2^{r(2^t - 1)} 2^{(p - r)t}.
\]

So at least one of these sequences is a \( r \)-scrambling family of subsets of \([t]\) provided

\[
\binom{p}{r} 2^{r(2^t - 1)} 2^{(p - r)t} < 2^{pt}.
\]

Clearly this inequality holds for \( p > c_t \) where \( c_t = c_r \sim e^{2^{2t-1}} \) is a constant larger than 1.

Here’s how the concept of scrambling families is used in provide upper bounds for \( \dim(q, r; n) \).

**Theorem 4.6.** (Spencer) If \( p = M(r, t) \) and \( n \geq 2^p \), then \( \dim(1, r; n) \leq t \).

**Proof.** Let \( \mathcal{F} \) be an \( r \)-scrambling family of subsets of \([t]\), say \( \mathcal{F} = \{A_1, A_2, \ldots, A_t\} \)
where \( p = M(r, t) \). Then set \( n = 2^p \) and let \( Q_1, Q_2, \ldots, Q_n \) be the subsets of \([p] \).

For each \( \alpha \in [t] \), define a linear order \( L_\alpha \) on the set \([n]\) by the following rules. Let \( x \) and \( y \) be distinct integers from \([n]\) and let \( u = \min(Q_x - Q_y) \cup (Q_y - Q_x) \).

Set \( x > y \) in \( L_\alpha \) if either

1. \( \alpha \in A_y \) and \( u \in Q_x - Q_y \), or
2. \( \alpha \notin A_y \) and \( u \notin Q_x - Q_y \).

It is not immediately clear why \( L_\alpha \) is a linear order on \([n]\) for each \( \alpha \in [t] \), but it is easy to check that this is so. Now let \( S \) be an \( r \)-element subset of \([n]\) and let \( x \in [n] - S \). We must show that \( x > S \) in \( L_\alpha \) for some \( \alpha \in [t] \). For each \( y \in S \), let \( u_y = \min(Q_x - Q_y) \cup (Q_y - Q_x) \).

Since \( \mathcal{F} \) is a \( r \)-scrambling family of subsets of \([t]\), there exists some \( \alpha \in [t] \) such that

\( \alpha \in A_{u_y} \) if and only if \( u_y \in Q_x \). It follows from the definition of \( L_\alpha \) that \( x > S \) in \( L_\alpha \).

By paying just a bit of attention to constants, the preceding results of Spencer actually yield the following upper bound on \( \dim(1, r; n) \).

**Theorem 4.7.** (Spencer) For all \( r \geq 2 \), \( \dim(1, r; n) \leq (1 + o(1)) \left(\frac{1}{\lg r} 2^{rt} \lg \lg n\right) \). \( \square \)

Of course, this bound is only meaningful if \( r \) is relatively small in comparison to \( n \), but in this range, it is surprisingly tight. The following lower bound is a quite recent result due to Kierstead.

**Theorem 4.8.** (Kierstead) If \( 2 \leq r \leq \lg \lg n - \lg \lg \lg n \), then

\[
\frac{(r + 2 - \lg \lg n - \lg \lg \lg n)^2 \lg n}{2 \lg (r + 2 - \lg \lg n + \lg \lg \lg n)} \leq \dim(1, r; n). \quad \square
\]

We will return to the issue of estimating \( \dim(1, r; n) \) in the next section.
5. The Dimension of Posets of Bounded Degree

Given a poset \( P = (X, P) \) and a point \( x \in X \), define the degree of \( x \) in \( P \), denoted \( \text{deg}_P(x) \), as the number of points in \( X \) which are comparable to \( x \). This is just the degree of the vertex \( x \) in the associated comparability graph. Then define \( \Delta(P) \) as the maximum degree of \( P \). Finally, define \( \text{Dim}(k) \) as the maximum dimension of a poset \( P \) with \( \Delta(P) \leq k \). Rödl and Trotter were the first to prove that \( \text{Dim}(k) \) is well defined. Their argument showed that \( \text{Dim}(k) \leq 2k^2 + 2 \). It is now possible to present a very short argument for this result by first developing the following idea due to Füredi and Kahn [12].

For a poset \( P = (X, P) \) and a point \( x \in X \), let \( U(x) = \{ y \in X : y > x \text{ in } P \} \) and let \( U[x] = U(x) \cup \{ x \} \). Dually, let \( D(x) = \{ y \in X : y < x \text{ in } P \} \) and \( D[x] = D(x) \cup \{ x \} \). The following proposition admits an elementary proof. In fact, something more can be said, and we will comment on this in the next section.

**Proposition 5.1.** (Füredi and Kahn) Let \( P = (X, P) \) be a poset and let \( L \) be any linear order on \( X \). Then there exists a linear extension \( L' \) of \( P \) so that if \( (x, y) \) is a critical pair and \( x > D[y] \) in \( L \), then \( x > y \) in \( L' \), so that \( x > D[y] \) in \( L' \). \( \square \)

**Theorem 5.2.** (Rödl and Trotter) If \( P = (X, P) \) is a poset with \( \Delta(P) \leq k \), then \( \text{Dim}(P) \leq 2k^2 + 2 \).

**Proof.** Define a graph \( G = (X, E) \) as follows. The vertex set \( X \) is the ground set of \( P \). The edge set \( E \) contains those two element subsets \( \{ x, y \} \) for which \( U[x] \cap U[y] \neq \emptyset \). Clearly, the maximum degree of a vertex in \( G \) is at most \( k^2 \). Therefore, the chromatic number of \( G \) is at most \( k^2 + 1 \). Let \( t = k^2 + 1 \) and let \( X = X_1 \cup X_2 \cup \ldots \cup X_t \) be a partition of \( X \) into subsets which are independent in \( G \). Then for each \( i \in \{ t \} \), let \( L_i \) be any linear order on \( X \) with \( X_i > X_i - X_i \) in \( L_i \). Finally, define \( L_{t+1} \) to be any linear order on \( X \) so that:

1. \( X_i > X_i - X_i \) in \( L_{t+1} \), and
2. The restriction of \( L_{t+1} \) to \( X_i \) is the dual of the restriction of \( L_i \) to \( X_i \).

We claim that for every critical pair \( (x, y) \in \text{crit}(P) \), if \( x \in X_i \), then either \( x > D[y] \) in \( L_i \) or \( x > D[y] \) in \( L_{t+1} \). This claim follows easily from the observation that any two points of \( D[y] \) are adjacent in \( G \) so that \( D[y] \cap X_i \leq 1 \).

Füredi and Kahn [12] made a dramatic improvement in the upper bound for \( \text{Dim}(k) \) by applying the Lovász Local Lemma [9]. We sketch their argument which begins with an application of random methods to provide an upper bound for \( \text{Dim}(1, r; n) \). In this sketch, we make no attempt to provide the best possible constants.

**Theorem 5.3.** (Füredi and Kahn) Let \( r \) and \( n \) be integers with \( 1 < r < n \). If \( t \) is an integer such that

\[
n \left( \frac{n - 1}{r} \right) \left( \frac{r}{r + 1} \right)^t < 1,
\]

then \( \text{Dim}(1, r; n) \leq t \). In particular, \( \text{Dim}(1, r; n) \leq r(r + 1) \log(n/r) \).

**Proof.** Let \( t \) be an integer satisfying the inequality given in the statement of the theorem. Then let \( \{ L_i : i \in \{ t \} \} \) be a sequence of \( t \) random linear orders on \( X \). The expected number of pairs \( (x, S) \) where \( S \) is an \( r \)-element subset of \( [n] \), \( x \in [n] - S \) and there is no \( i \in \{ t \} \) for which \( x > S \) in \( L_i \) is exactly the weight of the left hand side of this inequality is calculating. It follows that this quantity is less than one, so the probability that there are no such pairs is positive. This shows that \( \text{Dim}(1, r; n) \leq t \).

The estimate \( \text{Dim}(1, r; n) \leq r(r + 1) \log(n/r) \) follows easily.

**Theorem 5.4.** (Füredi and Kahn) If \( \text{dim}(P) \leq 100k \log^2 k \), then \( \text{dim}(P) \leq 100k \log^2 k \), i.e., \( \text{Dim}(k) \leq 100k \log^2 k \).

**Proof.** The inequality \( \text{dim}(P) \leq 100k \log^2 k \) is true for \( k \leq 1000 \), so we may assume that \( k \) is large. Using the Lovász Local Lemma, we let \( Y_1, Y_2, \ldots, Y_m \), with \( |D[x] \cap Y_i| \leq r \), for each \( i \), and let \( s = \text{dim}(1, r; q) \). We construct a graph \( G = (X, E) \) so that for every \( r \)-element subset \( [R] \) of \( X \) and some \( j \in [s] \), for which \( x > S \) in \( M_j \). For each \( j \in [s] \), define \( L_{i,j} \).

- If \( Y_i > X - Y_i \) in \( L_{i,j} \), and
- If \( a < b \in M_j \), then \( Y_{a,b} < Y_{b,a} \).

Finally, for each \( j \in [s] \), define \( L_{i,j} \).

- \( Y_i > X - Y_i \) in \( L_{i,j+1} \),
- If \( a < b \in M_j \), then \( Y_{a,b} < Y_{b,a} \),
- If \( a \in [n] \), then the restriction of \( L_{i,j+1} \) to \( Y_{i,a} \).

Next we claim that if \( (x, y) \in \text{crit}(P) \), then \( x > D[y] \) in \( L_{i,j} \) or \( x > D[y] \) in \( L_{i,j+1} \). This claim follows easily from the observation that any two points of \( D[y] \) are adjacent in \( G \) so that \( D[y] \cap X_i \leq 1 \).

Finally, we note that \( s = \text{dim}(1, r; n) \).

There are two fundamentally important theorems following inequality limiting the dimension of a poset. Unfortunately, the details of the proof also suggest that one could provide a better upper bound. Fortunately, the second approach will provide the following lower bound:

**Theorem 5.5.** (Kierstead) If \( \text{dim}(P) \leq 100k \log^2 k \), then \( \text{dim}(P) \leq 100k \log^2 k \).

As a consequence, it follows that the remaining challenge is to provide better upper bounds that seem to be our best hope. Here is a result by Kierstead [8] to show that \( \text{Dim}(k) \).
Bounded Degree

Define the degree of \( x \) in \( P \), denoted \( n(k) \), as the maximum dimension of a poset of width \( k \). We start by proving that \( \text{dim}(k) \leq 2k^2 + 2 \). It is now possible to select the first developing the following idea.

Let \( X \) be a poset and let \( L \) be any extension \( L' \) of \( P \) so that \( (x, y) \) is an element of \( L' \), so that \( x > D[y] \) in \( L' \).

The vertex set \( X \) is the ground set of subsets \( \{x, y\} \) for which \( \{x, y\} \notin \{x, y\} \in G \). Thus, \( k = k^2 + 1 \) and let \( X = X_1 \cup X_2 \cup \ldots \cup X_t \) be independent in \( G \). Then for each \( i \in \{1, \ldots, t\} \), let \( X_i \) be independent in \( G \). Finally, define \( L_{i+1} \) to be the restriction of \( L_i \) to \( X_i \).

If \( (x, y) \in \text{crit}(G) \), then \( x > Y \) in \( L_{i+1} \).

Proof. Let \( (x, y) \) be a critical pair and \( x \in X_i \), then either it follows easily from the observation that \( [D[y] \cap X_i] \leq 1 \).

Theorem 5.4. (Füredi and Kahn) If \( P = (X, P) \) is a poset for which \( \Delta(P) \leq k \), then \( \text{dim}(P) \leq 100k \log^2 k \), i.e., \( \text{Dim}(k) \leq 100k \log^2 k \).

Proof. The inequality \( \text{dim}(P) \leq 100k \log^2 k \) follows from the preceding theorem if \( k \leq 1000 \), so we may assume that \( k > 1000 \). Set \( m = [k/\log k] \) and \( r = [9 \log k] \).

Using the Lovász Local Lemma, we see that there exists a partition \( X = Y_1 \cup Y_2 \cup \ldots \cup Y_m \), with \( |D[x] \cap Y_i| \leq r \), for every \( x \in X \). Now fix \( i \in \{1, \ldots, m\} \), let \( q = rk + 1 \) and let \( s = \text{dim}(1, r, q) \). We construct a family \( R_{i} = \{L_{i,j} : j \in \{1, \ldots, f\}\} \) as follows.

Let \( G \) be the graph on \( X \) defined in the proof of Theorem 5.2. Then let \( G_i \) be the subgraph induced by \( Y_i \). Now it is easy to see that any point of \( Y_i \) is adjacent at most \( rk \) other points in \( Y_i \) in the graph \( G_i \). It follows that the chromatic number of \( G_i \) is at most \( k^2 + 1 \). Let \( Y_i = Y_{i,1} \cup \ldots \cup Y_{i,q} \) be a partition into subsets each of which is independent in \( G_i \). Then let \( R = \{M_j : j \in \{1, \ldots, f\}\} \) be a family of linear orders of \( \{y\} \) so that for every \( r \)-element subset \( S \subseteq \{y\} \) and every \( x \in \{y\} - S \), there is some \( j \in \{s\} \) for which \( x > S \) in \( M_j \).

Then for each \( i \in \{s\} \), define \( L_{i,q} \) as any linear order for which:

1. \( Y_i > Y_i \) in \( L_{i,q} \) and
2. \( a < b \) in \( M_j \), then \( Y_{i,a} < Y_{i,b} \) in \( L_{i,q} \).

Finally, for each \( i \in \{s\} \), define \( L_{i+1} \) as any linear order for which:

1. \( Y_i > Y_i \) in \( L_{i+1} \),
2. \( a < b \) in \( M_j \), then \( Y_{i,a} < Y_{i,b} \) in \( L_{i+1} \), and
3. \( a \in \{y\} \), then the restriction of \( L_{i+1} \) to \( Y_{i,a} \) is the dual of the restriction of \( L_{i+1} \) to \( Y_{i,a} \).

Next we claim that if \( (x, y) \) is a critical pair and \( x \in X_i \), then there is some \( j \in \{s\} \) so that \( x > D[y] \) in \( L(j) \). To see this observe that any two points in \( D[y] \) are adjacent in \( G \) so at most \( r \) points in \( D[y] \) belong to \( Y_i \), and all points of \( D[y] \) in \( Y_i \) belong to distinct subsets in the partition of \( Y_i \) into independent subsets. Let \( x \in Y_{i,j} \). Then there exists some \( j \in \{s\} \) so that \( j > j \) in \( M_j \) whenever \( j \neq j \) and \( D[y] \) in \( Y_{i,j} \). It follows that either \( x > D[y] \) in \( L(i,j) \) or \( x > D[y] \) in \( L(i,s+j) \).

Finally, we note that \( s = \text{dim}(1, r, q) \leq r(r + 1) \log(q/r) \), so that \( \text{dim}(P) \leq 100k \log^2 k \) as claimed.

There are two fundamentally important problems which leap out from the preceding inequality limiting the dimension of posets of bounded degree, beginning with the obvious question: Is the inequality \( \text{Dim}(k) = O(k \log^2 k) \) best possible? However, the details of the proof also suggest that the inequality could be improved if one could provide a better upper bound than \( \text{dim}(1, \log k; k) = O(\log^3 k) \). Unfortunately, the second approach will not yield much as Kierstead [15] has recently provided the following lower bound.

Theorem 5.5. (Kierstead) If \( \log \log n - \log \log \log \log n \leq r \leq 2k^{1/2} n \), then

\[
\left( \frac{r + 2 - \log \log n + \log \log \log n}{3 \log(2k^{1/2} \log n)} \right) \log n \leq \text{dim}(1, r; n) \leq \frac{2k^2 \log^2 n}{\log^2 k}.
\]
For a fixed positive integer $n$, consider a random poset $P_n$ having $n$ minimal elements $a_1, a_2, \ldots, a_n$ and $n$ maximal elements $b_1, b_2, \ldots, b_n$. The order relation is defined by setting $a_i < b_j$ with probability $p = p(n)$; also, events corresponding to distinct min-max pairs are independent.

Erdős, Kierstead and Trotter then determine estimates for the expected value of the dimension of the resulting random poset. The arguments are far too complex to be conveniently summarized here, as they make non-trivial use of correlation inequalities. However, the following theorem summarizes the lower bounds obtained in [8].

**Theorem 5.6.** (Erdős, Kierstead and Trotter)

1. For every $\epsilon > 0$, there exists $\delta > 0$ so that if
   \[ \frac{\log^{1+\epsilon} n}{n} < p \leq \frac{1}{\log n}, \]
   then
   \[ \dim(P) > \delta n \log n, \text{ for almost all } P. \]
2. For every $\epsilon > 0$, there exist $\delta, c > 0$ so that if
   \[ \frac{1}{\log n} \leq p < 1 - n^{-1+\epsilon}, \]
   then
   \[ \dim(P) > \max\{\delta n, n - \frac{cn}{p \log n}\}, \text{ for almost all } P. \]

The following result is then an easy corollary.

**Corollary 5.7.** (Erdős, Kierstead and Trotter) For every $\epsilon > 0$, there exists $\delta > 0$ so that if
   \[ n^{-1+\epsilon} < p \leq \frac{1}{\log n}, \]
   then
   \[ \dim(P) > \delta \Delta(P) \log n, \text{ for almost all } P. \]

Summarizing, we now know that
   \[ \Omega(k \log k) = D(k) = O(k \log^2 k). \tag{5.3} \]

It is the author’s opinion that the upper bound is more likely to be correct and that the proof of this assertion will come from investigating the dimension of a slightly different model of random height 2 posets. For integers $n$ and $k$ with $k$ large but much smaller than $n$, we consider a poset with $n$ minimal points and $n$ maximal points. However, the comparabilities come from taking $k$ random matchings.

The techniques used by Erdős, Kierstead and Trotter in [8] break down when $p = o(\log n/n)$. But this is just the point at which we can no longer guarantee that the maximum degree is $O(pn)$.

6. Fractional Dimension and Probability Spaces

In many instances, it is useful to consider a combinatorial parameter, as in many applications of the original problem. In [3], Brightwell and Scheinerman give a fractional dimension for posets. This offers new results, and many appealing questions and applications with immediate connections.

Let $P = (X,\leq)$ be a poset and let $\mathcal{F}$ be an ultrafilter on $X$. For each incomparable pair $(x,y)$, take $\mathcal{F}$ to reverse the pair $(x,y)$, i.e., $\{x < y\}$ if $x < y$, from $\mathcal{F}$.

**Theorem 6.1.** (Felsner and Trotter) For any poset $P = (X, \leq)$, let $\mathcal{F}$ be any ultrafilter on $X$.

Then $\mathcal{F}$-dim$(P) < k + 1$ unless one of the following holds. (Note that this theorem is a strengthening of Proposition 5.1.)

- There is a poset $P(1;r,n)$ with $\mathcal{F}$-dim$(P(1;r,n)) = r$ and a small fractional dimension. However, this case has a clear generalization in terms of fractional dimension.

**Theorem 6.2.** If $P = (X, \leq)$ is a poset with $|D(x)| \leq k$, for all $x \in X$, then $\mathcal{F}$-dim$(P) < k + 1$ unless one of the following holds.

- There is a poset $P(2;r,n)$ with $\mathcal{F}$-dim$(P(2;r,n)) = r$ and a small fractional dimension. However, this case has a clear generalization in terms of fractional dimension.

**Proof.** Let $\mathcal{F}$ be a multi-realizer of $P$. Let $(x,y) \in \text{crit}(P)$. Then take $t$ to be the number of times $t$ occurs in $\mathcal{F}$.

Then let $\{L_1, \ldots, L_t\}$ be a sequence of sets such that each set is chosen independently and each set is equally likely to be chosen. Then if $t$ is not reversed is less than one, so the probability that $t$ is positive is $\frac{1}{2}$. Felsner and Trotter [10] derive several results and these lead to some challenging problems similar to the one given in Theorem 5.1. The fractional dimension has produced a number of interesting results.
6. Fractional Dimension and Ramsey Theory for Probability Spaces

In many instances, it is useful to consider a fractional version of an integer valued combinatorial parameter, as in many cases, the resulting LP relaxation sheds light on the original problem. In [3], Brightwell and Scheinerman proposed to investigate fractional dimension for posets. This concept has already produced some interesting results, and many appealing questions have been raised. Here's a brief sketch of some questions with immediate connections to random methods.

Let \( P = (X, P) \) be a poset and let \( \mathcal{F} = \{ M_{i_1}, \ldots, M_{i_t} \} \) be a multiset of linear extensions of \( P \). Brightwell and Scheinerman [3] call \( \mathcal{F} \) a \( k \)-fold realizer of \( P \) if for each incomparable pair \( (x, y) \), there are at least \( k \) linear extensions in \( \mathcal{F} \) that reverse the pair \( (x, y) \), i.e., \(|\{ i : 1 \leq i \leq t, x > y \text{ in } M_i \}| \geq k \). The fractional dimension of \( P \), denoted by \( \text{fdim}(P) \), is then defined as the least real number \( q \geq 1 \) for which there exists a \( k \)-fold realizer \( \mathcal{F} = \{ M_1, \ldots, M_t \} \) of \( P \) so that \( k/t \geq 1/q \) (it is easily verified that the least upper bound of such real numbers \( q \) is indeed attained and is a rational number). Using this terminology, the dimension of \( P \) is just the least \( t \) for which there exists a 1-fold realizer of \( P \). It follows immediately that \( \text{fdim}(P) \leq \text{dim}(P) \), for every poset \( P \).

Note that the standard example of an \( n \)-dimensional poset also has fractional dimension \( n \). Brightwell and Scheinerman [3] proved that if \( P \) is a poset and \( |D(x)| \leq k \), for all \( x \in X \), then \( \text{fdim}(P) \leq k + 2 \). They conjectured that this inequality could be improved to \( \text{fdim}(P) \leq k + 1 \). This was proved by Felsner and Trotter [10], and the argument yielded a much stronger conclusion, a result with much the same flavor as Brooks' theorem for graphs.

**Theorem 6.1.** (Felsner and Trotter) Let \( k \) be a positive integer, and let \( P \) be any poset with \( |D(x)| \leq k \), for all \( x \in X \). Then \( \text{fdim}(P) \leq k + 1 \). Furthermore, if \( k \geq 2 \), then \( \text{fdim}(P) < k + 1 \) unless one of the components of \( P \) is isomorphic to \( S_{k+1} \), the standard example of a poset of dimension \( k + 1 \).

We do not discuss the proof of this result here except to comment that it requires a strengthening of Proposition 5.1, and to note that it implies that the fractional dimension of the poset \( P(1, r, n) \) is \( r+1 \). Thus a poset can have large dimension and small fractional dimension. However, there is one element bound which limits dimension in terms of fractional dimension.

**Theorem 6.2.** If \( P = (X, P) \) is a poset with \( |X| = n \) and \( \text{fdim}(P) = q \), then \( \text{dim}(P) \leq (2 + o(1))q \log n \).

**Proof.** Let \( \mathcal{F} \) be a multi-realizer of \( P \) so that \( \text{Prob}_{\mathcal{F}}[x > y] \geq 1/q \), for every critical pair \( (x, y) \in \text{crit}(P) \). Then take \( t \) to be any integer for which

\[
(n-1)(1-1/q)^t < 1.
\]

Then let \( \{ L_1, \ldots, L_t \} \) be a sequence of length \( t \) in which the linear extensions in \( \mathcal{F} \) are equally likely to be chosen. Then the expected number of critical pairs which are not reversed is less than one, so the probability that we have a realizer of cardinality \( t \) is positive.

Felsner and Trotter [10] derive several other inequalities for fractional dimension, and these lead to some challenging problems as to the relative tightness of inequalities similar to the one given in Theorem 6.1. However, the subject of fractional dimension has produced a number of challenging problems which are certain to
require random methods in their solutions. Here is two such problems, one of which has recently been solved.

A poset \( P = (X, P) \) is called an interval order if there exists a family \( \{[a_x, b_x] : x \in X \} \) of non-empty closed intervals of \( \mathbb{R} \) so that \( x < y \) in \( P \) if and only if \( b_x < a_y \) in \( \mathbb{R} \). Fishburn [11] showed that a poset is an interval order if and only if it does not contain \( 2 + 2 \) as a subposet. The interval order \( I_n \) consisting of all intervals with integer endpoints from \( \{1, 2, \ldots, n\} \) is called the canonical interval order.

Although posets of height 2 can have arbitrarily large dimension, this is not true for interval orders. For these posets, large height is a prerequisite for large dimension.

**Theorem 6.3.** (Füredi, Hajnal, Rödl and Trotter) If \( P = (X, P) \) is an interval order of height \( n \), then

\[
\dim(P) \leq \lg \lg n + (1/2 + o(1)) \lg \lg \lg n.
\]  

(6.1)

The inequality in the preceding theorem is best possible.

**Theorem 6.4.** (Füredi, Hajnal, Rödl and Trotter) The dimension of the canonical interval order satisfies

\[
\dim(I_n) = \lg \lg n + (1/2 + o(1)) \lg \lg \lg n.
\]  

(6.2)

Although interval orders may have large dimension, they have bounded fractional dimension. Brightwell and Scheinerman [3] proved that the dimension of any finite interval order is less than 4, and they conjectured that for every \( \epsilon > 0 \), there exists an interval order with dimension greater than \( 4 - \epsilon \). We believe that this conjecture is correct, but confess that our intuition is not really tested. For example, no interval order is known to have fractional dimension greater than 3.

Motivated by the preceding inequalities and the known bounds on the dimension and fractional dimension of interval orders and the posets \( P(1, r; n) \), Brightwell asked whether there exists a function \( f : Q \rightarrow \mathbb{R} \) so that if \( P = (X, P) \) is a poset with \( |X| = n \) and \( \dim(P) = g \), then \( \dim(P) \leq f(g) \lg \lg n \). If such a function exists, then the family \( P(1, r; n) \) shows that we would need to have \( f(g) = \Omega(2^g) \).

But we will show that there is no such function. The argument requires some additional notation and terminology. Fix integers \( n \) and \( k \) with \( 1 \leq k < n \). We call an ordered pair \((A, B)\) of \( k\)-element sets a \((k, n)\)-shift pair if there exists a \((k + 1)\)-element subset \( C = \{i_1 < i_2 < \cdots < i_{k+1}\} \subseteq \{1, 2, \ldots, n\} \) so that \( A = \{i_1, i_2, \ldots, i_k\} \) and \( B = \{i_2, i_3, \ldots, i_{k+1}\} \). We then define the \((k, n)\)-shift graph \( S(k, n) \) as the graph whose vertex set consists of all \( k\)-element subsets of \( \{1, 2, \ldots, n\} \) with a \( k\)-element set \( A \) adjacent to a \( k\)-element set \( B \) exactly when \((A, B)\) is a \((k, n)\)-shift pair. Note that the \((1, n)\) shift graph \( S(1, n) \) is just a complete graph. It is customary to call a \((2, n)\)-shift graph just a shift graph; similarly, a \((3, n)\)-shift graph is called a double shift graph. The formula for the chromatic number of the \((2, n)\)-shift graph \( S(2, n) \) is a folklore result of graph theory: \( \chi(S(2, n)) = \lfloor \log n \rfloor \). Several researchers in graph theory have told me that this result is due to András Hajnal, but András says that it is not. In any case, it is an easy exercise.

The following construction exploits the properties of the shift graph to provide a negative answer for Brightwell’s question.

**Theorem 6.5.** For every \( m \geq 3 \), there exists a poset \( P = (X, P) \) so that

1. \( |X| = m^2 \);
2. \( \dim(P) \geq \lg m \); and
3. \( \dim(P) \leq 4 \).

**Proof.** The poset \( P = (X, P) \) is composed of \( i, j \leq m \), so that \( |X| = m^2 \). The \( x(i, j) \) is the \( (i, j) \)-th element in \( P \), for each \( i \in [m] \), \( x(i_1, j) < x(i_2, j) \) for each \( i \in [m] \), \( x(i_1, j) < x(i_2, j) \).

We now show that \( \dim(P) \geq j \leq m, x(j, j - i), x(j, m) \). Let \( \dim(P) \) be the dimension of \( P \). For each \( i, j \) with \( 1 \leq i < j \leq m \), let \( \phi(i, j, x(i, j) \) be the \( (i, j) \)-th element in \( P \), and \( x(j, k) \). Our argument now divides into two cases:

1. \( x(i, j) > x(j, m) \), \( x(i, j) < x(j, m) \).

The inequalities in equation 6.3 cannot be proper coloring of the shift graph, \( \dim(P) \geq \lg m \).

Finally, we show that \( \dim(P) \) be the natural projection maps defined on the following.

Next, we claim that for each subset \( A \) of \( P \) so that \( x \geq \dim(P) \) in \( L(A) \) if:

1. \( x \geq |y| \) in \( P \);
2. \( p_x \in A \) and \( p_y \notin A \).

To show that such linear extension, see Chapter 2 of [23]. Let \( A \subseteq [m] \), \( p_x \in A \) and \( p_y \notin A \). Now \( p_y \) is an alternating cycle of length \( p_x \) and \( p_y \) (subscripts are interpreted cyclically). It follows that \( p_x < p_y \) in \( P \), for \( p_x, p_y \in P \). Thus \( p_x \in L(A) \) and \( p_y \notin L(A) \). The contradiction shows cycles. Thus the desired linear extension.

Finally, we note that if we take \( x > y \) in at least \( s/4 \) of the linear extensions of \( P \), we observe that there are exactly \( 2^s \) subsets \( A \) that contain \( p_z \). This shows that the proof of the theorem.

Now we turn our attention to the question of the least subset lattice \( D \subseteq X \) is called a down set. \( D \) always imply that \( x \in P \). The following is known to other researchers in the

**Theorem 6.6.** Let \( n \) be a positive integer. Then the least subset lattice \( L(n) \) is the least subset lattice \( 2^n \).
1. $|X| = m^2$; 
2. $\dim(X, P) \geq \log m$; and 
3. $\dim_f(X, P) \leq 4$.

**Proof.** The poset $P = (X, P)$ is constructed as follows. Set $X = \{x(i, j) : 1 \leq i, j \leq m\}$, so that $|X| = m^2$. The partial order $P$ is defined by first defining $x(i, j_1) < x(i, j_2)$ in $P$, for each $i \in [m]$ whenever $1 \leq j_1 < j_2 \leq m$. Furthermore, for each $i \in [m]$, $x(i, j_1) < x(i_2, j_2)$ in $P$ if and only if $(i_2 - i_1) + (j_2 - j_1) > m$.

We now show that $\dim(X, P) \geq \log m$. Note first that for each $i, j$ with $1 \leq i < j \leq m$, $x(i, j - i) > x(j, m)$. Let $\dim(X, P) = t$, and let $R = \{L_0, L_1, \ldots, L_t\}$ be a realization of $P$. For each $i, j$ with $1 \leq i < j \leq m$, choose an integer $d(i, j) \in \{1, 2, \ldots, t\}$ so that $x(i, j - d(i, j)) > x(j, m)$ in $L_d$. We claim that if $\phi$ is a proper coloring of the $(2, m)$ shift graph $S(2, m)$ using $t$ colors, which requires that $\dim(X, P) = t \geq \chi(S(2, m)) = [\log m]$. To see that $\phi$ is a proper coloring, let $i, j$ and $k$ be integers with $1 \leq i < j < k \leq m$, let $\phi(i, j) = \alpha$ and let $\phi(j, k) = \beta$. If $\alpha = \beta$, then $x(i, j - i) > x(j, m)$ in $L_d$ and $x(j, k - j) > x(k, m)$ in $L_d$. Also, $x(j, m) > x(j, k - j)$ in $P$. However, since $(k - i) + (m - j + i) > m$, it follows that $x(k, m) > x(i, j - i)$ in $P$. Thus, 

$$x(i, j - i) > x(j, m) > x(i, j - i) > x(k, m) > x(i, j - i)$$

in $P$.

The inequalities in equation (6.3) cannot all be true. The contradiction shows that $\phi$ is a proper coloring of the shift graph $S(2, m)$ as claimed. In turn, this shows that $\dim(X, P) \geq [\log m]$.

Finally, we show that $\dim_f(X, P) \leq 4$. For each element $x \in X$, let $p_1$ and $p_2$ be the natural projection maps defined by $p_1(x) = i$ and $p_2(x) = j$ when $x = x(i, j)$. Next, we claim that for each subset $A \subseteq [m]$, there exists a linear extension $L(A)$ of $P$ so that $x > y$ in $L(A)$ if:

1. $x \parallel y$ in $P$;
2. $p_1(x) \in A$ and $p_2(y) \notin A$.

To show that such linear extensions exist, we use the alternating cycle test (see Chapter 2 of [23]). Let $A \subseteq [m]$, and let $S(A) = \{(x, y) \in X \times X : x \parallel y \text{ in } P, p_1(x) \in A \text{ and } p_2(y) \notin A\}$. Now suppose that $\{u_k, v_k\} : 1 \leq k \leq p \subseteq S(A)$ is an alternating cycle of length $p$, i.e., $u_k \parallel v_k$ and $u_k \parallel v_{k+1}$ in $P$, for all $k \in [p]$ (subscripts are interpreted cyclically). Let $k \in [m]$. Then $p_1(u_k) \in A$ and $p_1(v_{k+1}) \in A$. It follows that $u_k < v_{k+1}$ in $P$, for each $k \in [p]$. It follows that $p_1(u_k) = p_1(v_{k+1}) = p_2(v_{k+1}) - p_2(u_k) > m$. Also, we know that $m \geq p_1(u_k) - p_1(u_k) + p_2(u_k) - p_2(u_k)$. Thus, $p_1(u_k) + p_2(u_k) > p_1(v_k) + p_2(v_k)$. Clearly, this last inequality cannot hold for all $k \in [p]$. The contradiction shows that $S(A)$ cannot contain any alternating cycles. Thus the desired linear extension $L(A)$ exists.

Finally, we note that if we take $F = \{L \in \mathcal{L}(A) : A \subseteq [\emptyset]\}$ and set $s = [F]$, then $x > y$ in at least $s/4$ of the linear extensions in $F$, whenever $x \parallel y$ in $P$. To see this, observe that there are exactly $2^s/4$ subsets of $[m]$ which contain $p_1(x)$ but do not contain $p_1(y)$. This shows that $\dim(X, P) \leq 4$ as claimed. It also completes the proof of the theorem.

Now we turn our attention to the double shift graph. If $P = (X, P)$ is a poset, a subset $D \subseteq X$ is called a down set, or an order ideal, if $x \leq y$ in $P$ and $y \in D$ always imply that $x \in D$. The following result appears in [13] but may have been known to other researchers in the area.

**Theorem 6.6.** Let $n$ be a positive integer. Then the chromatic number of the double shift graph $S(3, n)$ is the least $t$ so that there are at least $n$ down sets in the subset lattice $2^n$.  \[\square\]
The problem of counting the number of down sets in the subset lattice $2^i$ is a classic problem and is traditionally called Dedekind’s problem. Although no closed form expression is known, relatively tight asymptotic formulas have been given. For our purposes, the estimate provided by Kleitman and Markowsky [16] suffices. Theorem 6.6, coupled with the estimates from [16] permit the following surprisingly accurate estimate on the chromatic number $\chi(S(3,n))$ of the double shift graph [13].

**Theorem 6.7.** (Füredi, Hajnal, Rödl and Trotter)

$$\chi(S(3,n)) = \lg \lg n + (1/2 + o(1)) \lg \lg \lg n.$$  

Now that we have introduced the double shift graph, the following elementary observation can be made [13].

**Proposition 6.8.** For each $n \geq 3$, $\dim(1,2;n) \geq \chi(S(3,n))$, and $\dim(I_n) \geq \chi(S(3,n)).$

Although the original intent was to investigate questions involving the fractional dimension of posets, Trotter and Winkler [27] began to attack a ramsey theoretic problem for probability spaces which seems to have broader implications. Fix an integer $k \geq 1$, and let $n \geq k + 1$. Now suppose that $\Omega$ is a probability space containing an event $E_k$ for every $k$-element subset $S \subseteq \{1,2,\ldots,n\}$. We abuse terminology slightly and use the notation $\Prob(S)$ rather than $\Prob(E_S)$.

Now let $f(\Omega)$ denote the minimum value of $\Prob(A \bar{B})$, taken over all $(k,n)$-shift pairs $(A,B)$. Note that we are evaluating the probability that $A$ is true and $B$ is false. Then let $f(n,k)$ denote the maximum value of $f(\Omega)$ and let $f(k)$ denote the limit of $f(n,k)$ as $n$ tends to infinity.

Even the case $k = 1$ is non-trivial, as it takes some work to show that $f(1) = 1/4$. However, there is a natural interpretation of this result. Given a sufficiently long sequence of events, it is inescapable that there are two events, $A$ and $B$ with $A$ occurring before $B$ in the sequence, so that

$$\Prob(A \bar{B}) < \frac{1}{4} + \epsilon.$$  

The $\frac{1}{4}$ term in this inequality represents coin flips. The $\epsilon$ is present because, for finite $n$, we can always do slightly better than tossing a fair coin.

For $k = 2$, Trotter and Winkler [27] show that $f(2) = 1/3$. Note that this is just the fractional chromatic number of the double shift graph. This result is also natural and comes from taking a random linear order $L$ on $\{1,2,\ldots,n\}$ and then saying that a 2-element set $\{i,j\}$ is true if $i < j$ in $L$. Trotter and Winkler conjecture that $f(3) = 3/8$, $f(4) = 2/5$, and are able to prove that $\lim_{k \to \infty} f(k) = 1/2$. They originally conjectured that $f(k) = k/(2k + 2)$, but they have since been able to show that $f(5) \geq 27/64$, which is larger than $2/5$.

As an added bonus to this line of research, we are beginning to ask natural (and I suspect quite important) questions about patterns appearing in probability spaces.

**References**


den sets in the subset lattice $2^\mathbb{N}$ is a
soekend’s problem. Although no closed
asymptotic formulas have been given.
kleitman and Markovsky [16] suffices.
[16] permit the following surprisingly
results on $\chi(S(n,n))$ of the double shift graph [13].
(12) 
Duffus, H. Kierstead and W. T. Trotter, Fibres and ordered set coloring,
Duffus, B. Sands, N. Sauer and R. Woodrow, Two coloring all two-element
D. Dushnik, Concerning a certain set of arrangements, Proc. Amer. Math. Soc. 1
(1950), 788–796.
3. Erdős, H. Kierstead and W. T. Trotter, The dimension of random ordered
4. Erdős and L. Lovász, Problems and results on $3$-chromatic hypergraphs and
some related questions, in Infinite and Finite Sets, A. Hajnal et al., eds., North
Holland, Amsterdam (1975) 609–628.
S. Felsner and W. T. Trotter, On the fractional dimension of partially ordered
5. P. Fishburn, Intransitive indifference with unequal indifference intervals, J.
Comb. Theory Series 7 (1970), 144–149.
6. Z. Füredi and J. Kahn, On the dimensions of ordered sets of bounded degree,
7. Z. Füredi, P. Hajnal, V. Rödl, and W. T. Trotter, Interval orders and shift
8. D. Kelly and W. T. Trotter, Dimension theory for ordered sets, in Proceedings
of the Symposium on Ordered Sets, I. Rival et al., eds., Reidel Publishing (1982),
171–212.
10. D. Kleitman and G. Markovsky, On Dedekind’s problem: The number of
11. Z. Lonc and I. Rival, Chains, antichains and fibers, J. Comb. Theory Series
Hungar. 22, 349–353.
14. W. T. Trotter, Graphs and Partially Ordered Sets, in Selected Topics in Graph
15. W. T. Trotter, Problems and conjectures in the combinatorial theory of ordered
16. W. T. Trotter, Combinatorics and partially Ordered Sets: Dimension Theory,
17. W. T. Trotter, Progress and new directions in dimension theory for finite partially
ordered sets, in Extremal Problems for Finite Sets, P. Frankl, Z. Füredi, G.
A Bound of the Cardinality Containing \( \Delta \)-System

A. V. Kostochka*

Institute of Mathematics, Siberian Branch of the USSR Academy of Sciences

Dedicated to Professor Paul Erdős on his 70th birthday.

Summary. P. Erdős and R. Rado defined a \( \Delta \)-system \( \mathcal{H} \) if every two members have the same intersection. Let \( \varphi(n) \) (respectively, \( \gamma(n) \)) denote the maximum cardinality of an \( n \)-element family \( \mathcal{F} \) of \( \Delta \)-systems of cardinality 3. Namely, we prove that for any \( n \),

\[
\varphi(n) \leq 2^n
\]

and conjectured that

\[
\varphi(n) < 2^n \quad \text{for } n \geq 3.
\]

1. Introduction

P. Erdős and R. Rado [2] introduced the concept of a \( \Delta \)-system \( \mathcal{H} \) of finite sets if every two members have the same intersection. Let \( \varphi(n) \) (respectively, \( \gamma(n) \)) denote the maximum cardinality of an \( n \)-element family \( \mathcal{F} \) of \( \Delta \)-systems of cardinality 3. P. Erdős and R. Rado [2] proved

\[
\varphi(n) \leq 2^n
\]

and conjectured that

\[
\varphi(n) < 2^n \quad \text{for } n \geq 3.
\]

The best published upper bound for \( \varphi(n) \) is

\[
\varphi(n) < 2^n - 2
\]

Z. Füredi and J. Kahn (see [1]) proved

\[
\varphi(n) \leq 2^n - n - 1
\]

The aim of the present paper is to prove

Theorem 1.1. For any integer \( \alpha \geq 1 \),

\[
\varphi(n) \leq 2^n - n^\alpha
\]

* This work was partly supported by the Russian Foundation for Basic Research.