New Perspectives on Interval Orders and Interval Graphs

William T. Trotter

Summary Interval orders and interval graphs are particularly natural examples of two widely studied classes of discrete structures: partially ordered sets and undirected graphs. So it is not surprising that researchers in such diverse fields as mathematics, computer science, engineering and the social sciences have investigated structural, algorithmic, enumerative, combinatorial, extremal and even experimental problems associated with them. In this article, we survey recent work on interval orders and interval graphs, including research on on-line coloring, dimension estimates, fractional parameters, balancing pairs, hamiltonian paths, ramsey theory, extremal problems and tolerance orders. We provide an outline of the arguments for many of these results, especially those which seem to have a wide range of potential applications. Also, we provide short proofs of some of the more classical results on interval orders and interval graphs. Our goal is to provide fresh insights into the current status of research in this area while suggesting new perspectives and directions for the future.

1 Introduction

A complex process (manufacturing computer chips, for example) is often broken into a series of tasks, each with a specified starting and ending time. Task $A$ precedes Task $B$ if $A$ ends before $B$ begins. When $A$ precedes $B$, the output of $A$ can safely be used as input to $B$, and resources dedicated to the completion of $A$, such as machines or personnel, can now be applied to $B$. When $A$ and $B$ have overlapping time periods, they may be viewed as conflicting tasks, in the sense that they compete for limited resources.

This short paragraph is intended to motivate the formal definition of two of the most widely studied classes of discrete structures in all of combinatorial mathematics: interval orders and interval graphs. The main point to the discussion is that interval orders and interval graphs are important from an applications standpoint. This much is inescapable. They occur so naturally and with such frequency that they must be studied. Fortunately, the study of interval orders and interval graphs has yielded work of intrinsic interest and beauty, work that can be appreciated for its elegance independent of the fact that many find it useful and important.

The remainder of this section includes a brief summary of the notation and terminology necessary for the balance of the paper. For a more comprehensive treatment of background material, the reader is referred to Peter Fishburn's monograph Interval Orders and Interval Graphs [36]. Other recommended sources for background information are the author's survey articles [115], [116], [120], [121] and monograph [118] and the books by Golumbic [48] and
Roberts [98].

Throughout this paper, we consider a partially ordered set (or poset) \( P = (X, P) \) as a structure consisting of a set \( X \) and a reflexive, antisymmetric and transitive binary relation \( P \) on \( X \). We call \( X \) the ground set of the poset \( P \), and we call \( P \) a partial order on \( X \). The notations \( x \leq y \) in \( P \), \( y \geq x \) in \( P \) and \( (x, y) \in P \) are used interchangeably, and the reference to the partial order \( P \) is often dropped when its definition is fixed throughout the discussion. We write \( x < y \) in \( P \) and \( y > x \) in \( P \) when \( x \leq y \) in \( P \) and \( x \neq y \). When \( x, y \in X \), \( (x, y) \notin P \) and \( (y, x) \notin P \), we say \( x \) and \( y \) are incomparable and write \( x \parallel y \) in \( P \).

When \( P = (X, P) \) is a poset, we call the partial order \( P^d = \{(y, x) : (x, y) \in P\} \) the dual of \( P \) and we let \( P^d = (X, P^d) \).

When \( P \) is a binary relation on \( X \) and \( Y \subseteq X \), we denote the restriction of \( P \) to \( Y \) by \( P(Y) \). When \( P \) is a partial order on \( X \), \( Q = P(Y) \) is a partial order on \( Y \) and \( Q = (Y, Q) \) is called a subposet of \( P = (X, P) \). Also, we call \( Q \) the subposet determined by \( Y \). When \( X_1, X_2, \ldots, X_r \subseteq X \), we will find it convenient to denote the subposets they determine by \( X_1, X_2, \ldots, X_r \), respectively. In this article, we tend not to distinguish between isomorphic posets, so we abuse language slightly and say that a poset \( Q \) is contained in another poset \( P \) when \( Q \) is isomorphic to a subposet of \( P \).

Although we are concerned primarily with finite posets, i.e., those posets with finite ground sets, we find it convenient to use the familiar notation \( \mathbb{R}, \mathbb{Q}, \mathbb{Z} \) and \( \mathbb{N} \) to denote respectively the reals, rationals, integers and positive integers equipped with the usual orders. Note that these four infinite posets are total orders; in each case, any two distinct points are comparable. Total orders are also called linear orders, or chains. When \( X = X_1 \cup X_2 \cup \cdots \cup X_r \) is a partition and \( L_i \) is a linear order on \( X_i \) for each \( i = 1, 2, \ldots, r \), we let \( L = L_1 < L_2 < \cdots < L_r \) denote the linear order on \( X \) defined by \( x < y \) in \( L \) if and only if \( x \in X_i, y \in X_j \) and either \( i < j \) or both \( i = j \) and \( x < y \) in \( L_i \).

For a positive integer \( n \), we let \( [n] \) denote the \( n \)-element chain \( 0 < 1 < \cdots < n-1 \). Somewhat inconsistently, we let \( [n] \) denote the \( n \)-element set \( \{1, 2, \ldots, n\} \). Also, when \( X \) is a set, we let \( \binom{X}{n} \) denote the set of all \( n \)-element subsets of \( X \).

Let \( P = (X, P) \) be a poset, and let \( \mathcal{F} = \{Q_x = (Y_x, Q_x) : x \in X\} \) be a family of posets indexed by the elements of \( X \). Define the lexicographic sum of \( \mathcal{F} \) over \( P \), denoted \( \sum_{x \in P} \mathcal{F} \), as the poset \( Q = (Y, Q) \) where \( Y = \{(x, y) : x \in X, y \in Y_x \} \) and \( (x_1, y_1) < (x_2, y_2) \) in \( Q \) if and only if \( x_1 < x_2 \) in \( P \), or if both \( x_1 = x_2 \) and \( y_1 < y_2 \) in \( Q_x \). With this definition, a disjoint sum is just a lexicographic sum over a two-element antichain.

In the remainder of this article, we will assume some familiarity with the basic concepts for partially ordered sets. The author's survey article on partially ordered sets [120] provides a thorough overview of the combinatorial aspects. Other sources for background material on posets are Brightwell's survey article [17] and the author's other survey articles [115], [112], [117] and [121].

## 2 Interval orders and Interval Graphs

A poset \( P = (X, P) \) is called an interval order or interval representation if it is possible to assign to each element \( x \in X \) a closed interval \( I(x) = [a_x, b_x] \) in \( L \) such that \( x < y \) in \( P \) if and only if \( b_x < a_y \) in \( L \). We call \( L \) the length of the interval, i.e., \( |I(x)| = b_x - a_x \).

Note that end points of intervals used need not be distinct. In fact, distinct points \( x \) and \( y \) can be equal. We even allow degenerate intervals. On the other hand, \( P \) is said to be distinguishing if all intervals are distinct. It is easy to see that every interval order is an interval representation. In fact, since we are considering intervals, it could have just as well required that all intervals are distinct.

Analogously, a graph \( G = (V, E) \) is an interval representation if there is a function \( I \) which assigns to each vertex \( x \in V \) a closed interval \( I(x) = [a_x, b_x] \) from a linearly ordered set \( L \) so that \( \{x, y\} \in E \) if and only if \( I(x) \cap I(y) \neq \emptyset \). As before, we call \( I \) an interval representation if \( G \) is an interval graph.

Throughout this article, we will move between graphs in discussions about a family of intervals spanned by a family of intervals is just the independence of the set of vertices of a clique. Chains correspond to independent sets of vertices.

## 3 Classical representation theorems

A good fraction of the early research on interval orders was focused on characterization issues. Recent research has not contained a cycle on four or more vertices. Also, a vertex \( x \) in a graph \( G \) is simplicial if it is a part of a simplicial subgraph of \( G \). Thus, a simplicial subgraph is a graph that is triangle-free. Triangulated graphs are graphs that contain no induced cycles of length 4 or more. Obviously, interval graphs are triangulated, and the triangulated graphs are interval graphs. The class of interval graphs, e.g., the subdivision of a cycle.

Three distinct vertices \( x, y, z \) in a graph \( G \) form a split triple when for each two vertices in \( \{x, y, z\} \), there is no vertex on the path adjacent to the third.
2 Interval orders and interval graphs

A poset \( P = (X,P) \) is called an interval order if there is a function \( I \) assigning to each element \( x \in X \) a closed interval \( I(x) = [a_x,b_x] \) of a linearly ordered set \( L \) (usually, we take \( L \) as the real line \( \mathbb{R} \)) so that for all \( x, y \in X \), \( x < y \) in \( P \) if and only if \( b_x < a_y \) in \( L \). We call \( I \) an interval representation of \( P \), or just a representation for short. For brevity, whenever we say that \( I \) is a representation of an interval order \( P = (X, P) \), we will use the alternate notation \( [a_x,b_x] \) for the closed interval \( I(x) \). Also, we let \( |I(x)| \) denote the length of the interval, i.e., \( |I(x)| = b_x - a_x \).

Note that end points of intervals used in a representation need not be distinct. In fact, distinct points \( x \) and \( y \) from \( X \) may satisfy \( I(x) = I(y) \). We even allow degenerate intervals. On the other hand, a representation is said to be distinguishing if all intervals are non-degenerate and all end points are distinct. It is easy to see that every interval order has a distinguishing representation. In fact, since we are concerned only with finite posets, we could have just as well required that all intervals used in the representation be open.

Analogously, a graph \( G = (V,E) \) is an interval graph when there is a function \( I \) which assigns to each vertex \( x \in V \) a closed interval \( I(x) = [a_x,b_x] \) from a linearly ordered set \( L \) so that \( \{x,y\} \in E \) if and only if \( I(x) \cap I(y) \neq \emptyset \). As before, we call \( I \) an interval representation of \( G \) and note that, if desired, we may assume \( I \) is distinguishing.

Throughout this article, we will move back and forth between posets and graphs in discussions about a family of intervals. The interval graph determined by a family of intervals is just the incomparability graph of the interval order. Chains correspond to independent sets and antichains correspond to cliques.

3 Classical representation theorems

A good fraction of the early research on interval graphs and interval orders was focused on characterization issues. Recall that a graph is triangulated if it does not contain a cycle on four or more vertices as an induced subgraph. Also, a vertex \( x \) in a graph \( G \) is simplicial if its neighborhood is a complete subgraph of \( G \), so a graph is triangulated if and only if every induced subgraph has a simplicial vertex. Triangulated graphs are a well studied class of perfect graphs (see [58] and Chapter 4 of Golumbic’s monograph [48], for example). Obviously, interval graphs are triangulated, but it is natural to ask whether all triangulated graphs are interval graphs. This is not true. In fact, not all trees are interval graphs, e.g. the subdivision of \( K(1,3) \) is not an interval graph.

Three distinct vertices \( x, y \) and \( z \) in a graph \( G \) are said to form an asteroidal triple when for each two vertices in \( \{x,y,z\} \), there is a path joining them, with no vertex on the path adjacent to the third. For example, the three leaves in a
subdivision of $K(1, 3)$ form an asteroidal triple. In [83], Lekkerkerker and Boland proved that a triangulated graph is an interval graph if and only if it does not contain any asteroidal triples. They used this characterization theorem to provide a minimum list of forbidden subgraphs for interval graphs. This list includes the cycles on four or more vertices, three other infinite families and two isolated examples. One of these is the subdivision of $K(1, 3)$.

Other characterizations of interval graphs in terms of forbidden substructures have been provided by Gilmore and Hoffman [47] and by Ghouila-Houri [46]. Characterizations of interval graphs by forbidden subgraphs or forbidden substructures provide important structural information about the properties of interval graphs but do not necessarily yield a useful algorithm. Using a special kind of data structure called a PQ-tree, Booth and Lueker [15] produced an $O(n^2)$ algorithm for testing whether a graph $G$ on $n$ vertices is an interval graph and producing the representation when it is.

Characterization problems for interval graphs are closely related to characterization problems for comparability graphs. The classic paper of Gallai [45] provides a forbidden subgraph (again, in terms of induced subgraphs) characterization of comparability graphs with a minimum list including eight infinite families and 10 isolated examples. A comparability graph may have many different transitive orientations, but Gallai shows that if $T_1$ and $T_2$ are transitive orientations of the same comparability graph, then $T_1$ may be transformed into $T_2$ by a finite sequence of reversals applied to autonomous sets. Gallai’s paper remains one of the deepest and most important contributions to this subject.

Next, we discuss three important representation theorems which are essential to understanding the material which follows. First, a finite poset $P = (X, P)$ is called a weak order if there exists a function $f : X \rightarrow \mathbb{R}$ so that for all $x, y \in X$ with $x \neq y$,

1. $x < y$ in $P$ if and only if $f(x) < f(y)$ in $\mathbb{R}$, and

2. $x \parallel y$ if and only if $f(x) = f(y)$.

The following elementary result is left as an exercise (see [36], e.g.).

**Proposition 3.1** Let $P = (X, P)$ be a poset. Then the following are equivalent.

1. $P$ is a weak order.

2. $P$ does not contain $2 + 1$ as a subposet.

3. $P$ is the lexicographic sum of a family of antichains over a chain. ■

Given a representation $I$ of an interval order $P = (X, P)$, there are two natural weak orders defined on $X$ by the end points. The ordering by left end points $L$ defined by $x < y$ in $L$ if and only if $a_x < a_y$ in $\mathbb{R}$ and the ordering by right end points $R$ defined by $x < y$ in $R$ if and only if $b_x < b_y$ in $\mathbb{R}$, the representation is distinguishing, these vary.

Next, we have the following characterization due to Fishburn [35]. Our argument is motivated by Scott and Suppes [101] and Greenough [51] and requires the following.

For a poset $P = (X, P)$ and a subset $S$ of $X$, there exists some $x \in S$ with $y < x$ in $P$. Also, let $y < x$ in $P$. Then, say $S = \{x\}$, we write $D(x)$ and $D[x]$ rather than $D_{\leq}(x)$ and $D_{\leq}[x]$ for a subset $S \subseteq X$, we define $U(S) = \{y \in X : y > x \text{ in } P\}$. As before, set $U[S] = U(S) \cup S$.

**Theorem 3.2** Let $P = (X, P)$ be a poset.

1. $P$ is an interval order.

2. $P$ does not contain $2 + 2$ as a subposet.

3. Whenever $x < y$ and $z < w$ in $P$, then $x < z$ in $P$.

4. For every $x, y \in X$, either $D(x) \subseteq D(y)$ or $D(y) \subseteq D(x)$.

5. For every $x, y \in X$, either $U(x) \subseteq U(y)$ or $U(y) \subseteq U(x)$.

**Proof** The equivalence of the last four statements show that statement 1 is equivalent to 4. For an interval order and that if $I$ is an interval representation in $P$, without loss of generality, we may assume $a_x < a_y < \cdots < a_m$.

Now suppose that statement 4 holds for a poset $P$.

A linear order $L$ on $Y$ by $D$ if $D'$ so that $D_1 < D_2 < \cdots < D_m$ in $L$. For each $D(x) = D_i$ and $j = m$ if $x$ is maximal, and otherwise.

One advantage to the proof given here for interval orders is that the total number of construction is minimal. Also, note that $\{D(x) : x \in X\}$ is an interval representation, as pointed out above.

An interval order $P = (X, P)$ is called a semi-order for which it has an interval representation $F$ such that $F(x)$ is exactly $c$, for every $x \in X$. From here, it may be more natural if these objects we consider; however, after rescaling, it may be a matter of intervals used in the representation of a semi-order.

For semi-orders, we have the following representation part of which is due to Scott and Suppes [101] by several authors, including Bogart [4], Fishburn [35].
right end points \( R \) defined by \( x < y \) in \( R \) if and only if \( b_x < b_y \) in \( \mathbb{R} \). When the representation is distinguishing, these weak orders are linear orders.

Next, we have the following characterization theorem for interval orders due to Fishburn [35]. Our argument is motivated by proofs due to Bogart [5] and Greenough [51] and requires the following notation.

For a poset \( P = (X, \leq) \) and a subset \( S \subseteq X \), let \( D(S) = \{ y \in X : \text{there exists } x \in S \text{ with } y < x \text{ in } P \} \). Also, let \( D[S] = D(S) \cup S \). When \( |S| = 1 \), say \( S = \{ x \} \), we write \( D(x) \) and \( D[x] \) rather than \( D(\{ x \}) \) and \( D(\{ x \}) \). Dually, for a subset \( S \subseteq X \), we define \( U(S) = \{ y \in X : \text{there exists } x \in X \text{ with } y > x \text{ in } P \} \). As before, set \( U[S] = U(S) \cup S \).

**Theorem 3.2** Let \( P = (X, \leq) \) be a poset. Then the following are equivalent.

1. \( P \) is an interval order.
2. \( P \) does not contain \( 2 + 2 \) as a subposet.
3. Whenever \( x < y \) and \( z < w \) in \( P \), then either \( x < w \) or \( z < y \) in \( P \).
4. For every \( x, y \in X \), either \( D(x) \subseteq D(y) \) or \( D(y) \subseteq D(x) \).
5. For every \( x, y \in X \), either \( U(x) \subseteq U(y) \) or \( U(y) \subseteq U(x) \).

**Proof** The equivalence of the last four statements is immediate. We now show that statement 1 is equivalent to 4. Suppose first that \( P = (X, \leq) \) is an interval order and that \( I \) is an interval representation of \( P \). Let \( x, y \in X \); without loss of generality, we may assume \( a_x \leq a_y \) in \( \mathbb{R} \). Then \( D(x) \subseteq D(y) \).

Now suppose that statement 4 holds for a poset \( P = (X, \leq) \). We show that \( P \) is an interval order. Let \( Y = \{ D(x) : x \in X \} \), and let \( m = |Y| \). Then define a linear order \( L \) on \( Y \) by \( D < D' \) in \( L \) if \( D \subseteq D' \). Then label the sets in \( Y \) so that \( D_1 < D_2 < \cdots < D_m \) in \( L \). For each \( x \in X \), let \( F(x) = [i, j] \), where \( D(x) = D_i \) and \( j = m \) if \( x \) is maximal, and \( D_{j+1} = \bigcap \{ D(y) : x < y \text{ in } P \} \), otherwise.

One advantage to the proof given here for Fishburn’s representation theorem for interval orders is that the total number of end points used in the representation is minimal. Also, note that \( |\{ D(x) : x \in X \}| = |\{ U(x) : x \in X \}| \), when \( P = (X, \leq) \) is an interval order, as pointed out by Greenough [51].

An interval order \( P = (X, \leq) \) is called a semi-order if there is a constant \( c \) for which it has an interval representation \( F \) such that the length of the interval \( F(x) \) is exactly \( c \), for every \( x \in X \). From a modern perspective, it would perhaps be more natural if these objects were called constant length interval orders; however, after rescaling, it may be assumed that the constant length of intervals used in the representation of a semi-order is 1.

For semi-orders, we have the following representation theorem, the principal part of which is due to Scott and Suppes [104]. Simple proofs have been given by several authors, including Bogart [4], Fishburn [36] and Rabinovitch [92, 91].
Theorem 3.3 Let $P = (X, P)$ be an interval order. Then the following statements are equivalent.

1. $P$ is a semi-order.
2. $P$ does not contain $3 + 1$ as a subposet.
3. Whenever $x < y < z$ and $w || y$ in $P$, then either $x < w$ or $w < z$ in $P$.
4. The binary relation $W$ is a weak order on $X$, where

$$W = \{(x, y) \in X \times X : x = y\} \cup \{(x, y) \in X \times X : D(x) \subseteq D(y), U(y) \subseteq U(x)\} \cup \{(x, y) \in X \times X : D(x) \subseteq D(y), U(y) \subseteq U(x)\}.$$

Proof The equivalence of statements 2, 3 and 4 is immediate. We show that statements 1 and 4 are equivalent. First let $P = (X, P)$ be a semi order and let $I$ be an interval representation in which all intervals have length $c$. Let $I(x) = [a_x, a_x + c]$, for every $x \in X$. Then $(x, y) \in W$ if and only if $a_x < a_y$ in $R$, so that $W$ is a weak order on $X$.

Now suppose that statement 4 holds for a poset $P = (X, P)$. We show that $P$ is a semi-order. We actually prove something stronger. Let $L$ be any linear order on $X$ extending the weak order $W$. Proceeding by induction on $|X|$, we show that there exists a distinguishing interval representation $I$ of $P$ which assigns to each $x \in X$ a unit length interval $I(x) = [a_x, a_x + 1]$ such that for all $x, y \in X$, $a_x < a_y$ in $R$ if and only if $x < y$ in $L$.

Noting that the claim holds trivially when $|X| = 1$, consider the inductive step. Suppose that $L$ orders $X$ as $x_1 < x_2 < \cdots < x_n$. Let $Y = X \setminus \{x_n\}$, let $Q = P(Y)$ and $L' = L(Y)$. In the poset $Q = (Y, Q)$, let $W''$ be the binary relation defined in statement 4 for the subposet $Q$. Then let $W' = W(Y)$. Then $W'' \subseteq W' \subseteq L'$.

It follows that $Q$ is a semi-order and that there exists a distinguishing representation $I'$ of $Q$ so that for all $y, z \in Y$, $a_y < a_z$ if and only if $y < z$ in $L'$. Also, $y < z$ in $Q$ if and only if $a_y + 1 < a_z$. We now show that this representation can be extended by an appropriate choice of an interval $I(x_0) = [a_{x_0}, a_{x_0} + 1]$ for $x_0$. If $y < x_n$ for every $y \in Y$, let $a = \max\{a_y : y \in Y\}$ and set $a_{x_n} = 2 + a$.

So we may assume that $S = \{y \in Y : y || x_n\} = \emptyset$. It follows that there is a positive integer $i$ so that $S = \{x_i, x_{i+1}, \ldots, x_{i-1}\}$, and $[a, b] = \cap \{I(y) : y \in S\}$ is a nondegenerate interval. If $D(x_n) = \emptyset$, set $a' = a$; otherwise, set $a' = \max\{a_y + 1 : y < x_n\}$. In either case, note that $a' < b$ in $R$. It follows that we may take $a_n$ as any real number between $a'$ and $b$ distinct from any end point previously chosen.

There is an important corollary to the Scott-Suppes theorem for semi-orders, a result which was first noted by Roberts [97]. Call an interval order $P = (X, P)$ proper if it admits an interval representation and $x \neq y$, then $I(x) \not\subseteq I(y)$ and $I(y) \not\subseteq I(x)$ proper interval order, but as Roberts pointed out, also a semi-order. We will revisit this later.

4 Dilworth's theorem for interval orders

The width of a finite poset is the maximum cardinality of a chain, and the height is the maximum cardinality of an antichain. Theorem 4.1 asserts that a poset of width $w$ can be partitioned into $w + 1$ chains, and that a poset of height $h$ can be partitioned into $h$ antichains. Some results have been provided by many authors, and we provide one for the second result.

Theorem 4.1 Let $P = (X, P)$ be a poset. Any linear partition

$$X = A_1 \cup A_2 \cup \cdots \cup A_k,$$

with $A_j$ an antichain, for each $j \in [h]$.

Proof For each $x \in X$, let $d(x)$ denote the number of chains containing $x$. Then for each $j \in [h]$, let $A_j = \{x \in X : d(x) = j\}$.

Here is a sketch of how a bipartite matching can be used to find a partition of a poset $P$ and a partition of its chains. Given a poset $P = (X, P)$, form the bipartite graph $G$ with $V = \{a_x : x \in X\} \cup \{b_x : x \in X\}$ and $E = \{(a_x, b_y) : x < y\}$. Then let $\mathcal{M}$ be a maximum matching in $G$ by putting distinct $x, y \in X$ in the same chain if and only if $x < y$ in $P$.

For interval orders, we can be more explicit about the connection with graph coloring. All we need to know is that we can find a coloring of the edges of a bipartite graph by putting distinct $x, y \in X$ in the same chain if and only if $x < y$ in $P$.

Theorem 4.2 Interval graphs are perfect, i.e., they can be colored to the maximum clique size. Furthermore, we can find such a coloring by applying First Fit to the vertices in the graph in a distinguishing representation.

Proof Let $I$ be a distinguishing representation $(X, E)$, and let $L$ be the linear order on $X$ defined by $x < y$ in $L$ if and only if $b_x < b_y$ in $G$. Then $\mathcal{N}_L(x)$ is a clique in $G$. There is a natural coloring of $E$ by putting distinct $x, y \in X$ in the same chain if and only if $x < y$ in $P$.
An interval order. Then the following states:

1. Let $[y]$ be a subposet.
2. If $y \in P$, then either $x < y$ or $w < z$ in $P$.
3. A weak order on $X$, where
   \begin{align*}
   &X : x = y \\
   &\text{and } X : D(x) \subseteq D(y), U(y) \subseteq U(x) \\
   &\times X : D(x) \not\subseteq D(y), U(y) \not\subseteq U(x) \}
   
4. Elements 2, 3 and 4 is immediate. We show that 1.

   First let $P = (X, P)$ be a semi-order and

   in which all intervals have length $e$. Then $y \in P$. Then $(x, y) \in W$ if and only if $a_x \leq a_y$ in $X$.

   It holds for a poset $P = (X, P)$. We show that $a_y$ is something stronger. Let $L$ be any linear order $W$. Proceeding by induction on $|X|$, we

   have the following interval representation $I$ of $P$ which

   for length $e$ and $I(x) = [a_x, a_x + 1]$ such that for

   if $x < y$ in $L$.

   trivially when $|X| = 1$, consider the inductive

   is $x_1 < x_2 < \ldots < x_n$. Let $Y = X - \{x_n\}$, the poset $Q = (Y, Q)$, let $W'$ be the binary

   the subposet $Q$. Then let $W' = W(Y)$.

   order and that there exists a distinguishing

   for all $y, z \in Y$, $a_y < a_z$ if and only if $y < z$ in $Q$.

   only if $a_y + 1 < a_z$. We now show that

   is led by an appropriate choice of an interval

   $x_n$ for every $y \in Y$, let $a = \max\{a_y : y \in Y\}$

   $y \in Y : y|x_n| \neq \emptyset$. It follows that there is a

   $e_{i+1}, \ldots, e_{n-1}$, and $\{a, b\} = \bigcap \{I(y) : y \in S\}$

   $D(x_n) = \emptyset$, set $a' = a$; otherwise, set $a' = b$,

   b$ distinct from any end point.

   primary to the Scott-Suppes theorem for semi-

   noted by Roberts [97]. Call an interval order

   $P = (X, P)$ proper if it admits an interval representation $I$ so that if $x, y \in X$ and $x \neq y$, then $I(x) \not\subseteq I(y)$ and $I(y) \not\subseteq I(x)$. A semi-order is obviously a proper interval order, but as Roberts pointed out, a proper interval order is also a semi-order. We will revisit this theme in Section 19.

   4 Dilworth's theorem for interval orders

   The width of a finite poset is the maximum cardinality of an antichain, and the height is the maximum cardinality of a chain. Dilworth's theorem [26] asserts that a poset of width $w$ can be partitioned into $w$ chains. Dually [58], a poset of height $h$ can be partitioned into $h$ antichains. Short proofs of these results have been provided by many authors, e.g. see [120] or [118]. Here is one for the second result.

   **Theorem 4.1** Let $P = (X, P)$ be a poset of height $h$. Then there exists a partition

   \[ X = A_1 \cup A_2 \cup \cdots \cup A_h, \]

   with $A_j$ an antichain, for each $j \in [h]$.

   **Proof** For each $x \in X$, let $d(x)$ denote the height of the subposet determined by $D[x]$. Then for each $j \in [h]$, let $A_j = \{x \in X : d(x) = j\}$.

   Here is a sketch of how a bipartite matching algorithm can be used to find the width $w$ of a poset $P$ and a partition into $w$ chains. See Brightwell [17] for details. Given a poset $P = (X, P)$, form a bipartite graph $G = (V, E)$ with $V = \{a_x : x \in X\} \cup \{b_x : x \in X\}$ and $E = \{(a_x, b_y) : x \not< y \text{ in } P\}$. Then let $\mathcal{M}$ be a maximum matching in $G$. Define a chain partition of $X$ by putting distinct $x, y \in X$ in the same chain when there exists a sequence

   $x = u_0, u_1, \ldots, u_s = y$ so that $\{a_{u_{i-1}}, b_{u_i}\} \in \mathcal{M}$, for every $i \in [s]$.

   For interval orders, we can be more explicit by taking advantage of an important connection with graph coloring. Although this material is well known, we summarize it briefly as the concepts will be used later in this article. The argument comes from Hajnal and Suranyi's paper [58] on classes of perfect graphs.

   **Theorem 4.2** Interval graphs are perfect, i.e., the chromatic number is equal to the maximum clique size. Furthermore, an optimal coloring always results from applying First Fit to the vertices in the order their right end points occur in a distinguishing representation.

   **Proof** Let $I$ be a distinguishing representation of an interval graph $G = (X, E)$, and let $L$ be the linear order on $X$ determined by the right end points, i.e., $x < y$ in $L$ if and only if $b_x < b_y$ in $R$. For each $x \in V$, let $N_L(x) = \{y \in X : \{x, y\} \in E\}$. Note that $N_L(x)$ is a complete subgraph of $G$, for each
$x \in X$, as all the intervals corresponding to intervals in $N_L(x)$ contain the left end point $a_x$ of $I(x)$.

When First Fit assigns color $\alpha$ to a vertex $x$, then $x$ belongs to a complete subgraph of size $\alpha$ consisting of $x$ and $\alpha - 1$ vertices from $N_L(x)$. It follows that if the maximum clique size of $G$ is $k$, then First Fit will color $G$ in exactly $k$ colors. □

Equivalently, First Fit partitions the associated interval order $P = (X, \preceq)$ for which $I$ is a distinguishing representation into $w$ chains, where $w$ is its width.

In material to follow, we will find it convenient to be even more explicit than the argument used for Theorem 4.1 in applications of the dual to Dilworth’s theorem. Set $A_0 = \emptyset$ and $X_0 = X$. Now suppose that we have defined $A_j$ and $X_j-1$ for some $j \geq 1$. Set $X_j = X_{j-1} - A_{j-1}$. If $X_j \neq \emptyset$, let $y_j$ be the unique element of $X_j$ for which the right end point $r_j = b_{y_j}$ is minimum. Then let $A_j = \{x \in X_j : b_{y_j} \in I(x)\}$. When the algorithm halts, we have a partition $X = A_1 \cup A_2 \cup \cdots \cup A_d$ into $d$ antichains, and we have a chain $C = \{y_1, y_2, \ldots, y_h\}$ of cardinality $h$. Furthermore, every interval in the representation intersects the end point of at least one interval in $C$. We call $C$ the lexicographically least maximum chain of $P$, and we call the associated partition into antichains the canonical minimum partition.

5 Linear extensions and dimension

When $P$ and $Q$ are binary relations on a set $X$, we say $Q$ is an extension of $P$ when $P \subseteq Q$; a linear order $L$ on $X$ is called a linear extension of a partial order $P$ on $X$ when $P \subseteq L$. A set $\mathcal{R}$ of linear extensions of $P$ is called a realizer of $P$ when $P = \bigcap \mathcal{R}$, i.e., for all $x, y \in X$, $x < y$ in $P$ if and only if $x < y$ in $L$, for every $L \in \mathcal{R}$. The minimum cardinality of a realizer of $P$ is called the dimension of $P$ and is denoted $\dim(P)$. Note that if $P$ contains $Q$, then $\dim(Q) \leq \dim(P)$.

It is natural to ask what causes a poset to have large dimension. Here is a partial answer. For integers $n \geq 2$ and $k \geq 0$, define the crown $S^k_n$ as the poset of height two with $n + k$ minimal elements $a_1, a_2, \ldots, a_{n+k}$, $n+k$ maximal elements $b_1, b_2, \ldots, b_{n+k}$ and ordering $a_i < b_j$ if and only if $j \in \{i + k + 1, i + k + 2, \ldots, i - 1\}$. In this definition, the subscripts are interpreted cyclically, so that $b_{n+k+1} = b_1$, etc. When $n \geq 3$, the dimension of the crown $S^k_n$ is given by the following formula [110]:

$$\dim(S^k_n) = \left\lceil \frac{2(n + k)}{k + 2} \right\rceil$$ (2)

For each $k \geq 0$, the poset $S^k_2$ is the disjoint sum of $k + 2$ two-element chains, so these posets have dimension 2. When $n \geq 3$, the crown $S^k_n$ always has dimension at least 3. Posets in the family $\mathcal{S} = \{S^0_n : n \geq 2\}$ are referred to as

interval orders and interval graphs.

Standard examples. Note that the dimension for each $n \geq 3$, $S^0_n$ is $n$-irreducible, i.e., the subposet having dimension $n - 1$. Also note that $S^0_n$ is isomorphic to the family of 1-element subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. The same is true of a special case. It has dimension two and consists of two 2-element chains, but it is not irreducible.

We summarize some basic facts about dimension, referring the reader to [118] for proofs.

Proposition 5.1 Let $P = (X, \preceq)$ and $Q = (Y, \preceq)$.

1. $\dim(P + Q) = \max\{2, \dim(P), \dim(Q)\}$.
2. $\dim(P \times Q) \leq \dim(P) + \dim(Q)$, with greatest and least elements.
3. The removal of a point from $P$ decreases $\dim(P)$.
4. If $A$ is a maximum antichain in $P$, then
   $$\max\{2, |X - A|\}.$$ 5. $\dim(P) = \dim(P^d)$. □

Note that if the family of standard examples of Proposition 5.1 are best possible. We will use inequality 1 in the preceding theorem in a general formula for dimension and lexicographic

Proposition 5.2 Let $P = (X, \preceq)$ be a poset and $x \in X$ be a family of posets. Then

$$\dim\left(\sum_{x \in P} F\right) = \max\left\{\dim(P), \max_{x \in P} \dim(F)\right\}$$

For additional background information on the author’s monograph [118], the survey articles [115] and [119] also discuss combinatorial problems of dimension for posets and a wide range of connections in [123], with greater detail provided in the

6 Linear extensions of interval orders

When $P = (X, \preceq)$ is a poset, $A, B \subseteq X$ are called an extension of $P$, we say $B$ is over $A$ in $L$ with $b \in B$ and $a \in B$ in $P$. In applying this definition, we do not require that $b > a$ in $L$, for all $a \in A$ if $b = a$. The following elementary result was first
standard examples. Note that the dimension of $S_0^n$ is exactly $n$. Furthermore, for each $n \geq 3$, $S_0^n$ is $n$-irreducible, i.e., the removal of any point leaves a subposet having dimension $n - 1$. Also note that when $n \geq 3$, the standard example $S_0^n$ is isomorphic to the family of 1-element and $(n-1)$-element subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. The standard example $S_0^n$ is somewhat of a special case. It has dimension two and is isomorphic to the disjoint sum of two 2-element chains, but it is not irreducible.

We summarize some basic facts about dimension in the following proposition, referring the reader to [118] for proofs and references.

**Proposition 5.1** Let $P = (X, P)$ and $Q = (Y, Q)$ be posets. Then:

1. $\dim(P \times Q) = \max\{\dim(P), \dim(Q)\}$.

2. $\dim(P \times Q) \leq \dim(P) + \dim(Q)$, with equality holding if $P$ and $Q$ have greatest and least elements.

3. The removal of a point from $P$ decreases $\dim(P)$ by at most one.

4. If $A$ is a maximum antichain in $P$, then $\dim(P) \leq |A|$ and $\dim(P) \leq \max\{2, |X - A|\}$.

5. $\dim(P) = \dim(P^d)$. \hfill \blacksquare

Note that the family of standard examples shows that inequalities 3 and 4 of Proposition 5.1 are best possible. We will also find it convenient to put inequality 1 in the preceding theorem in a more general setting. Here is the general formula for dimension and lexicographic sums (see [118]).

**Proposition 5.2** Let $P = (X, P)$ be a poset, and let $F = \{Q_x = (Y_x, P_x) : x \in X\}$ be a family of posets. Then

$$\dim\left(\sum_{x \in P} F\right) = \max\{\dim(P), \max\{\dim(Q_x) : x \in X\}\}.$$  \hfill (3)

For additional background information on dimension, the reader is referred to the author's monograph [118], the survey article [63] on dimension by Kelly and Trotter and the survey articles [115] and [121]. The articles [112], [117] and [119] also discuss combinatorial problems for posets. Connections between dimension for posets and a wide range of combinatorial problems are discussed in [123], with greater detail provided in the monograph [118].

6 Linear extensions of interval orders

When $P = (X, P)$ is a poset, $A, B \subseteq X$ and $A \cap B = \emptyset$, and $L$ is a linear extension of $P$, we say $B$ is over $A$ in $L$ when $b > a$ in $L$, whenever $a \in A$, $b \in B$ and $a | b$ in $P$. In applying this definition, it is important to note that we do not require that $b > a$ in $L$, for all $a \in A$ and $b \in B$, only the incomparable pairs. The following elementary result was first discovered by Rabinovich [94].
Theorem 6.1 Let $P = (X, P)$ be an interval poset, and let $A, B \subset X$ with $A \cap B = \emptyset$. Then there exists a linear extension $L$ of $P$ with $B$ over $A$ in $L$.

Proof Let $I$ be a distinguishing interval representation of $P$. For each $x \in X$, let $p_x = a_x$ if $x \in A$ and $p_x = b_x$, otherwise. Then define a linear extension $L$ by setting $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$. 

More generally, the following proposition, first noted by Felsner in [28], is an easy exercise.

Proposition 6.2 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

7 Dimension of interval orders

It is natural to ask whether an interval order can have large dimension. If the answer is yes, it cannot be due to the presence of large standard examples, as no interval order contains any of them. Note that for each $n \geq 2$, the subposet of $S_n^0$ determined by $a_1, a_2, b_1$ and $b_2$ is isomorphic to $2 + 2$.

Nevertheless, interval orders may have large dimension, and to explain how this may occur, we introduce a standard example of an interval order. For an integer $n \geq 2$, let $I_n = \binom{[n]}{2}, P_n$ denote the interval order defined by the representation $I([i, j]) = [i, j]$. To avoid confusion with the family of standard examples discussed previously, we call the interval orders in the family $\{I_n : n \geq 2\}$ canonical interval orders.

The following result is due to Bogart, Rabinovitch, and Trotter [10].

Theorem 7.1 For every integer $t$, there exists an integer $n_0$ so that if $n \geq n_0$, then the dimension of the canonical interval order $I_n$ is larger than $t$.

Proof Evidently, $\dim(I_n) = n$ is a non-decreasing function of $n$. We assume that $\dim(I_n) \leq t$, for all $n \geq 2$ and obtain a contradiction when $n$ is sufficiently large in terms of $t$. Let $i, j, k$ be distinct integers with $1 \leq i < j < k \leq n$. Then $\{i, j\} \cup \{j, k\} \in P_n$, so if $C = \{L_1, L_2, \ldots, L_t\}$ is a realization of $P_n$, then we may choose $\alpha \in \{1, 2, \ldots, t\}$ so that $\{i, j\} \cup \{j, k\} \in L_{\alpha}$. This is a coloring of the 3-element subsets of $\{1, 2, \ldots, n\}$ with $t$ colors. If $n$ is sufficiently large, there exists a 4-element subset $S = \{i < j < k < l\}$ and an integer $\alpha \in \{1, 2, \ldots, t\}$ so that all 3-element subsets of $S$ are mapped to $\alpha$. This implies that $\{i, j\} \cup \{j, k\} \cup \{k, l\} \cup \{i, j\} \in L_{\alpha}$, which is a contradiction.

Now that we know that interval orders can have large dimension, we pause to discuss some of the special properties interval orders exhibit.

Let $P = (X, P)$ be a poset and let $X = X_1 \cup X_2$ be a partition of $X$ into disjoint non-empty subsets. It is natural to ask whether one can say anything about the dimension of $\dim(P)$ given information about the dimension of $\dim(P_1)$, $\dim(P_2)$, and $\dim(P_3)$.

Proposition 7.2 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

Lemma 7.2 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

Theorem 7.3 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

Interval Orders and Interval Graphs

Now it is natural to ask whether we can say anything about the dimension of $\dim(P)$ given information about the dimension of $\dim(P_1)$, $\dim(P_2)$, and $\dim(P_3)$.

For posets in general, the answer is no. For example, the partition of the point set of the standard example of an interval order into antichains. Note that if $x < y$ in $P$, then $x \leq y$ in $\mathbb{R}$.

For interval orders, things are different. The following is due to Felsner [28].

Proposition 7.2 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

Lemma 7.2 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

Theorem 7.3 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

Now it is natural to ask whether we can say anything about the dimension of $\dim(P)$ given information about the dimension of $\dim(P_1)$, $\dim(P_2)$, and $\dim(P_3)$.

For posets in general, the answer is no. For example, the partition of the point set of the standard example of an interval order into antichains. Note that if $x < y$ in $P$, then $x \leq y$ in $\mathbb{R}$.

For interval orders, things are different. The following is due to Felsner [28].

Proposition 7.2 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

Lemma 7.2 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.

Theorem 7.3 Let $P = (X, P)$ be an interval order, and let $I$ be any distinguishing interval representation of $P$. If $L$ is a linear extension of $P$, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in $L$ if and only if $p_x < p_y$ in $\mathbb{R}$.
Interval Orders and Interval Graphs

Let \( P \) be an interval poset, and let \( A, B \subseteq X \) with \( B \cap \text{l}(A) \neq \emptyset \), then there exists a linear extension \( L \) of \( P \) with \( B \) over \( A \) in \( L \) if \( x < y \) for \( x \in A \) and \( y \in B \). Otherwise, define a linear extension \( L \) of \( P \) such that \( x < y \) in \( L \). If \( p_x < p_y \) in \( L \), then \( x < y \) in \( L \).

Theorem 7.3 Let \( P = (X, \leq) \) be an interval order, and let \( X = X_1 \cup X_2 \) be a partition of \( X \) into disjoint non-empty subsets. Then

\[
\dim(P) \leq 2 + \max\{\dim(X_1), \dim(X_2)\}.
\]

Proof Let \( t = \max\{\dim(X_1), \dim(X_2)\} \). From Lemma 7.2, we know that there exists a family \( \mathcal{F} = \{L_1, L_2, \ldots, L_t\} \) of linear extensions of \( P \) so that \( \mathcal{F} = \{L_1(X_i), L_2(X_i), \ldots, L_t(X_i)\} \) is a realizer of \( X_i \), for \( i = 1, 2 \). Then let \( M_1 \) and \( M_2 \) be linear extensions of \( P \) so that \( X_1 \) is over \( X_2 \) in \( M_1 \) and \( X_2 \) is over \( X_1 \) in \( M_2 \). It follows that \( \{M_1, M_2\} \cup \mathcal{F} \) is a realizer of \( P \).

When one of the two sets in the partition is the set of maximal elements, we can do a little better. We leave the proof as an exercise.

Theorem 7.4 Let \( P = (X, \leq) \) be an interval order which is not an antichain. If \( X_1 \) is the set of all maximal elements, and \( X_2 = X - X_1 \), then

\[
\dim(P) \leq 1 + \dim(X_2).
\]

Now it is natural to ask whether we can say anything about the dimension of \( \dim(P) \) given information about \( \dim(X_1) \) and \( \dim(X_2) \). For posets in general, the answer is no. For example, for each \( n \geq 2 \), consider the partition of the point set of the standard example \( S_n^2 \) into minimal elements and maximal elements. The dimension of \( S_n^2 \) is \( n \) but the two antichains have dimension 2.

For interval orders, things are different. The next result follows easily from Proposition 6.2.

Lemma 7.2 Let \( P = (X, \leq) \) be an interval order, and let \( X = X_1 \cup X_2 \) be a partition of \( X \) into disjoint non-empty subsets. If \( L_1 \) and \( L_2 \) are linear extensions of the subposets \( P_1 \) and \( P_2 \) induced by \( X_1 \) and \( X_2 \), respectively, then there exists a linear extension \( L \) of \( P \) so that \( L_1 = L(X_1) \) and \( L_2 = L(X_2) \).
Although interval orders can have large dimension, this is not true for semi-orders. The following result is due to Rabinovitch [93].

**Theorem 7.6** If $P = (X, P)$ is a semi-order, then $\dim(P) \leq 3$.

**Proof** Let $P = (X, P)$ be a semi-order, let $I$ be a distinguishing representation of $P$ and let $X = A_1 \cup A_2 \cup \cdots \cup A_h$ be the canonical partition into antichains.

Let $\mathcal{O} = \bigcup \{A_j : 1 \leq j \leq h, \ j \text{ odd}\}$ and $\mathcal{E} = \bigcup \{A_j : 1 \leq j \leq h, \ j \text{ even}\}$. Let $L_1$ and $L_2$ be linear extensions of $P$ with $\mathcal{O}$ over $\mathcal{O}$ in $L_1$ and $\mathcal{O}$ over $\mathcal{E}$ in $L_2$. Then let $L_3 = L_1^A(A_1) < L_2^A(A_2) < \cdots < L_1^A(A_h)$. 

A semi-order has bounded dimension, not just because it has a representation in which all the intervals have the same length, but rather because there is no element incomparable with all the points in a long chain. We leave the following lemma as an exercise.

**Lemma 7.7** For every $k \geq 1$, there exists an integer $s_k$ so that if $P = (X, P)$ is an interval order in which the maximum size of a chain $C$ for which there exists a point $x$ incomparable to all points of $C$ is at most $k$, then $\dim(P) \leq s_k$.

## 8 Critical pairs and alternating cycles

In arguments to follow, we will find it convenient to take advantage of a technical detail in the proof of Theorem 7.6. Let $L$ be an arbitrary linear order on $X$. Define linear extensions $L_d$ and $L_u$ of $P$ as follows. Set $x < y$ in $L_d$ if and only if one of the following conditions holds:

1. $D(x) \subseteq D(y)$.
2. $D(x) = D(y)$ and $U(y) \subseteq U(x)$.
3. $D(x) = D(y)$, $U(y) = U(x)$, and $x < y$ in $L$.

Dually, set $x < y$ in $L_u$ if and only if one of the following conditions holds:

1. $U(y) \subseteq U(x)$.
2. $U(y) = U(x)$ and $D(x) \subseteq D(y)$.
3. $U(y) = U(x)$, $D(x) = U(x)$, and $x > y$ in $L$.

Now let $\mathcal{F}$ be a family of linear extensions of $P$. Then $\{L_d, L_u\} \cup \mathcal{F}$ is a realizer of $P$ if and only if for every $x, y \in X$ with $x \parallel y$ in $P$, $D(x) \subseteq D(y)$, and $U(y) \subseteq U(x)$, there exists $L \in \mathcal{F}$ with $x > y$ in $L$.

This last observation is a special case of a somewhat more general situation. For an arbitrary poset $P = (X, P)$, let $\text{inc}(P) = \{(x, y) \in X \times X : x \parallel y \text{ in } P\}$. Then a family $\mathcal{R}$ of linear extensions of $P$ is a realizer of $P$ if and only if for every $(x, y) \in \text{inc}(P)$, there exists $L \in \mathcal{R}$ so that $x > y$ in $L$. Call a pair $(x, y) \in \text{inc}(P)$ a critical pair if $u < x$ in $P$ implies $v > x$ in $P$, for all $u, v \in X$. All critical pairs. It follows that $\mathcal{R}$ is a realizer of $P$ if and only if there exists some $L \in \mathcal{R}$ so that $(x, y) \in \text{crit}(P)$.

We say $L$ reverses the incomparable pair $(x, y) \in \text{inc}(P)$. We say that $L$ reverses $S$ when it reverses all its members. For an integer $k \geq 2$, a subset $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ of $\text{inc}(P)$ is an alternating cycle when $x_i \leq y_{i+1}$ in $P$ for all $i$. Last definition, the substrings are interpreted as alternating cycle $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ if $y_j = x_{j+1}$ for all $i, j = 1, 2, \ldots, k$. When following three statements hold:

1. The elements in $\{x_1, x_2, \ldots, x_k\}$ form an interval.
2. The elements in $\{y_1, y_2, \ldots, y_k\}$ form an interval.
3. If $i, j \in [k]$ and $x_i \geq y_j$, then $j = i + 1$.

The following elementary result is due to Trotter for a short proof and a number of applications.

**Theorem 8.1** Let $P = (X, P)$ be a poset. The following statements are equivalent.

1. There exists a linear extension $L$ of $P$.
2. $S$ does not contain an alternating cycle.
3. $S$ does not contain a strict alternating cycle.

## 9 Interval orders and shift graphs

Although it has been known for many decades that semi-orders have relatively tight bounds have been found. The dimension in interval orders is best explained by the following coloring problem. Fix integers $n$ and $k$ with $k \leq n$. A pair $(A, B)$ of $k$-element sets a $(k, n)$-shift pair. A subset $C = \{i_1, i_2, \ldots, i_{k+1}\} \subseteq \{1, 2, \ldots, n\}$ and $B = \{i_2, i_3, \ldots, i_{k+1}\}$. We then define the shift graph whose vertex set consists of all $k$-element set $A$ adjacent to a $k$-element set $B$ in the shift. The shift graph $S(1, n)$ is just a graph on $2n$ vertices. It is customary to call a $(2, n)$-shift graph a shift graph or a double shift graph.
In an interval order, let $I$ be a distinguishing representation of $P$ and $A_i$ be the canonical partition into antichains corresponding to $i$ (odd) and $E = \bigcup \{A_j : 1 \leq j \leq h, j \text{ even}\}$, where $h$ is the number of $\subseteq$ relations of $P$ with order $O$ in $L_1$ and $O$ over $O \subseteq E \{A_2 \} \subseteq \cdots \subseteq L^h(A_h)$. 

If a dimension, not just because it has a representative the same length, but rather because there are all the points in a long chain. We leave the problem as an exercise.

There exists an integer $s_k$ so that if $P = (X, P)$ is the minimum size of a chain $C$ for which there exist elements of $C$ is at most $k$, then $\dim(P) \leq s_k$.

### 7.7 Alternating cycles

You will find it convenient to take advantage of a theorem 7.6. Let $L$ be an arbitrary linear order of $A_i$ and $L_u$ of $P$ as follows. Set $x < y$ in $L_d$ if relations hold:

1. $D(x)$.
2. $x < y$ in $L_d$.
3. $y < x$ in $L_d$.

If one of the following conditions holds:

- $D(y)$.
- $x < y$ in $L_u$.

Then $x, y \in X$ with $x < y$ in $P$, $D(x) \subseteq D(y)$, and $x > y$ in $L_u$.

In the special case of a somewhat more general situation (a), let $\text{inc}(P) = \{(x, y) \in X \times X : x < y \text{ in } P\}$, where

$x \in P$ is a realization of $P$ if and only if for each $L \in \mathcal{R}$ so that $x > y$ in $L$. Call a pair

$$(x, y) \in \text{inc}(P)$$

a critical pair if $u < x$ in $P$ implies $u < y$ in $P$ and $v > y$ in $P$ implies $v > x$ in $P$, for all $u, v \in X$. Then let $\text{crit}(P)$ denote the set of all critical pairs. It follows that $\mathcal{R}$ is a realization of $P$ if and only if for every $(x, y) \in \text{crit}(P)$, there exists some $L \in \mathcal{R}$ so that $x > y$ in $L$.

We say $L$ reverses the incomparable pair $(x, y)$ when $x > y$ in $L$. Let $S \subseteq \text{inc}(P)$. We say that $L$ reverses $S$ when $x > y$ in $L$, for each $(x, y) \in S$. For an integer $k \geq 2$, a subset $S = \{(x_i, y_i) : 1 \leq i \leq k\} \subseteq \text{inc}(P)$ is called an alternating cycle when $x_i \leq y_{i+1}$ in $P$, for all $i = 1, 2, \ldots, k$. In this last definition, the subscripts are interpreted cyclically, i.e., $y_{k+1} = y_1$. An alternating cycle $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ is strict if $x_i \leq y_j$ in $P$ if and only if $j = i + 1$, for all $i, j = 1, 2, \ldots, k$. When an alternating cycle is strict, the following three statements hold:

1. The elements in $\{x_1, x_2, \ldots, x_k\}$ form a $k$-element antichain.
2. The elements in $\{y_1, y_2, \ldots, y_k\}$ form a $k$-element antichain.
3. If $i, j \in [k]$ and $x_i \geq y_j$, then $j = i + 1$ and $x_i = y_j$.

The following elementary result is due to Trotter and Moore [124]. See [118] for a short proof and a number of applications.

**Theorem 8.1** Let $P = (X, P)$ be a poset and let $S \subseteq \text{inc}(P)$. Then the following statements are equivalent.

1. There exists a linear extension $L$ of $P$ which reverses $S$.
2. $S$ does not contain an alternating cycle.
3. $S$ does not contain a strict alternating cycle.

### 9 Interval orders and shift graphs

Although it has been known for many years that interval orders of large height must contain long chains, it has only been in the last few years that relatively tight bounds have been found. The relationship between height and dimension in interval orders is best explained via a connection with a graph coloring problem. Fix integers $n$ and $k$ with $1 \leq k < n$. We call an ordered pair $(A, B)$ of $k$-element sets a $(k, n)$-shift pair if there exists a $(k + 1)$-element subset $C = \{i_1 < i_2 < \cdots < i_{k+1}\} \subseteq \{1, 2, \ldots, n\}$ so that $A = \{i_1, i_2, \ldots, i_k\}$ and $B = \{i_2, i_3, \ldots, i_{k+1}\}$.

We then define the $(k, n)$-shift graph $S(k, n)$ as the graph whose vertex set consists of all $k$-element subsets of $\{1, 2, \ldots, n\}$ with a $k$-element set $A$ adjacent to a $k$-element set $B$ exactly when $(A, B)$ is a $(k, n)$-shift pair. The shift graph $S(1, n)$ is just a complete graph on $n$ vertices. It is customary to call a $(2, n)$-shift graph just a shift graph; similarly, a $(3, n)$-shift graph is called a double shift graph.
One of the folklore results of graph theory is the following formula for the chromatic number of the shift graph (throughout this paper, we use the notation \( \lg n \) to denote the base 2 logarithm of \( n \)).

**Theorem 9.1** The chromatic number of the shift graph \( S(2, n) \) is \( \lceil \lg n \rceil \).

The proof of Theorem 7.1 establishes the following lower bound.

**Proposition 9.2** The dimension of the canonical interval order \( I_n \) is at least as large as the chromatic number of the double shift graph \( S(3, n) \).

In turn, the next result relates the determination of the dimension of the family of canonical interval orders to the classical enumeration problem known as Dedekind’s problem: estimate the number of antichains in the poset \( 2^4 \), the cartesian product of \( t \) two-element chains. This poset is just the subset lattice, the family of all subsets of \([t]\) partially ordered by inclusion. The next four results are due to Füredi, Hajnal, Rödl and Trotter [44].

**Theorem 9.3** The chromatic number of the double shift graph \( S(3, n) \) is the least positive integer \( t \) for which there are at least \( n \) antichains in the subset lattice \( 2^t \).

**Proof** Suppose that there are \( n \) antichains in the subset lattice \( 2^t \). We show that the chromatic number of \( S(3, n) \) is \( \leq t \). Let \( Q \) be the partial order defined on the antichains of \( 2^k \) by setting \( A \leq B \) in \( Q \) if and only if for every \( S \in A \), there exists \( B \in B \) so that \( A \subseteq B \). Then let \( L \) be any linear extension of \( Q \), and suppose that \( A_1 < A_2 < \cdots < A_n \) in \( L \). Each \( i, j \) with \( 1 \leq i < j \leq n \), let \( B(i, j) \in A_i \) be a set so that there is no set \( A \in A_i \) with \( A \subseteq B(i, j) \). Then for each \( i, j, k \) with \( 1 \leq i < j < k \leq n \), choose an element \( \alpha \in B(j, k) - B(i, j) \), and set \( \phi \{ i, j, k \} = \alpha \). Then \( \phi \) is a coloring of \( S(3, n) \).

Conversely, suppose that \( \phi : \binom{[n]}{3} \to [t] \) is a coloring, define for each \( i, j \) with \( 1 \leq i < j \leq n \), the set \( B(i, j) = \{ \phi \{ i, j, k \} : j < k \leq n \} \). Then for each \( i \in [n] \), set \( A_i = \{ B(i, j) : i < j \leq n \} \). Partial order each \( A_i \) by inclusion and let \( A_i \) be the maximal elements. Then each \( A_i \) is an antichain in \( 2^t \) and \( A_i \neq A_j \) when \( i \neq j \).

Although no closed form solution to Dedekind’s problem has been found, relatively tight estimates are known (see [77], e.g.). For our purposes, we may use the estimate which results as follows. There are \( \binom{\lfloor t/2 \rfloor}{3} \) subsets of size \( \lfloor t/2 \rfloor \). Any subset of these sets forms an antichain in \( 2^t \).

**Theorem 9.4** The chromatic number of the double shift graph satisfies:

\[
\chi(S(3, n)) = \lceil \lg n \rceil + (1/2 + o(1)) \lceil \lg \lg n \rceil.
\]

Interval Orders and Interval Graphs

Hopefully, the reader has noticed the following problem for canonical interval orders. The lack of use of repeated endpoints, a phenomenon well developed in the representation. After the modification, it seems that the problem is much harder. However, this turns out to be false.

**Theorem 9.5** Let \( n \) and \( t \) be positive integers, and let \( n \leq 2^{t(n/3)} \). Then the dimension of the canonical interval order \( I_n \) is at least \( 2^{t(n/3)+1} \).

**Proof** We let \( M_1 \) and \( M_2 \) be the following partial orders: let \( [i_1, j_1] \) and \( [i_2, j_2] \) be distinct elements of \( [t] \), with \( i_1 < i_2 \), or if both \( i_1 = i_2 \) and \( j_1 > j_2 \). Define \( i_1 < j_2 \), or if both \( j_1 = j_2 \) and \( i_1 > i_2 \). It remains to show that \( L \) is an interval graph.

Let \( s = \binom{[n]}{3} \) and let \( S_1, S_2, \ldots, S_s \) be the set of \( \binom{[t]}{3} \) subsets of \([t]\). Note that \( A = \{ S_1, S_2, \ldots, S_s \} \) is a lattice on the set of all \( [t] \) elements of \( X \) are functions from \( [s] \) to \( [t] \) under inclusion order on \( X \). By this, we mean that if \( f, g \) are functions from \([s] \) to \([t] \), then \( f < g \) in \( L \) if and only if \( f(j) < g(j) \) for all \( j \in [s] \).

Now let \( S_1, S_2, \ldots, S_s \) be the set of \( \binom{[t]}{3} \) subsets of \([t]\). Note that \( A = \{ S_1, S_2, \ldots, S_s \} \) is a lattice on the set of all \( [t] \) elements of \( X \) are functions from \( [s] \) to \([t] \) under inclusion order on \( X \). By this, we mean that if \( f, g \) are functions from \([s] \) to \([t] \), then \( f < g \) in \( L \) if and only if \( f(j) < g(j) \) for all \( j \in [s] \).

Now let \( [i_1, j_1] \) and \( [i_2, j_2] \) be elements of \( L \) that allow the possibility that \( i_2 = j_1 \). Select either 3 or 4 elements. Choose the least integer \( n \) such that \( f_{i_1}(k_1) = 0 \) and \( f_{i_2}(k_1) = 1 \). Further statements hold:

1. For every \( i \in E \) with \( i \neq i_1 \), \( f_{i_1}(i_1) = 1 \).
2. For every \( i \in E \) with \( i \neq i_2 \), \( f_{i_2}(i_2) = 0 \).
3. \( f_{i_1}(k_1) = 0 \) and \( f_{i_2}(k_1) = 1 \).

When the third of these statements holds, we know that two may occur with either \( |E| = 3 \) or \( |E| = 4 \), but not both. How could we prove that \( [i_1, j_1] > [i_2, j_2] \) in \( L \)?

When the first statement holds, set \( E' = E \setminus \{ j_2 \} \). In either case, where \( \{|f_{i_1}(k_1) : i \in E'\} > 1 \). If the first statement holds, we will require that \( [i_1, j_1] > [i_2, j_2] \) in \( L \).

We leave it as an exercise that such linear orders exist and are canonical.
Interval Orders and Interval Graphs

Hopefully, the reader has noticed the following subtext to the dimension problem for canonical interval orders. The lower bound depends heavily on the use of repeated end points, a phenomenon which we can eliminate by modifying the representation. After the modification, it is conceivable that the dimension problem is much harder. However, this turns out not to be the case.

**Theorem 9.5** Let \( n \) and \( t \) be positive integers with \( n \geq 2 \). If

\[
n \leq 2^\left( \frac{t}{2n} \right),
\]

then the dimension of the canonical interval order \( I_n \) is at most \( t + 3 \).

**Proof** We let \( M_1 \) and \( M_2 \) be the following two linear extensions of \( I_n \). Let \([i_1, j_1]\) and \([i_2, j_2]\) be distinct elements of \( I_n \). Set \([i_1, j_1]\) \( \leq \) \([i_2, j_2]\) in \( M_1 \) if \( i_1 < i_2 \), or if both \( i_1 = i_2 \) and \( j_1 > j_2 \). Dually, set \([i_1, j_1]\) \( \leq \) \([i_2, j_2]\) in \( M_2 \) if \( j_1 < j_2 \), or if both \( j_1 = j_2 \) and \( i_1 > i_2 \). It remains to find \( t+1 \) additional linear extensions \( L_1, L_2, \ldots, L_{t+1} \) so that when \( i-1 < i_2 \leq j_1 < j_2 \), there is at least one \( L_0 \) so that \([i_1, j_1]\) \( \leq \) \([i_2, j_2]\) in \( L_0 \).

Let \( s = \left( \frac{t}{n} \right) \) and let \( S_1, S_2, \ldots, S_s \) be a listing of all the \( \left[ \frac{t}{2} \right] \)-element subsets of \([t]\). Note that \( A = \{S_1, S_2, S_3, \ldots, S_s\} \) is an antichain in the subset lattice \( 2^t \). Also let \( X \) denote the set of all 0-1 sequences of length \( s \), i.e., the elements of \( X \) are functions from \([s]\) to \([0,1]\). We let \( L \) be the lexicographic order on \( X \). By this, we mean that if \((f,g) \in X \) and \( j \) is the least integer for which \( f(j) \neq g(j) \), then \( f < g \) in \( L \) if and only if \( f(j) < g(j) \). This implies \( f(j) = 0 \) and \( g(j) = 1 \). Now let \( L' = \{f_1, f_2, \ldots, f_n\} \) be the restriction of \( L \) to \( n \) distinct elements of \( X \).

Now let \([i_1, j_1]\) and \([i_2, j_2]\) be elements of \( I_n \) with \( i_1 < i_2 \leq j_1 < j_2 \). Note that we allow the possibility that \( i_2 = j_1 \). Set \( E = \{i_1, i_2, j_1, j_2\} \) so that \( E \) has either 3 or 4 elements. Choose the least integer \( k \) so that \( |\{f_i(k) : i \in E\}| > 1 \). Note that \( f_{i_2}(k_1) = 0 \) and \( f_{j_2}(k_1) = 1 \). Furthermore, exactly one of the following statements holds:

1. For every \( i \in E \) with \( i \neq i_1 \), \( f_i(j_1) = 1 \).
2. For every \( i \in E \) with \( i \neq j_2 \), \( f_i(j_1) = 0 \).
3. \( f_{i_2}(k_1) = 0 \) and \( f_{j_2}(k_1) = 1 \).

When the third of these statements holds, we must have \( |E| = 4 \), but the first two may occur with either \( |E| = 3 \) or \( |E| = 4 \). Also, when the third statement holds, we will require that \([i_1, j_1] > [i_2, j_2]\) in \( L_{t+1} \).

When the first statement holds, set \( E' = E - \{i_1\} \), and when the second statement holds, set \( E' = E - \{j_2\} \). In either case, let \( k_0 \) be the least element where \(|\{f_i(k) : i \in E'\}| > 1 \). If the first statement holds, choose \( \alpha \in S_{k_0} - S_{i_2} \) and require \([i_1, j_1] > [i_2, j_2]\) in \( L_0 \). If the second statement holds, choose \( \alpha \in S_{k_0} - S_{j_2} \) and require \([i_1, j_1] > [i_2, j_2]\) in \( L_0 \).

We leave it as an exercise that such linear extensions exist.
The preceding theorem shows that the lower bound provided in inequality (6) is also an upper bound. With a little more work, the same kind of estimate works for arbitrary interval orders (see [44] for details).

**Theorem 9.6** The maximum dimension \( d(h) \) of an interval order of height \( h \) satisfies:

\[
  d(h) = \log \log h + (1/2 + o(1)) \log \log \log h. \tag{8}
\]

Before closing this section, we comment that the dimension problem for interval orders is closely related to the problem of determining the dimension of the poset consisting of all 1-element and 2-element subsets of \( \{1, 2, \ldots, n\} \), partially ordered by inclusion. Spencer [107] was the first to establish the connection between this problem and the classic result of Erdős and Szekeres concerning monotonic subsequences of a sequence of integers. In recent years, there has been rapid progress in estimating the dimension of posets consisting of layers of the subset lattice. A summary of this work together with additional references is provided by Trotter in [121].

### 10 Interval orders and overlap graphs

A graph \( G = (V, E) \) is called an overlap graph when there exists a function \( I \) assigning to each vertex \( x \in V \) a closed interval \( I(x) = [a_x, b_x] \) of \( \mathbb{R} \) so that for all \( x, y \in V \), \( \{x, y\} \in E \) if and only if \( I(x) \cap I(y) \neq \emptyset \), \( I(x) \not\subseteq I(y) \) and \( I(y) \not\subseteq I(x) \), i.e., the intervals intersect, but neither is contained in the other. Again, we call the function \( I \) a representation of the overlap graph \( G \). If required, we may assume that a representation of an overlap graph is distinguishing.

In general, overlap graphs need not be perfect, e.g., a cycle on 5 vertices in an overlap graph. However, when all the intervals used in the representation intersect, then the graph is perfect, and it is easy to color such graphs.

**Proposition 10.1** Let \( I \) be a distinguishing representation for an overlap graph \( G = (V, E) \). If \( I(x) \cap I(y) \neq \emptyset \), for all \( x, y \in V \), then \( G \) is the comparability graph of a poset \( P = (V, P) \) with \( \dim(P) \leq 2 \), so \( G \) is perfect. Furthermore, the First Fit algorithm will provide an optimal coloring of \( G \) if the vertices are colored in the order determined by the left end points.

**Proof** Let \( L_1 \) and \( L_2 \) be the linear orders on \( V \) determined by left and right end points, respectively. Then let \( P = L_1 \cap L_2 \). Clearly, \( \{x, y\} \) is an edge of \( G \) if and only if \( x \) and \( y \) are comparable in the poset \( P = (V, P) \). From its definition, we know that \( \dim(P) \leq 2 \). The First Fit algorithm applied to the vertices in the order of their left end points is the minimum antichain partition, described in the proof of Theorem 4.1.

When the intervals used in the representation do not share a common point, it is not immediately clear that there is any bound on the chromatic number of an overlap graph in terms of the maximum clique size. This fact is due to Gyárfás [55] and the best bounds are due to Kratochvíl [78]. Recall that the notation \( \chi(Q) \) denotes the chromatic number of a graph, with a positive integer \( k \) being the size of the clique. For all \( k > k_0 \).

**Theorem 10.2** Let \( m(k) = \max\{\chi(Q) : Q \text{ is a graph with maximum clique size } k\} \). Then

1. \( m(k) = \Omega(k \log k) \).
2. \( m(k) = O(2^k) \).

Quite recently, the concepts used in the proof of Theorem 10.2 were applied by Kierstead and Trotter [75] to derive a lower bound on the dimension theory for interval orders. We omit the proof for the best possible constants.

**Theorem 10.3** For every interval order \( Q \) and integer \( k \), there exists an interval order \( P = (X, P) \) such that if \( P \) is any interval order isomorphic to \( Q \), then \( P \) contains a subposet isomorphic to \( Q \).

Since every interval order \( Q \) is isomorphic to an interval order \( I_n \), provided \( n \) is sufficiently large, this result implies that for every integer \( n \geq 2 \), there is an interval order with \( \dim(P) > t_n \), then \( P \) contains the canonical interval order \( I_n \).

Let \( P = (X, P) \) be an interval order, and let \( T \) be a subposet of \( P \) an \( n \)-tower. Then

1. \( T \) contains an \( m \)-element chain \( Z = \{z_1, z_2, \ldots, z_m\} \).
2. For every pair \( i, j \) with \( 1 \leq i < j \leq m \), \( z_i \) and \( z_j \) are incomparable in \( P \), which is incomparable with \( z_k, z_{k+1}, \ldots, z_m \) for all \( k \).

It is an easy exercise to show that if \( P \) contains a subposet isomorphic to \( I_n \), then \( P \) contains an \( m \)-tower. So, Theorem 10.3 provides somewhat more technical result.

**Theorem 10.4** For every integer \( m \), there exists an interval order \( Q \) with \( \dim(Q) \geq m \) such that if \( P \) is an interval order, then \( P \) is isomorphic to an \( m \)-tower.

**Proof** We proceed by induction on \( m \). If \( m = 1 \), then \( P \) contains a 2-tower, a weak order and hence \( P \) contains an \( 1 \)-tower. Now consider a value of \( m > 1 \). Construct an interval order \( Q \) with \( \dim(Q) \geq m \) so that \( P \) contains an \( m \)-tower.
Interval Orders and Interval Graphs

253

an overlap graph in terms of the maximum clique size. The first proof of this
fact is due to Gyárfás [55] and the best bounds to date are due to Kostochka
and Kratochvíl [78]. Recall that the notation $f(k) = \Omega(g(k))$ means that
there exists a positive constant $c$ and an integer $k_0$ so that $f(k) \geq cg(k)$, for
all $k > k_0$.

**Theorem 10.2** Let $m(k) = \max\{\chi(G) : G \text{ is an overlap graph with max-
imum clique size } k\}$. Then

1. $m(k) = \Omega(k \log k)$.

2. $m(k) = O(2^k)$.  

Quite recently, the concepts used in the proof of Theorem 10.2 have been 
used by Kierstead and Trotter [75] to solve a long standing problem in 
dimension theory for interval orders. We outline this work, but we do not aim 
for the best possible constants.

**Theorem 10.3** For every interval order $Q = (Y, Q)$, there exists an integer $t_0$
so that if $P = (X, P)$ is any interval order with $\dim(P) > t_0$, then $P$ contains 
a subposet isomorphic to $Q$.

Since every interval order $Q$ is isomorphic to a subposet of the canonical 
interval order $I_n$, provided $n$ is sufficiently large, Theorem 10.3 is equivalent to 
showing that for every integer $n \geq 2$, there exists an integer $t_n$ so that if $P$ 
is an interval order with $\dim(P) > t_n$, then $P$ contains a subposet isomorphic to 
the canonical interval order $I_n$.

Let $P = (X, P)$ be an interval order. For an integer $m \geq 2$, we call a 
subposet $T$ of $P$ an $m$–tower $T$ when

1. $T$ contains an $m$–element chain $Z = \{z_1 < z_2 < \cdots < z_m\}$, and

2. For every pair $i, j$ with $1 \leq i < j \leq m$, $T$ contains an element $w(i, j)$ 
which is incomparable with $z_i, z_{i+1}, \ldots, z_j$ and comparable with all other 
elements of $Z$.

It is an easy exercise to show that if $P$ contains a 3–tower, it contains 
a subposet isomorphic to $I_3$. So, Theorem 10.3 is also equivalent to the following 
statement which provides somewhat more technical result.

**Theorem 10.4** For every integer $m \geq 2$, there exists an integer $t_m$ so that 
if $P$ is an interval order with $\dim(P) \geq t_m$, then $P$ contains a subposet 

isomorphic to an $m$–tower.

**Proof** We proceed by induction on $m$. An interval order which does not 
contain a 2–tower is a weak order and has dimension at most 2. So it suffices 
to take $t_2 = 3$. Now consider a value of $m \geq 3$ and assume that there exists 
an integer $t_{m-1}$ so that any interval order whose dimension is at least $t_{m-1}$
contains an \((m - 1)\)-tower. Now let \(P = (X, P)\) be any interval order whose dimension \(t\) is at least \(t_{m-1} + 9\). We show that \(P\) contains an \(m\)-tower.

The key idea in the remainder of the proof is the notion of distance in the overlap graph. Let \(I\) be a distinguishing representation of \(P = (X, P)\). We proceed to build a realizer of \(P\), starting with the two linear extensions \(M_1\) and \(M_2\), the orderings determined by the left and right end points respectively in the representation \(I\). The important thing to notice is that it only remains to reverse critical pairs of the form \((x, y)\), where \(a_x < a_y < b_x < b_y\). In particular, \(x\) and \(y\) are adjacent in the overlap graph.

Let \(G\) denote the overlap graph determined by \(I\), and let \(G_1, G_2, \ldots, G_s\) denote the components of \(G\). Then let \(X_i = (X_i, P_i)\) be the subposet determined by the vertex set of \(G_i\). For each \(i \in [s]\), define the root of \(G_i\) to be the unique vertex in \(G_i\) whose left end point is minimal. We denote the root of \(G_i\) by \(r_i\). For every vertex \(x \in G_i\), let \(d(x, r_i)\) denote the distance from \(x\) to \(r_i\) in \(G_i\). The following key lemma is due to Gyárfás [55]. We leave the proof as an exercise.

**Lemma 10.5** Let \(i \in [s]\) and let \(j \geq 0\). Then let \(x, y\) and \(z\) each be at distance \(j\) from the root \(r_i\) of a component \(G_i\) of \(G\). If \(I(z) \subseteq I(x) \cap I(y)\), and \(r_i = u_0, u_1, \ldots, u_t = z\) is a shortest path from \(r_i\) to \(z\) in \(H\) then \(I(u_{i-1}) \not\subseteq I(u_i) \cup I(x) \cup I(y)\). 

Next we classify all vertices of \(G\) as either left or right, and we denote the set of all left vertices by \(L\) and the set of all right vertices by \(R\). A vertex \(x\) belongs to \(L\) if and only if there exists a shortest path \(r_i = u_0, u_1, \ldots, u_t = x\) from the root of the component to which \(x\) belongs to \(x\) so that the left end point of \(I(u_{i-1})\) is less than the left end point of \(I(x)\). Then set \(R = X - L\).

Similarly, we classify all vertices as either even or odd and denote these two sets by \(E\) and \(O\), respectively. A vertex \(x\) belongs to \(E\) if and only if the distance from \(x\) to the root of the component to which it belongs is even. Then set \(O = X - E\).

Then let \(M_3, M_4, M_5\) and \(M_6\) be linear extensions of \(P\) with

1. \(L\) over \(R\) in \(M_3\);
2. \(R\) over \(L\) in \(M_4\);
3. \(E\) over \(O\) in \(M_5\); and
4. \(O\) over \(E\) in \(M_6\).

It follows that we may assume that there is a pair \(i, j\) with \(i \in [s]\) and \(j \geq 2\) so that the subposet \(Q = (Y, Q)\) determined by all left vertices at distance \(j\) from the root \(r_i\) of component \(G_i\) has dimension at least \(t_{m-1} + 3\).

Consider the following recursive definition. Set \(Y_0 = Y\). If \(Y_k\) has already been defined for some \(k \geq 0\) and the dimension of the subposet \(Y_k\) is less than \(t_{m-1} + 1\), set \(Z_{k+1} = Y_k\) and halt.

By assumption of \(Y_k\) is at least \(t_{m-1} + 1\), let \(y_k + 1\) \(\dim(W(y_k + 1, Y_k))\geq t_{m-1} + 1\) whose left and right end points \(Y_{k+1} = Y_k - W(y_k + 1, Y_k)\). It follows that \(\dim(Z_{k+1}) = t_{m-1}\).

Suppose this recursive definition halts. Let \(Z = Z_1 \cup Z_2 \cup Z_s\) and \(B = B_1 \cup B_2 \cup B_s\). If \(\dim(B) \geq t_{m-1} + 1\). Also, note that for each hypothesis implies that \(Z_i\) contains an \((m - 1)\)-towers.

Since the dimension of \(B\) is large, it follows that there are integers \(k_1\) and \(k_2\) with \(1 < k_1 < k_2 - 3\) and \(b_k||b_{k+1} \in P\) so that \(b_k||b_{k+1}\) in \(P\). It follows that the interval from two disjoint \((m - 1)\)-towers, one from \(Z_{k_1+1}\), and consider \(U\).

For each vertex \(y \in T\), the interval corresponding on the shortest path from \(r_i\) to \(y\) proper of the lemma, this interval also has a left end point of \(b_{k_1}\). Thus this interval also intersects an \(m\)-tower.

11 Semi-orders and balancing pairs

Let \(P = (X, P)\) be a poset and let \(\mathcal{F}\) be the set of linear extensions of \(P\). Consider the linear orderings in the uniform space form. For a distinct pair \(x, y\) in \(\mathcal{F}\), denoted \(\text{Prob}_x[y > y]\), is defined as

\[
\text{Prob}_x[y > y] = \frac{1}{t} \sum_{i=1}^{t} \frac{1}{\binom{n}{k}}
\]

In this section, we are concerned with the linear extensions of \(P\). We let \(\lambda(P) = |\Lambda|\), where \(\Lambda\) is the subposet of \(P\) not containing \(x, y\). Note that \(\lambda(P)^{\perp}\) and \(\lambda(P)^2\) are some one of the most intriguing problems.

**Conjecture 11.1** If \(P = (X, P)\) is a finite poset, there exists an incomparable pair \(x, y \in X\)

\[
1/3 \leq \text{Prob}_x[y > y]
\]

This conjecture was made independently by many papers on this subject attribute...
Let \( P = (X, P) \) be any interval order whose 0-tower is \( m \)-dimensional. We shall show that \( P \) contains an \( m \)-tower.

One way of the proof is the notion of distance in the decomposing representation of \( P = (X, P) \). We start with the two linear extensions \( M_1 \) and \( M_2 \) by the left and right end points respectively in the overlap graph.

The most important thing to notice is that it only remains to find \( (x, y) \), where \( a_x < a_y < b_x < b_y \). In the overlap graph. We leave the proof as an exercise.

11 Semi-orders and balancing pairs

Let \( P = (X, P) \) be a poset and let \( \mathcal{F} = \{M_1, \ldots, M_t\} \) be a multiset of linear extensions of \( P \). Consider the linear extensions of \( \mathcal{F} \) as outcomes in a uniform sample space. For a distinct pair \( x, y \in X \), the probability that \( x > y \) in \( \mathcal{F} \), denoted \( \Pr(x > y) \), is defined by

\[
\Pr(x > y) = \frac{1}{t} \{i : 1 \leq i \leq t, x > y \text{ in } M_i\}.
\]

In this section, we are concerned with the family \( \Lambda(P) \) of all linear extensions of \( P \). We let \( \lambda(P) = |\Lambda(P)| \). For this family, we drop the subscript and just write \( \Pr(x > y) \). Note that \( \Pr(x > y) = 0 \), when \( x < y \in P \); \( \Pr(x > y) = 1 \), when \( x > y \in P \) and \( 0 < \Pr(x > y) < 1 \), when \( x \parallel y \in P \). In 1969, S. S. Kisilitsyn [76] made the following conjecture, which remains one of the most intriguing problems in the combinatorial theory of posets.

**Conjecture 11.1** If \( P = (X, P) \) is a finite poset which is not a chain, then there exists an incomparable pair \( x, y \in X \) so that

\[
\frac{1}{3} \leq \Pr(x > y) \leq \frac{2}{3}.
\]

This conjecture was made independently by both M. Fredman and N. Linial, and many papers on this subject attribute the conjecture to them. It is now
known as the 1/3–2/3 conjecture. If true, the conjecture would be best possible, as shown by 2 + 1.

The first major breakthrough in this area came in 1984, when Kahn and Saks [62] used the Alexandrov/Penckel inequalities for mixed volumes to prove the following result.

**Theorem 11.2** If \( P = (X, P) \) is a finite poset which is not a chain, then there exists an incomparable pair \( x, y \in X \) so that
\[
\frac{3}{11} < \text{Prob}[x > y] < \frac{8}{11}. \quad \blacksquare
\] (11)

Recently, there has been a slight improvement in this result using a special case of a conjecture called the Cross-product conjecture. The result is due to Brightwell, Felsner and Trotter [18].

**Theorem 11.3** If \( P = (X, P) \) is a finite poset which is not a chain, then there exists an incomparable pair \( x, y \in X \) so that
\[
\frac{5 - \sqrt{5}}{10} < \text{Prob}[x > y] < \frac{5 + \sqrt{5}}{10}. \quad \blacksquare
\] (12)

As pointed out in [18], there is an infinite semi-order for which the inequality in Theorem 11.3 is best possible, so that the 1/3--2/3 conjecture is false if one attempts to extend it to infinite posets. However, for finite semi-orders, we can do even better. For a poset \( P = (X, P) \), we say \( x \) covers \( y \) and write \( x \triangleright y \) in \( P \) when \( x > y \) in \( P \) and if \( x \geq z \geq y \) in \( P \), then either \( x = z \) or \( y = z \). The next result is due to Brightwell [16].

**Theorem 11.4** If \( P = (X, P) \) is a finite semi-order which is not a chain, then there exists an incomparable pair \( x, y \in X \) so that
\[
\frac{1}{3} < \text{Prob}[x > y] < \frac{2}{3}. \quad \blacksquare
\] (13)

**Proof** Suppose that the theorem is false. Choose a counterexample \( P = (X, P) \) with \(|X| = n \) minimum. Then let \( I \) be a distinguishing representation. Label the points of \( X \) as \( x_1, x_2, \ldots, x_n \) in the order determined by \( \min \) and points. Define a linear order \( L \) on \( P \) by setting \( x < y \) in \( L \) if and only if \( \text{Prob}[x > y] < 1/3 \). Clearly, \( L \) is a linear extension of \( P \). Furthermore, \( L \) orders \( X \) as \( x_1 < x_2 < \cdots < x_n \).

We claim that \( x_i \triangleright x_{i+1} \), for all \( i = 1, 2, \ldots, n - 1 \). To the contrary, suppose \( x_i < x_{i+1} \) in \( P \). Then \( P \) is the lexicographic sum over a two-element chain of the subposets determined by \( \{x_1, x_2, \ldots, x_i \} \) and \( \{x_{i+1}, x_{i+2}, \ldots, x_n \} \). One of these posets is not a chain, and we immediately contradict our choice of \( P = (X, P) \) as a minimum counterexample.

We say that a point \( x \) separates \( x_i \) and \( x_{i+1} \) from above if \( x \triangleright x_i \) and \( x \triangleright x_{i+1} \) in \( P \). Dually, we say \( x \) separates \( x_i \) and \( x_{i+1} \) from below if \( x \triangleright x_i \) and \( x \triangleright x_{i+1} \).

Interval Orders and Interval Graphs and \( x_i \| x_j \) in \( P \). Finally, we say \( x_j \) separates \( x_i \) from above or from below. Note that if \( x_i \triangleright x_j \) then \( x_k < x_j \) in \( P \), for all \( k = 1, 2, \ldots, i \). But if \( x_i < x_k \) then \( x_j < x_k \) in \( P \), for all \( k = 1, 2, \ldots, i+1 \).

It follows that at most \( 2(n - 1) \) of the pairs \( x_i \) and \( x_{i+1} \) do not separate pairs from below, while \( x_i \| x_j \) above. If \( 2(n - 1) > 2 \) then \( x_j \) separates \( x_i \) and \( x_{i+1} \). From this we conclude that there are at least two such values for which \( x_i \) separates \( x_i \) and \( x_{i+1} \).

Let \( A(P) \) be the set of all linear extensions \( \Lambda_1 = \{L \in \lambda : x_i < x_{i+1} \text{ in } L \}, \Lambda_2 = \{L \in \lambda : x_i = x_{i+1} \text{ between them in } L \}, \Lambda_3 = \{L \in \lambda : x_i > x_{i+1} \text{ between them in } L \}\). Then \( |\Lambda_3|/|\Lambda_1| = \text{Prob}[x_i \triangleright x_{i+1}] \). Clearly, the map \( x_i \mapsto x_{i+1} \) is order-isomorphic to \( x_i \mapsto x_{i+1} \). For a linear order \( x_i \mapsto x_{i+1} \), there exists a unique element \( x_j \) such that \( x_j \) separates \( x_i \) from above, \( x_{i+1} \) from below, then \( 1/3 < \text{Prob}[x_i > x_{i+1}] \).

There are some other special classes of semi-orders in which the conjecture is known to be true. For example, Fishburn and Trotter showed that it is valid for all posets of height one.

In a poset \( P = (X, P) \), a sequence \( (x_i) \) called a linear extension majority cycle, or \( LEM \) cycle, is one for which \( \text{Prob}[x_i > x_{i+1}] > 1/2 \), for all \( i \in [n] \). It is an interesting observation that LEM cycles do not contain LEM cycles, but Brightwell has shown that LEM cycles can exist in interval orders having dimension at most two.

**12 Interval orders and extremal combinatorics**

Here are two interesting extremal problems. The first problem is investigated by Fishburn and Trotter [56] and \( Q(n, k) \), let \( Q(n, k) \) denote the family of all \( n \)-element \( k \)-comparable pairs. Then set \( e(n, k) = \max_{Q(n, k)} |Q(n, k)| \).

**Theorem 12.1** Every poset \( P = (X, P) \) is a semi-order.

**Proof** Suppose that \( P = (X, P) \in Q(n, k) \) is a semi-order. Suppose further that \( P \) is not a subposet isomorphic to \( 2 + 2 \). Label the elements \( \{x, y, u, v\} \), so that \( u \in D(x) - D(y) \) and \( v \in D(x) - D(y) \).
If true, the conjecture would be best possible unless this area came in 1984, when Kahn and Lovasz [11] proved inequalities for mixed volumes to prove

$$\text{finite poset which is not a chain, then there is an X so that}$$

$$|x > y| < \frac{8}{11}. \quad \Box \quad (11)$$

An improvement in this result using a special I-product conjecture. The result is due to

$$\text{finite poset which is not a chain, then there is an X so that}$$

$$|x > y| < \frac{5 + \sqrt{5}}{10}. \quad \Box \quad (12)$$

An infinite semi-order for which the inequality holds, so that the 1/3 - 2/3 conjecture is false for finite posets. However, for finite semi-orders, if $P = (X, P)$, we say $x$ covers $y$ and write $x \geq y$ if $x \geq z \geq y$ in $P$, then either $x = z$ or

$$\text{finite semi-order which is not a chain, then there is an X so that}$$

$$\text{prob}[x > y] \leq \frac{2}{3}. \quad (13)$$

is false. Choose a counterexample $P = (X, P)$, then let $I$ be a distinguishing representation, $x_1, x_2, \ldots, x_n$ in the order determined by left end on $P$ by setting $x < y$ in $L$ if and only if $x_i \neq y_i$ for $1 \leq i \leq n - 1$. To the contrary, suppose the lexicographic sum over a two-element chain $x_1, x_2, x_3$ and $\{x_{i+1}, x_{i+2}, \ldots, x_n\}$. One and immediately contradict our choice of example.

$$\text{sequences x}_i \text{ and x}_{i+1} \text{ from above if } x_j := x_i \text{ and separates x}_i \text{ and x}_{i+1} \text{ from below if } x_{i+1} := x_j$$

and $x_i || x_j$ in $P$. Finally, we say $x_j$ separates $x_i$ and $x_{i+1}$ if it separates them from above or from below. Note that if $x_j$ separates $x_i$ and $x_{i+1}$ from above, then $x_k < x_j$ in $P$, for all $k = 1, 2, \ldots, i$. Dually, if $x_k$ separates $x_i$ and $x_{i+1}$ from below, then $x_j < x_k$ in $P$, for all $k = i+1, i+2, \ldots, n$. So each $x_j$ separates at most two pairs, one from above and one from below. Furthermore, $x_1$ and $x_2$ do not separate pairs from below, while $x_{n-1}$ and $x_n$ do not separate pairs from above. It follows that there at most $2(n - 4) + 4 = 2n - 4$ pairs $(i, j)$ so that $x_j$ separates $x_i$ and $x_{i+1}$. From this we conclude that there is an integer $i$ (in fact, there are at least two such values) for which there is at most one integer $j$ so that $x_j$ separates $x_i$ and $x_{i+1}$. We show that $1/3 \leq \text{Prob}[x_i > x_{i+1}] \leq 2/3$.

Let $\Lambda(P)$ be the set of all linear extensions of $P$, and let $|\Lambda(P)| = t$. Set $\Lambda_1 = \{L \in \Lambda : x_i < x_{i+1} \text{ in L}\}$; $\Lambda_2 = \{L \in \Lambda - \Lambda_1 : x_i < x_{i+1} \text{ in L}\}$; and $\Lambda_3 = \Lambda - (\Lambda_1 + \Lambda_2)$. Then $|\Lambda_3|/t = \text{Prob}[x_i > x_{i+1}] < 1/3$. Consider the map $h: \Lambda_1 \rightarrow \Lambda_2$ defined as follows. For a linear extension $L \in \Lambda_1$, form $h(L)$ by exchanging $x_i$ and $x_{i+1}$. Clearly, the map $h$ is an injection. It follows that $|\Lambda_1| \leq |\Lambda_2|$. Furthermore, $|\Lambda_3|/t = \text{Prob}[x_i > x_{i+1}] < 1/3$, so $|\Lambda_2| > t/3$. In particular, there exists a unique element $x_j$ which separates $x_i$ and $x_{i+1}$. If $x_j$ separates from above, $1/3 < \text{Prob}[x_{i+1} > x_j] < 2/3$. If $x_j$ separates from below, then $1/3 < \text{Prob}[x_j > x_i] < 2/3$. \Box

There are some other special classes of posets for which the 1/3 - 2/3 conjecture is known to be true. For example, Fishburn, Gehrlein and Trotter [39] showed that it is valid for all posets of height 2.

In a poset $P = (X, P)$, a sequence $(x_1, x_2, \ldots, x_n)$ of length $n \geq 3$ is called a linear extension majority cycle, or just a LEM cycle for short, when $\text{Prob}[x_i > x_{i+1}] > \frac{1}{2}$, for all $i \in [n]$. It is an easy exercise to show that semi-orders do not contain LEM cycles, but Brightwell, Fishburn and Winkler [19] show that LEM cycles can exist in interval orders—in fact, even in interval orders having dimension at most two.

12 Interval orders and extremal problems

Here are two interesting extremal problems involving semi-orders. The first problem is investigated by Fishburn and Trotter in [41]. For integers $n$ and $k$ with $0 \leq k \leq \binom{n}{2}$, let $Q(n, k)$ denote the family of all posets with $n$ points and $k$ comparable pairs. Then set $e(n, k) = \max \{|\Lambda(P)| : P = (X, P) \in Q(n, k)\}$.

**Theorem 12.1** Every poset $P = (X, P) \in Q(n, k)$ with $|\Lambda(P)| = e(n, k)$ is a semi-order.

**Proof** Suppose that $P = (X, P) \in Q(n, k)$, $|\Lambda(P)| = e(n, k)$, but that $P$ is not a semi-order. Suppose further that $P$ is not an interval order. Then $P$ contains a subposet isomorphic to $2 + 2$. Label the 4 points in the copy of $2 + 2$ as $\{x, y, u, v\}$, so that $u \in D(x) - D(y)$ and $v \in D(y) - D(x)$. Of all copies of
2 + 2 in \( P \), we may assume that we have chosen one so that \( |U(x)| + |U(y)| \)
is minimum. It follows that one of \( U(x) \) and \( U(y) \) is a subset of the other. Without
loss of generality, we assume that \( U(x) \subseteq U(y) \). Let \( P' = (X, P') \) be
the poset obtained from \( P \) by replacing the relations \( z < y \) by \( z < x \) for all
\( z \in D(y) - D(x) \). Then \( P' \in Q(n, k) \).

Interchanging the points and \( y \) transforms a linear extension from \( \Lambda(P) - \Lambda(P') \)
into a linear extension from \( \Lambda(P') - \Lambda(P) \). Furthermore, this map is an
injection. It is not a surjection, because any linear extension with \( y < u < v < x \) is not in the image of the map. The contradiction shows that \( P \) is an
interval order.

Now assume that \( P \) contains a subposet isomorphic to \( 3 + 1 \), and label the
elements in the 3-element chain so that \( x < y < z \in P \). Label the element
incomparable to these three points as \( w \). Now form a poset \( P'' = (X, P'') \in Q(n, k) \)
by replacing relations \( t < y \) by \( t < w \) for all \( t \in D(y) - D(w) \). Then
\( P' \in Q(n, k) \). As before, \( P'' \) has more linear extensions than \( P \). \( \blacksquare \)

The second problem sounds similar. It was posed to me by Peter Winkler.
Define the flexibility of a poset \( P = (X, P) \), denoted \( \text{flex}(P) \), by

\[
\text{flex}(P) = \sum_{x \in X} |U(x) + D(x)|^2.
\]

Then the same kind of argument used to prove Theorem 12.1 can be used to show
that among all posets with \( n \) points and \( k \) comparable pairs, those
with maximum flexibility are semi-orders. Despite our knowledge about the
structure of the extremal posets, little progress has been made in solving either
of these problems in full generality.

Now here is an interesting extremal problem for posets on which significant
results have been obtained for interval orders. When \( L \) is a linear extension of
\( P = (X, P) \), let \( j(L, P) \) count the number of consecutive pairs of elements in \( L \)
which are incomparable in \( P \). The jump number of \( P \) is then the minimum
value of \( j(L, P) \) taken over all linear extensions of \( P \). In [89], Mitas shows that
determining the jump number of an interval order is NP-complete. However,
Mitas [89], Felsner [29] and Syslo [108] have (independently) given a polynomial
algorithm for approximating the jump number within a ratio of 3/2.

Bogart and Stellflug [11] define the representation length of a semi-order
as the least positive integer \( k \) for which it has a representation using intervals
of length \( k \) with integer endpoints. For each \( k \geq 1 \), they provide a forbidden
subposet characterization of semi-orders with representation length \( k \).

For interval orders, we have the following natural extremal problem posed by Peter Fishburn in [36]. Given an interval order \( P \), find the least positive
integer \( k \) for which \( P \) has a representation using intervals having \( k \) distinct
lengths. This parameter is called the interval count of \( P \). Two interesting
questions are immediate. First, what is the maximum value of the interval
count of an interval order on \( n \) points? Second, can the removal of a single
point drop the interval count by an arbitrarily large amount?

Interval Orders and Interval Graphs

13 Interval orders and Hamiltonian Graphs

Considered as a graph, the diagram of a chromatic number exceeding \( h \). However, this
and Rodl [90] shows that for every integer \( h \), there is a graph whose chromatic
number of the diagonal graph.

For interval orders, the situation is considerably different. The open intervals with integer end points of height \( h \).

Furthermore, the diagonal graph \( S(2, h+1) \), a graph whose chromatic

Surprisingly, this is not from best possible.

Let \( t \) be a positive integer, and let \( S(h) \) be an\( \alpha \)-sequence of sets. Felsner and Trotter [34] called
\( S(h) \) a subset of \([t]\). For example, \( \alpha(3) = 5 \) and
\( \{0, \{1, \{2, \{3, \{1, \{3, \{1, \{2, \{3\}\}\}\}\}\}\}\}\}. Note that
the terms from an \( \alpha \)-sequence is again an \( \alpha \)-sequence.

Let \( D(h) \) denote the maximum chromatic number of a graph of height \( h \). Clearly, \( D(1) = 1 \) and \( D(2) = 2 \).

Theorem 13.1 For each \( h \geq 2, D(h) = h \).

Proof We first show that if \( \alpha(t) > h \), then \( D(h) = h \).

Then let \( I \) be a distinguishing representation of \( C \) as an \( \alpha \)-sequence of subsets of \([t]\), and let \( \alpha(C) = t \).

For each \( x \in X \), set \( i(x) = \max \{i \mid j \leq i \in I(x) \} \). Note that
\( i(x) \) is a proper coloring of \( \phi(x) \).

We claim that \( \alpha(C) \) is a proper coloring of \( \phi(x) \).

We now sketch the proof that if \( D(h) \)

Finally, we note that for each \( h \geq 2 \), there exists an interval order graph which has
the chromatic number of the diagram of \( P \).

We show that the choice \( m_0 = 2h^2 \) works.
we have chosen one so that $|U(x)| + |U(y)|$ of $U(x)$ and $U(y)$ is a subset of the other
such that $U(x) \subseteq U(y)$. Let $P' = (X, P')$ be
replacing the relations $z < y$ by $z < z$ for all
$k$.

$\Lambda(x, y) \Lambda(P') - \Lambda(P)$). Furthermore, this map is an
because any linear extension with $y < u < x$ is a map. The contradiction shows that $P$ is an

subposet isomorphic to $3 + 1$, and label the
so that $x < y < z$ in $P$. Label the element
as $w$. Now form a poset $P'' = (X, P'')$ of
$b$ by $b < w$ for all $t \in D(y) - D(w)$. Then
more linear extensions than $P$.

Theorem 12.1 can be used to prove Theorem 12.1. It is posed to me by Peter Winkler.

\begin{equation}
\sum_{x \in X} |U(x) + D(x)|^2.
\end{equation}

\begin{equation}
D(h) \text{ denote the maximum chromatic number of an interval order of height } h. \text{ Clearly, } D(1) = 1 \text{ and } D(2) = 2.
\end{equation}

\section{Interval orders and hamiltonian paths}

Considered as a graph, the diagram of a poset of height $h$ cannot have
chromatic number exceeding $h$. However, the “partite” construction of Neidert
and Rödl [90] shows that for every integer $h$, there exists a poset $P$ of height $h$
so that the chromatic number of the diagram of $P$ is exactly $h$.

For interval orders, the situation is completely different, and the chromatic
number of the diagram of an interval order of height $h$ is much less than $h$.
The open intervals with integer end points in $\{1, 2, \ldots, h+1\}$ form an interval
order of height $h$. Furthermore, the diagram of this interval order is just the
shift graph $S(2, h+1)$, a graph whose chromatic number is exactly $\lfloor \log(h+1) \rfloor$.

Surprisingly, this is not far from best possible.

Let $t$ be a positive integer, and let $S = (S_0, S_1, \ldots, S_h)$ be a sequence
of sets. Felsner and Trotter [34] called $F$ an $\alpha$-sequence if $S_1 \nsubseteq S_0$
and $S_j - (S_i \cup S_{i-1}) \neq \emptyset$, for all $i, j$ with $1 \leq i < j \leq h$. Define $\alpha(t)$ to be
the maximum $h$ for which there exists an $\alpha$-sequence $(S_0, S_1, \ldots, S_h)$, with
each $S_i$ a subset of $[t]$. For example, $\alpha(3) = 5$ as evidenced by the $\alpha$-sequence $A = \emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}$. Note that any subsequence of consecutive
early, we have $\alpha(t) \geq h$.

\begin{proof}
We first show that if $\alpha(t) > h$, then $D(h) \leq t$. Let $S = (S_0, S_1, \ldots, S_h)$
be an $\alpha$-sequence of subsets of $[t]$, and let $P$ be an interval order of height $h$.
Then let $I$ be a distinguishing representation of $P$. Let $b$ be the lexicographically
least maximum chain be $B = \{b_1 < b_2 < \cdots < b_h\}$, and let the canonical
partition into antichains be $C = A_1 \cup A_2 \cup \cdots \cup A_h$. For each $i \in [h]$, let
$r_i = a_i$. Then let $r_0$ be any real number with $r_0 < a_x$ in $\mathbb{R}$, for every $x \in X$.

For each $x \in X$, set $i(x) = \max\{i : 0 \leq i \leq h, r_i < a_x\}$ and $j_x = \max\{j : 1 \leq j \leq h, r_j \in I(x)\}$. Note that $i_x < j_x$, for every $x \in X$. We then
define a coloring $\phi_x : X \rightarrow [t]$ as follows. If $i_x = 0$, choose $\phi(x) \in S_{j_x} - S_0$. If
$i_x > 1$, choose $\phi(x) \in S_{j_x} - (S_{i_x-1} \cup S_{i_x})$.

We claim that $\phi$ is a proper coloring of the diagram of $P$. Suppose that
$x < y$ in $P$. Then either $i_y = j_x$ or $i_y = 1 + j_x$. In either case, note that
$\phi(x) \neq \phi(y)$.

We now sketch the proof that if $D(h) \leq t$, then $\alpha(t) \geq h$. For integers $h, m \geq 2$, let $P(h, m)$ denote the interval order determined by the family of
all closed intervals with length at least $m - 1$ having integer end points from
$\{1, 2, \ldots, m(h+1) - 1\}$. Note that the height of $P(h, m)$ is $h$. We now show
that for each $h \geq 2$, there exists an integer $m_0$ so that if $m > m_0$ and the
chromatic number of the diagram of $P(h, m)$ is $t$, then $\alpha(t) \geq h$. In fact, we
show that the choice $m_0 = 2^{h^2}$ works.
Fix $h \geq 2$ and then let $m$ be any integer with $m > m_0$. Suppose that the chromatic number of the diagram of $P(h, m)$ is $t$. Note $t \leq h$. Now suppose that $\phi$ is a coloring of the diagram of $P(m, h)$ using colors from $[t]$. For each $j = 1, 2, \ldots, m(h + 1) - 1$, let $A_j = \{\phi([i, j]) : 1 \leq i \leq j - m + 1\}$. Then for each $i = 0, 1, \ldots, m - 1$, let $V_i = (A_{m+i}, A_{2m+i}, \ldots, A_{hm+i})$. Each $V_i$ is a vector of length $h$ with each entry a subset of $[t]$. Since there are at most $2^{ht}$ such vectors, it follows that there exist integers $i_1, i_2$ with $0 \leq i_1 < i_2 \leq m - 1$ for which $V_{i_1} = V_{i_2}$.

Set $S_0 = \emptyset$ and $S_k = A_{nk+i_1}$, for $k = 1, 3, \ldots, h$. We claim that the sequence $(S_0, S_1, \ldots, S_h)$ is an $\alpha$-sequence. Clearly, $S_1 \neq \emptyset$ as $S_1$ contains the color $\phi$ assigns to the interval $[1+i_1, m+i_1]$. Thus $S_1 \not\subseteq S_0$. Now suppose that $1 < i < j < h+1$ and $S_j \subseteq S_{i-1} \cup S_{i}$. Suppose that $\phi$ assigns color $\beta \in [t]$ to the interval $x = [1+mi+i_1, m(j+i_2)]$. Then there is an interval $y \in S_{i-1} \cup S_i$ with $\phi(y) = \beta$. Since $S_i = A_{m+i_1} = A_{mi+i_2}$, and $S_{i-1} = A_{m(i-1)+i_1} = A_{m(i-1)+i_2}$, there is an interval $y$ with $y \subseteq (mi+i_1, m(i-1)+i_2)$ so that $\phi(y) = \phi(x) = \beta$. This is a contradiction, since $x > y$.

Felsner and Trotter [34] conjecture that

$$\alpha(t) = 2^{t-1} + \left\lfloor \frac{t-1}{2} \right\rfloor. \quad (15)$$

If this conjecture is true, then an $\alpha$-sequence $S$ of subsets of $[t]$ of maximum size has the following property. If we form a new sequence $\mathcal{H}$ from $S$ by inserting between two consecutive sets in $S$ their union, when the first set is not a subset of the second, then we get a listing of all $2^{t}$ subsets of $[t]$. For example, from the 6 term $\alpha$-sequence of subsets of $[3]$ given above, this listing is $(\emptyset, \{1\}, \{1, 2\}, \{2\}, \{2, 3\}, \{3\}, \{1, 3\}, \{1, 2, 3\})$. This listing is a special kind of hamiltonian path in the $t$-cube. Whenever a set appears in the list, all of its subsets, with at most a single exception, appear previously. If there is an exception, it is listed next.

We call such a path an order-preserving hamiltonian path in the $t$-cube. This is a slight abuse of the concept of order-preserving, but it is the strongest notion that makes sense. It is known that there are order-preserving hamiltonian paths in the $t$-cube for $1 \leq t \leq 8$, but the general question is open.

We should point out that attempts to settle whether equation (15) is always valid have produced the best known partial result on the well known "middle two levels" problem. The origins of the problem are a bit unclear, but it was first told to me by Ivan Havel during a visit to Prague.

**Problem 13.2** Is the diagram of the poset consisting of all $k$-element and $(k+1)$-element subsets of a $(2k+1)$-element set, partially ordered by inclusion, a hamiltonian graph? 

We refer the reader to [34] and [99] for details.

Interval Orders and Interval Graphs

14 On-line and un-cooperative coloring

An on-line optimization problem, such as coloring, can be considered as a two-person game involving a builder and a colorer. The game is played in a series of rounds with the players taking turns. On-line coloring involves two parameters. A builder has $n$ vertices, the game lasts at most $n$ rounds. A Builder presents the vertex $v_i$ of $G$ and the colorer has to select a color for $v_i$. The colorer has to select a color for $v_i$ in such a way that no two adjacent vertices have the same color. If the colorer is able to respond with a legitimate choice of a color for the new vertex, then Colorer is able to respond with a legitimate choice of a color for the new vertex. If not, then Colorer has lost the game. The on-line chromatic number of $G$ is the least $t$ for which Colorer has a winning strategy.

In [71], Kierstead and Trotter prove the following theorem.

**Theorem 14.1** The on-line chromatic number of a graph $G$ is $k$.

**Proof** Here's how the strategy works. For each $i$, build a new vertex $v_i$ and two edges: $v_i$ is adjacent to $v_{i+1}$ and $v_{i+2}$.

We refer the reader to [34] and [99] for details.
14 On-line and un-cooperative coloring

An on-line optimization problem, such as on-line graph coloring, can be considered as a two-person game involving a Builder and a Colorer. The game is played in a series of rounds with the players alternating turns. Each instance of on-line coloring involves two parameters: an integer \( t \) and a graph \( G \). If \( G \) has \( n \) vertices, the game lasts at most \( n \) rounds. In Round 1, where \( 1 \leq i \leq n \), Builder presents the vertex \( v_i \) of \( G \) and describes all edges joining \( v_i \) with vertices in \( \{ v_j : 1 \leq j < i \} \). This information is complete and correct. In particular, if the game lasts all \( n \) rounds, then Builder must have correctly specified the entire graph.

After receiving the information for the new vertex \( v_i \), Colorer must then assign to \( v_i \) a color from the set \( \{1, 2, \ldots, t\} \) so that this color is distinct from those previously assigned to neighbors of \( v_i \). These assignments are permanent.

The \((t, G)\) game ends at Round \( n \) if Builder is the winner if Colorer has no legitimate choice of a color for the new vertex \( v_n \). If on the other hand, Colorer is able to respond with a legitimate color for each of the \( n \) vertices of \( G \), then Colorer is the winner. The on-line chromatic number of a graph \( G \) is then the least \( t \) for which Colorer has a winning strategy for the \((t, G)\) game—regardless of the strategy employed by Builder.

In [71], Kierstead and Trotter prove the following foundational result.

**Theorem 14.1** The on-line chromatic number of an interval graph of maximum clique size \( k \) is at most \( 3k - 2 \).

**Proof** Here’s the winning strategy for Colorer. Given a new vertex \( x \) by Builder, Colorer assigns \( x \) to a set \( S_i \) where \( i \) is the least positive integer for which there is no complete subgraph of size \( i + 1 \) containing \( x \) and \( i \) other vertices previously assigned to \( S_1 \cup S_2 \cup \cdots \cup S_i \). Note that \( S_i \) is just an independent set, so it can be colored with a single color. For each \( i \geq 2 \), we show that First Fit will color \( S_i \) with the 3 colors from the set \( \{3i - 4, 3i - 3, 3i - 2\} \).

We accomplish this by showing that for \( i \geq 2 \), \( S_i \) is the disjoint sum of paths.

Fix \( i \geq 2 \). When a vertex \( u \) is presented by Builder and assigned by Colorer to \( S_i \), as opposed to \( S_{i-1} \), there is a clique \( K_u \) consisting of \( u \) and \( i - 1 \) vertices from \( S_1 \cup S_2 \cup \cdots \cup S_{i-1} \). Then the intersection of the intervals from \( K_u \) is a nonempty interval \( I_u \), which is contained in the interval corresponding to \( u \). Now let \( u \) and \( v \) be adjacent vertices from \( S_i \). Then it is easy to see that \( I_u \) does not intersect the interval corresponding to \( v \) and \( I_v \) does not intersect the interval corresponding to \( u \). From this, it follows easily that \( S_i \) is triangle free and that each vertex from \( S_i \) has at most 2 neighbors in \( S_i \).

The algorithm presented in the preceding theorem has one feature in common with the First Fit algorithm discussed in Theorem 4.2: it is not necessary to know the maximum clique size in advance. If First Fit is used to color an interval graph, and the vertices are not processed in the order of left end points,
then it is not clear how many colors will be used. In [65], Kierstead showed that First Fit will use at most $40k$ colors on an interval graph with maximum clique size $k$, regardless of the order in which the vertices are processed. Subsequently, Kierstead and Qin [70] improved this upper bound to $26k$. From below, Chrobak and Slusarek [23] showed that no on-line algorithm can color all interval graphs with maximum clique size $k$ with fewer than $4.4k$ colors.

Kierstead's analysis of the performance of First Fit in coloring interval graphs provided a solution to an important long standing problem in computer science called the Dynamic Storage Allocation problem. The standard two dimensional bin packing problem is to pack a family of rectangles in $\mathbb{R}^2$, with sides parallel to the coordinate axes, into a region of minimum area. The Dynamic Storage Allocation problem is to pack the rectangles into a region of minimum height—when the projections of the rectangles onto the horizontal coordinate axis form a fixed interval graph. Of course, by “pack,” we mean that the rectangles are to be placed so that their interiors are disjoint. So if the maximum sum of the heights of rectangles whose projections have a common point is $t$, then $t$ is a lower bound on the height required for a packing, and it was conjectured that a height of $O(t)$ would suffice.

One proposed approach to finding a reasonably good packing was to assume all rectangles had height a power of 2. This assumption would at most double the optimal height required for a packing. These rectangles would then be partitioned into subrectangles of height one. Finally, First Fit would be used to color the rectangles (intervals) with all intervals formed from the same rectangle colored consecutively. The number of colors used by First Fit would then be an upper bound on the minimum height required for a packing. Accordingly, Kierstead and Qin’s bound implies that the rectangles can in fact be packed into a region of height $52t$.

We refer the reader to [66] for a full discussion. As an added bonus, this paper provides an alternative approach to the dynamic storage allocation problem which stands as the best solution to date. This approach uses the same partition of rectangles, but colors them with a modified version of the on-line algorithm used in Theorem 14.1 rather than with First Fit. The end result is to show that the rectangles can be packed into a height of $6t - 4$.

Ironically, the research which led to the proof of Theorem 14.1 was motivated, not by the Dynamic Storage Allocation problem, but by the on-line version of Dilworth’s theorem. In [64], Kierstead proved that there is an on-line algorithm which will partition a poset built one point at a time into $(5^w - 1)/4$ chains, where $w$ denotes the width of the poset. When the poset is known to be an interval order, then the preceding theorem asserts that $3w - 2$ chains suffice.

Kierstead’s on-line chain partitioning algorithm requires knowledge of the order. Just knowing whether points are comparable is not enough. However, for interval graphs, our algorithm only makes use of the comparability graph. For many years, it remained an open problem to determine whether a comparability graph of independence number $\alpha$ has a bounded number of complete subgraphs. Answered by Kierstead, Penrice and Trotter in [69].

In [68], Kierstead, McNulty and Trotter define the persistence of a poset. Here the game is between a Realizer and a family $\mathcal{R}$ of linear extensions of a poset $\mathcal{P}$. One does not play at a point at a time. They show that the on-line posets is infinite. However, the posets in this $\mathcal{R}$ have width defined, provided that the $\mathcal{R}$ do not contain any 3-dimensional crown $S^3_3$. They then proceed to show that the bounded width is well defined, provided that the $\mathcal{R}$ do not contain any 3-dimensional crown $S^3_3$.

**Theorem 14.2** The on-line dimension of $\mathcal{R}$ is $\alpha_i$, where $t = (5^{k+1} - 1)/4$.

On the surface, this result has nothing to do with posets. The proof makes use of an auxiliary order at a critical point where this structure gains from being an interval order.

Other sources of information about on-line posets include a more recent survey by Kierstead [67]. In part, this treatment of the recent breakthroughs begins by showing that for all $k \geq 3$, there exists an $\mathcal{R}$ of $n$-ary posets of any $k$-colorable graph on $n$ vertices. This argument shows that $\epsilon = O(1/k!)$. Probably, another way to put this is that the game of on-line posets gains from being a graph on $n$ vertices.

Another good source of problems is [27], [Faigle, Kern, Kierstead and Trotter] and this list goes on.

**Theorem 14.3** The on-line chromatic number of the on-line posets is $\alpha_i$, where $t = (5^{k+1} - 1)/4$.

**Proof** Let $I$ be a distinguishing representation for the poset. Then $I$ is an interval order. When it is her turn...
Interval Orders and Interval Graphs

parability graph of independence number $k$ can be partitioned on-line into a bounded number of complete subgraphs. An affirmative answer was provided by Kierstead, Penrice and Trotter in [69].

In [68], Kierstead, McNulty and Trotter investigate on-line dimension. Here the game is between a Realizer and a Builder, with Realizer building a family $\mathcal{R}$ of linear extensions of a poset $\mathcal{P}$ which Builder is constructing one point at a time. They show that the on-line dimension of a class of width 4 posets is infinite. However, the posets in this class all contain the 3-dimensional crown $S_3^2$. They then proceed to show that the on-line dimension of posets of bounded width is well defined, provided that the posets are crown-free, i.e., do not contain any 3-dimensional crown $S_3^2$.

**Theorem 14.2** The on-line dimension of a crown-free poset of width $k$ is at most $t!$, where $t = (5^{k+1} - 1)/4$.

On the surface, this result has nothing to do with interval orders, but the proof makes use of an auxiliary order at a critical point in the argument. This structure turns out to be an interval order, and the order-theoretic properties this structure gains from being an interval order are key elements of the proof.

Another source of information about on-line coloring include [72] and the more recent survey by Kierstead [67]. In particular, this last paper contains a concise treatment of the recent breakthrough where Kierstead succeeded in showing that for all $k \geq 3$, there exists an $\epsilon > 0$ so that the on-line chromatic number of any $k$-colorable graph on $n$ vertices is at most $n^{1-\epsilon}$. Kierstead’s argument shows that $\epsilon = O(1/k!)$. Probably, this can be improved to $O(1/k)$. Another good source of problems (some of which are on-line) concerning interval graphs and other classes of perfect graphs in Gyárfás’s survey paper [54].

In [27], Faigle, Kern, Kierstead and Trotter consider the following game theoretic problem for graphs. Two players, Alice and Bob, color a graph $G$ using elements of the set $[t]$ as colors. They alternate turns with Alice having the first move. Alice wins if the graph is eventually colored and Bob (an uncooperative partner) wins if at some point before the graph is colored, there is no legitimate move. The *game chromatic number* of $G$ is the least $t$ for which Alice has a winning strategy. For example, it is shown in [27] that the game chromatic number of a tree is at most 4; furthermore, this result is best possible. In [74], Kierstead and Trotter show that a planar graph has game chromatic number at most 33; they also show that there exists a planar graph with game chromatic number at least 8.

For interval graphs, the following result, given in [27], provides the best known bound on the game chromatic number of an interval graph.

**Theorem 14.3** The game chromatic number of an interval graph $G = (V, E)$ with maximum clique size $k$ is at most $3k - 2$.

**Proof** Let $f$ be a distinguishing representation of an interval graph $G$ with maximum clique size $k$. When it is her turn to color, Alice prefers to color a
vertex \( x \) adjacent to the vertex just colored by Bob. Among such, she prefers
those whose intervals intersect the interval corresponding to the vertex just
colored by Bob. Finally among such vertices, Alice prefers the one whose
interval has right end point as large as possible. She then colors this vertex by
First Fit.

We claim that Alice and Bob can never reach an impasse if the number of
colors is \( 3k - 2 \). It suffices to show that the strategy given for Alice can be
used by either player. Let \( x \) be the vertex to be colored. It suffices to show
that \( x \) has at most \( 3k - 3 \) colored neighbors. Split the colored neighbors into
three sets \( N_1 \cup N_2 \cup N_3 \), where

1. \( N_1 \) is the set of colored neighbors of \( x \) whose intervals contain the right
   end point of \( I(x) \);
2. \( N_2 \) is the set of colored neighbors of \( x \) whose intervals are properly con-
   tained in \( I(x) \); and
3. \( N_3 \) is the set of colored neighbors of \( x \) whose intervals contain the left
   end point of \( I(x) \) but not the right.

Clearly, \( |N_1| \leq k - 1 \) and \( |N_3| \leq k - 1 \), so our claim follows if we can show
that \( |N_2| \leq k - 1 \). Now our strategy for Alice insures that she will not have
colored any of the vertices in \( N_2 \), since she will always prefer to color \( x \). So all
vertices in \( N_2 \) are colored by Bob, and at every turn—except possibly the last
one—Alice has selected a vertex other than \( x \) to color. Such a vertex must
have an interval containing the interval corresponding to the vertex in \( N_2 \) just
colored by Bob, and its right end point is greater than the right end point
of \( x \). Therefore Alice’s response was to color a vertex from \( N_1 \). It follows that
\( |N_2| \leq k \). Now suppose that \( |N_2| = k \). Among the vertices in \( N_2 \), let \( y \) be the
unique vertex whose right end point is as large as possible. Then \( y, x \) and the
\( k - 1 \) vertices in \( N_2 \) form a clique of size \( k + 1 \).

The reader should note that it is just a coincidence that the expression
\( 3k - 2 \) appears in both the preceding two theorems. In the first case, we know
that it is best possible, but in the second, we believe it is not. We leave it as an
exercise to show that, for each \( k \geq 2 \), there exists an interval graph \( G \) whose
maximum clique size is \( k \) and whose game chromatic number is at least \( 2k \).

15 Fractional dimension and ramsey theory for probability spaces

It is often useful to consider a fractional version of an integer valued combi-
natorial parameter as, in many cases, the resulting LP relaxation sheds light
on the original problem. In [20], Brightwell and Scheinerman proposed to
investigate fractional dimension for posets.

Interval Orders and Interval Graphs

Let \( P = (X, P) \) be a poset, and let \( 2 \) linear extensions of \( P \). Brightwell and Scheinerman
of \( P \) if, for each incomparable pair \((x, y)\), there is a \( \mathcal{F} \) which reverse the pair \((x, y)\), i.e., \([i \leq k]/\)
fractional dimension of \( P \), denoted by \( \text{fdim}(P) \), for which there exists a \( q \geq 1 \) for which there exists a \( 1 \)-fold realization of \( P \) so that \( k/t \geq 1/q \) (it is easily verified that \( r \) is indeed attained and is the
this terminology, the dimension of \( P \) is just
1-fold realization of \( P \). It follows immediately from \( P \).

The dimension or fractional dimension of \( P \) is the least upper bound of \( \text{fdim}(P) \) (respectively) in the class. We have seen that \( \dim(X) = \infty \) for
right order and \( \text{dim}(P) = \infty \) if \( \text{dim}(P) = \infty \) is an interval order and \( \text{dim}(P) = \infty \) if \( P \) has \( x > y \) in \( L \) for any incomparable
Building a realization from one such \( L \) for every \( P \) gives \( \text{fdim}(P) < 4 \).

Brightwell and Scheinerman conjectured though no example of an interval order of fractional dimension
was then known. In the remainder of this section, Trotter and Winkler in [126] to settle the
First, the following preliminary results, the
true and events being false. See [126] for the

**Theorem 15.1** For every \( \epsilon \geq 0 \), there exists a \( \mathcal{U}_i \) \( U_i \) is some sequence of \( i \) and \( j \) with \( 1 \leq i < j \leq m \).

With this result in hand, we can now show that our argument makes extensive use of ramsey theory about
sets hold in a uniform manner. To be small errors, and the argument takes some time to fit under control. In this sketch, we ignore

**Theorem 15.2** For every \( \epsilon > 0 \), there exists a \( \mathcal{U}_i \) fractional dimension of the canonical isomorphic:

**Proof** Let \( \epsilon > 0 \), and suppose that \( \text{fdim}(P) \leq n \). We argue to a contradiction, provided

Let \( S = \{s_1, s_2, \ldots, s_{2m}\} \) be a \( 2m \)-element
\( \cdots < s_{2m} \). Then let \( U(S) \) denote the even
Let $P = (X, P)$ be a poset, and let $F = \{M_1, \ldots, M_t\}$ be a multiset of linear extensions of $P$. Brightwell and Scheinerman [20] call $F$ a $k$-fold realizer of $P$ if, for each incomparable pair $(x, y)$, there are at least $k$ linear extensions in $F$ which reverse the pair $(x, y)$, i.e., $|\{i : 1 \leq i \leq t, \ x > y \ \text{in} \ M_i\}| \geq k$. The fractional dimension of $P$, denoted by $\text{fdim}(P)$, is then defined as the least real number $q \geq 1$ for which there exists a $k$-fold realizer $F = \{M_1, \ldots, M_t\}$ of $P$ so that $k/t \geq 1/q$ (it is easily verified that the least upper bound of such real numbers $q$ is indeed attained and is therefore a rational number). Using this terminology, the dimension of $P$ is just the least $t$ for which there exists a 1-fold realizer of $P$. It follows immediately that $\text{fdim}(P) \leq \text{dim}(P)$, for every poset $P$.

The dimension or fractional dimension of a class of posets is defined to be the least upper bound of $\text{dim}(P)$ (respectively $\text{fdim}(P)$) over all posets $P$ in the class. We have seen that $\text{dim}(I) = \infty$ for the class $I$ of interval orders, but Brightwell and Scheinerman showed that $\text{fdim}(I) \leq 4$. To see this, observe that if $P = (X, P)$ is an interval order and $A \subseteq X$, there is a linear extension $L$ of $P$ with $x > y$ in $L$ for any incomparable pair $(x, y)$ with $x \in A$ and $y \not\in A$. Building a realizer from one such $L$ for each subset $A$ of $X$ of size $|X|/2$ gives $\text{fdim}(P) < 4$.

Brightwell and Scheinerman conjectured in [20] that $\text{fdim}(I) = 4$, even though no example of an interval order of fractional dimension even as high as 3 was known. In the remainder of this section, we sketch the proof approach taken by Trotter and Winkler in [126] to settle this conjecture in the affirmative.

First, the following preliminary result is required. Intuitively, this theorem asserts that in a sufficiently long sequence of events, one cannot do substantially better than toss a fair coin in trying to balance between events being true and events being false. See [126] for the proof.

**Theorem 15.1** For every $\epsilon > 0$, there exists an integer $m$ so that if $m \geq m_0$ and $\{U_i : 1 \leq i \leq m\}$ is any sequence of events in a probability space, then there exist integers $i$ and $j$ with $1 \leq i < j \leq m$ so that $\text{Prob}[U_i \cap \neg U_j] < \frac{1}{2} + \epsilon$.

With this result in hand, we can now sketch the proof of the solution. The argument makes extensive use of Ramsey theory to make certain statements about sets hold in a uniform manner. To be precise, these statements involve small errors, and the argument takes some care to show that the errors can be kept under control. In this sketch, we ignore these errors.

**Theorem 15.2** For every $\epsilon > 0$, there exists an integer $n_0$ so that if $n > n_0$, the fractional dimension of the canonical interval order $I_n$ is at least $4 - \epsilon$.

**Proof** Let $\epsilon > 0$, and suppose that $\text{fdim}(I_n) < 4 - \epsilon$, regardless of the size of $n$. We argue to a contradiction, provided $n$ is sufficiently large.

Let $S = \{s_1, s_2, \ldots, s_{2m}\}$ be a $2m$-element subset of $[n]$, with $s_1 < s_2 < \cdots < s_{2m}$. Then let $U(S)$ denote the event that for some $i$ with $1 \leq i \leq m$,
[\sigma_1, \sigma_{m+1}] > [\sigma_i, \sigma_{m+1}]. Using Ramsey theory, it is relatively easy to see that for fixed \( m \), if \( n \) is sufficiently large, we may assume that the probability of \( U(S) \) is constant, for all \( 2m \)-element subsets of \([n]\). But Trotter and Winkler show more. They show that one may also assume that the event \( U(S) \) depends only on \( \sigma_1 \) and \( \sigma_{m+1} \). We denote this event by \( U(x, y) \), where \( x = \sigma_1 \) and \( y = \sigma_{m+1} \).

Dually, let \( D(S) \) denote the event that for some \( j \) with \( 1 \leq j < m \), \([\sigma_j, \sigma_{m+j}] > [\sigma_m, \sigma_{2m}]\). This time, the event \( D(S) \) depends only on \( \sigma_m \) and \( \sigma_{2m} \). So we can just write \( D(x, y) \), where \( x = \sigma_m \) and \( y = \sigma_{2m} \).

It follows that one can find a large homogeneous subset \( H \) so that \( U(x, y) \cap D(x, y) = \emptyset \), for every \( x, y \in H \) with \( x < y \in \mathbb{R} \). Furthermore, if \( x < y < z < w \) in \( H \), then \( U(x, z) \cap U(y, w) = \emptyset \). If the homogeneous set \( H \) has more than \( 2m_0 \) elements, the result follows from Theorem 15.1.

The dimension problem for interval orders is closely related to the graph coloring problem for shift graphs, a subject of independent interest. Similarly, the research on the fractional dimension of interval orders has led to many new and interesting concepts. We give hints to one of these in the sketch of the proof of Theorem 15.2, namely the development of a general Ramsey theory for probability spaces. However, there are several concrete combinatorial problems which are also quite attractive.

Fix integers \( n \) and \( k \) with \( 1 \leq k < n \). Suppose we have a probability space containing an event \( E_k \) for every \( k \)-element subset \( S \) of \([n]\). We abuse notation and just refer to this event as \( S \). Now consider the minimum probability \( \text{Prob}(A \bar{B}) \) taken over all \((k, n)\)-shift pairs. In turn, take the maximum value of this probability over all probability spaces and let \( n \) go to infinity. The resulting value is called \( f(k) \). For example, from Theorem 15.1, it follows that \( f(1) = \frac{1}{2} \). In [126], Trotter and Winkler prove that \( f(2) = \frac{1}{3} \), \( f(3) \geq \frac{3}{8} \) and \( f(4) \geq \frac{5}{16} \). In general, they prove that \( f(k) \) is strictly increasing and converges to \( \frac{1}{2} \).

The relaxation of dimension to fractional dimension is an appealing concept. In [33], Felsner and Trotter show that the fractional dimension of a poset in which each point is comparable with at most \( k \) others is at most \( k + 1 \). They also prove several other inequalities linking fractional dimension with width and cardinality. Nevertheless, there are many challenging open questions in this area.

16 Higher-dimensional analogues for graphs

In recent years, there has been a steady stream of results providing higher dimension analogues of interval graphs. Perhaps the first of these is due to Roberts [96] who defined the boxicity of a graph \( G = (V, E) \) as the least \( t \) for which there exists a function \( B \) assigning to each vertex \( x \in V \) a sequence \((I_x(1), I_x(2), \ldots, I_x(t))\) of closed intervals of \( \mathbb{R} \) so that \( \{x, y\} \in E \) if and only if \( I_x(i) \cap I_y(i) \neq \emptyset \), for all \( i \in [t] \). Equivalently, the boxicity of a graph is just the least \( t \) so that the graph is the intersection graph of \( t \)-dimensional boxes. Moreover, boxicity one graphs are graphs with boxicity one. Robertson's graph on \( n \) vertices is at most \([n/2]\), when \( n \geq 2 \). Boxes are obtained by taking the complement of a line with boxicity \( n \). In [127], Wittenshausen showed that there is only graph with \( 2n \) vertices and boxicity \( n \).

However, when a graph has \( 2n+1 \) vertices, it is more complicated. For example, in [128], shows that a graph \( G \) on \( 2n+1 \) vertices has boxicity \( m \), if the following conditions hold:

1. \( G \) contains \( H_m \).
2. \( G \) contains the join of \( C_5 \) and \( H_{n-2} \).
3. \( G \) contains the join of \( W_3 \) and \( H_{n-2} \).

In [109], Thomassen showed that the boxicity is exactly 3, in fact, the boxes corresponding to adjacent vertices are disjoint.

Many of the basic concepts for interval orders can be extended to digraphs. In [3], Beineke and Zafiropoulo (under different terminology) of an interval digraph, that is, a digraph \( D \), there are two integers \( i \) and \( j \) such that \( D \) contains a directed arc from \( x \) to \( y \) if and only if the arc \((i, j)\) is in \( E(D) \). Similarly, the question of interval digraphs is studied in [3].

Define the interval number of a graph \( G \) as \( \chi(G) \). It is the intersection graph of a family of sets \( X \) of \( t \) pairwise disjoint closed intervals of \( \mathbb{R} \). The maximum degree of \( G \) is \( d \), then the interval number is \([d + 1]/n\). This inequality is tight if \( G \) is a complete graph. Scheinerman [100] also showed that there exists an absolute constant \( r \) such that the interval number of a graph with \( q \) edges is at most \([q/2] \). Kotzig and West showed that the interval number of a cycle is \([n/2] \). Scheinerman [100] showed that there exists an absolute constant \( r \) such that the interval number of a graph of genus \( \gamma \) is at most \( [2\gamma/3] \).

In [122], Trotter and Harary show that the bipartite graph \( K(m, n) \) is \((m + 1)/(m + n) \) and that the number of \( K(m, n) \) in \([m + n] \) is at least this large for any graph \( G \). The intersection number of a triangle with \( q \) edges is at least \([q + 1]/n\). This implies that if the intersection number of the graph is \( m \), then the representation form the intersection graph of a graph with at least \( m \) edges. This requires \( q \leq m - 1 \).
Interval Orders and Interval Graphs

For the theory, it is relatively easy to see that for any given graph, we may assume that the probability of $U(S)$ is the intersection of $[n]$ with some set $S$. But Trotter and Winkler showed that the event $U(S)$ depends only on the event $U(x, y)$, where $x = s_1$ and $y = s_{m+1}$. Therefore, the event $D(S)$ depends only on $s_m$ and $s_{2m}$. For some $j < m$, the event $D(S)$ depends only on $s_m$ and $s_{2m}$.

Let $H$ be any homogeneous subset of $S$ where $U(x, y) \cap H = \emptyset$ for any $x < y$ in $S$. Furthermore, if $U(x, y) \cap H = \emptyset$, then the homogenous set $H$ must have more than $2m$ elements.

Theorem 15.1: Interval orders are closely related to the graph of a subset of independent interest. Similarly, the study of interval orders has led to many new techniques and results. For example, the development of a general Ramsey theory for intervals.

If $S$ is a minimal example, from Theorem 15.1, it follows that $f(2) = \frac{1}{3}$, $f(3) \geq \frac{3}{8}$ and $f(k)$ is strictly increasing and converges.

The fractional dimension of a poset is an appealing concept that the fractional dimension of a poset in which at most $k$ others is at most $k + 1$. They give lower bounds linking fractional dimension with width and height. There are many challenging open questions in this area.

### Analogues for graphs

A steady stream of results providing higher lower bounds for graphs. Perhaps the first of these is due to Trotter and Winkler for the graph $G = (V, E)$ as the least $t$ for which any tree $T \subseteq G$ assigns to each vertex $x \in V$ a sequence of disjoint intervals of $\mathbb{R}$ so that $\bigcup \{x, y\} \in E$ if and only if $x \neq y$. Equivalently, the boxtivity of a graph is just the least $t$ so that the graph is the intersection of boxes in $\mathbb{R}^t$. So interval graphs are graphs with boxtivity one. Roberts showed that the boxtivity of a graph $G$ on $n$ vertices is at most $\lceil n/2 \rceil$, where $n \geq 2$. For example, the graph $H_n$, obtained by taking the complement of a matching on $n$ edges has $2n$ vertices and boxtivity $n$. In [127], Wittenshausen showed that for all $n \geq 1$, $H_n$ is the only graph with $2n$ vertices and boxtivity $n$.

However, when a graph has $2n + 1$ vertices and boxtivity $n$, the situation is modestly more complicated. For example, the cycle $C_5$ on $5$ vertices has boxtivity $2$. Also, the graph $W_n$ with vertex set $\{1, 2, \ldots, 7\}$ and edges joining $i$ to $i + 2$, $i + 3$, $i + 4$ and $i + 5$ (cyclically) has boxtivity $3$. In [113], Trotter showed that a graph $G$ on $2n + 1$ vertices has boxtivity $n$ if and only if one of the following conditions holds:

1. $G$ contains $H_n$.
2. $G_n$ contains the join of $C_5$ and $H_n$.
3. $G_n$ contains the join of $W_3$ and $H_n$.

In [109], Thomassen showed that the boxtivity of a planar graph is at most $3$; in fact, the boxes corresponding to adjacent vertices may be required to intersect on a face.

Many of the basic concepts for interval graphs have natural interpretation for digraphs. In [3], Beineke and Zambon introduced the notion (with different terminology) of an interval digraph. By this we mean that for each vertex $x$ in a digraph $D$, there are two intervals of the real line $R_x$ and $S_x$, so that $D$ contains a directed arc from $x$ to $y$ if and only if $R_x \cap S_y \neq \emptyset$. Structural questions for interval digraphs are studied in [86], [85], [105] and [106].

Define the interval number of a graph $G = (V, E)$ as the least $t$ for which $G$ is the intersection graph of a family of sets, with each set being the union of $t$ pairwise disjoint closed intervals of $\mathbb{R}$. In [53], Griggs and West show that if the maximum degree of $G$ is $d$, then the interval number of $G$ is at most $\lceil d/2 \rceil$. This inequality is tight if $G$ is triangle-free. Griggs and West also showed that there exists an absolute constant $c > 0$ so that the interval number of a graph with $q$ edges is at most $c \sqrt{q}$. In [103], Scheinerman and West showed that the interval number of a planar graph is at most $3$, and Scheinerman [100] showed that there exists an absolute constant $c > 0$ so that the interval number of a graph of genus $\gamma$ is at most $c' \sqrt{\gamma}$.

In [122], Trotter and Harary show that the interval number of a complete bipartite graph $K(m, n)$ is at least $\max(\min(m, n) + 1, m + n)$. The fact that the interval number of $K(m, n)$ is at least this large follows from the following elementary observation. The interval number of a triangle-free graph $G$ with $n$ vertices and $q$ edges is at least $\lceil (q + 1)/n \rceil$. This inequality follows from the fact that if the interval number of the graph is $t$, then the $t$ intervals used in a representation form the intersection graph of a forest on $nt$ vertices and at least $q$ edges. This requires $q \leq nt - 1$. 


Somewhat surprisingly, the determination of the interval numbers for complete multipartite graphs proved to be more challenging. The interval number of the complete multipartite graph $K(n_1, n_2, \ldots, n_s)$, with $n_1 \geq n_2 \geq \cdots \geq n_s \geq 2$, is at least as large as the interval number of $K(n_1, n_2)$. Call this quantity $t_0$. In [59], Hopkins, Trotter and West show the interval number of $K(n_1, n_2, \ldots, n_s)$ is at most $t_0 + 1$. Furthermore, they show that it is equal to $t_0$, except possibly for the two cases $(n_1, n_2) = (7, 5)$ and $n_1 = n_2^2 - n_2 - 1$. In both these exceptional cases, the interval number of $K(n_1, n_2, \ldots, n_s)$ may equal $t_0 + 1$, provided there are enough other parts of appropriate size.

Motivated by the formula for complete bipartite graphs, Trotter and Harary [122] conjectured that the maximum interval number of a graph on $n$ vertices is $\lceil (n + 1)/4 \rceil$. This conjecture was proved by Griggs in [52].

In [22], Chang and West introduced the concept of interval number for digraphs. For a digraph $D$, the interval number of $D$ is just the least positive integer $t$ for which there exists a function $F$ assigning to each vertex $x$ two subsets $R_x, S_x$ of the real numbers so that:

1. For each node $x$ in $D$, $R_x$ and $S_x$ are each the union of at most $t$ pairwise disjoint intervals of $\mathbb{R}$, and

2. $D$ contains an arc from $x$ to $y$ if and only if $R_x \cap S_y \neq \emptyset$.

Chang and West showed that the maximum interval number of a digraph on $n$ nodes is $\Theta(n/\log n)$. They also defined the concept of boxicity for digraphs and showed that the maximum boxicity of a digraph on $n$ nodes is $\lceil n/2 \rceil$.

Aigner and Andreea [1] introduced an interesting variation of interval number. For an graph $G = (V, E)$, they defined the total interval number of $G$ as the least positive integer $t$ for which there exists a function $F$ assigning to each vertex $x$ of $G$ a set $F(x)$ which is the union of $t_x$ pairwise disjoint closed intervals of $\mathbb{R}$ so that:

1. For every $x, y \in V$, $\{x, y\} \in E$ if and only if $F(x) \cap F(y) \neq \emptyset$, and

2. $\sum_{x \in X} t_x = t$.

Aigner and Andreea [1] produced upper bounds on total interval number for several classes of graphs. For example, they showed that the maximum total interval number of a tree on $n$ nodes is $\lceil (9n - 3)/4 \rceil$. In [80], Kratzie and West showed that the maximum total interval number of an outerplanar graph on $n$ nodes is $\lceil 3n/2 - 1 \rceil$ while the maximum total interval number of a general graph on $n$ nodes is $\lceil (n^2 + 1)/4 \rceil$. These results settled conjectures made by Aigner and Andreea in [1]. Other results on total interval number are given by Kostochka and West in [79]; in particular, they bound the total interval number in terms of the maximum degree, and characterize graphs for which the bound is sharp. The components of these graphs are balanced complete bipartite graphs.

Interval Orders and Interval Graphs

In [81], Kratzie and West provide a linear-time algorithm to determine the total interval number of a tree, and then extend this result to test whether the total interval number of a graph is at most a given number of edges, even for the class of triangle-free graphs.

Given a poset $P = (X, P)$ and points $x, y \in X$, the interval $[x, y]$ is the set $\{u \in X : x \leq u \leq y\}$. The interval number of a graph $G$ is the least $t$ for which there exists a total ordering of the vertices of $G$ such that each interval $[u, v]$ contains at most $t$ edges.

In [56], Gyárfás and West consider the interval number of a graph as the least $t$ for which the graph is an interval graph. They will discuss analogous concepts for posets in this paper.

17 Higher dimensional analogues

The investigation of higher dimensional analogues of the interval number has produced a steady stream of results. First, however, we consider analogues of the linear intervals which contains the linear orders. The interval number of a poset $P = (X, P)$ as the least $t$ for which $P$ is the interval graph of a graph $G$. The hereditary property serves to define the interval number of a graph $G$ as the least positive integer $t$ for which there exists a function $F$ assigning to each vertex $x$ of $G$ a set $F(x)$ which is the union of $t_x$ pairwise disjoint closed intervals of $\mathbb{R}$ so that:

Interval Orders and Interval Graphs

In [81], Kratzie and West provide a linear-time algorithm to determine the total interval number of a tree, and then extend this result to test whether the total interval number of a graph is at most a given number of edges, even for the class of triangle-free graphs.

Given a poset $P = (X, P)$ and points $x, y \in X$, the interval $[x, y]$ is the set $\{u \in X : x \leq u \leq y\}$. The interval number of a graph $G$ is the least $t$ for which there exists a total ordering of the vertices of $G$ such that each interval $[u, v]$ contains at most $t$ edges.

In [56], Gyárfás and West consider the interval number of a graph as the least $t$ for which the graph is an interval graph. They will discuss analogous concepts for posets in this paper.

17 Higher dimensional analogues

The investigation of higher dimensional analogues of the interval number has produced a steady stream of results. First, however, we consider analogues of the linear intervals which contains the linear orders. The interval number of a poset $P = (X, P)$ as the least $t$ for which $P$ is the interval graph of a graph $G$. The hereditary property serves to define the interval number of a graph $G$ as the least positive integer $t$ for which there exists a function $F$ assigning to each vertex $x$ of $G$ a set $F(x)$ which is the union of $t_x$ pairwise disjoint closed intervals of $\mathbb{R}$ so that:

Interval Orders and Interval Graphs

In [81], Kratzie and West provide a linear-time algorithm to determine the total interval number of a tree, and then extend this result to test whether the total interval number of a graph is at most a given number of edges, even for the class of triangle-free graphs.

Given a poset $P = (X, P)$ and points $x, y \in X$, the interval $[x, y]$ is the set $\{u \in X : x \leq u \leq y\}$. The interval number of a graph $G$ is the least $t$ for which there exists a total ordering of the vertices of $G$ such that each interval $[u, v]$ contains at most $t$ edges.

In [56], Gyárfás and West consider the interval number of a graph as the least $t$ for which the graph is an interval graph. They will discuss analogous concepts for posets in this paper.
In [81], Kratcke and West provide a linear time algorithm for computing the total interval number of a tree, and they show that it is NP-complete to test whether the total interval number of a graph is exactly one more than the number of edges, even for the class of triangle-free, 3-regular planar graphs.

Given a poset $P = (X, P)$ and points $x, y \in X$, with $x \leq y$ in $P$, the interval $[x, y]$ is just the set $\{u \in X : x \leq u \leq y \}$ in $P$. The poset boxicity of a graph $G$ is the least $t$ for which there exists a $t$-dimensional poset $P$ for which $G$ is the intersection graph of intervals in $P$. In [125], Trotter and West show that there exists an absolute constant $c > 0$ so that the poset boxicity of a graph $G$ on $n$ vertices is at most $c \log \log n$. They also show that there exist graphs with arbitrarily large poset boxicity.

In [56], Gyárfás and West consider the multitrack interval number of a graph as the least $t$ for which the graph is the union of $t$ interval orders. We will discuss analogous concepts for posets in Sections 17 and 19.

17 Higher dimensional analogues for orders

The investigation of higher dimensional analogues of interval orders has also produced a steady stream of results. First, let $\mathcal{P}$ be any hereditary class of orders which contains the linear orders. Then we can define the $\mathcal{P}$-dimension of a poset $P = (X, \preceq)$ as the least $t$ for which $P$ is the intersection of $t$ orders from $\mathcal{P}$. The hereditary property serves to ensure that the $\mathcal{P}$-dimension of $P$ is at most the $\mathcal{P}$-dimension of $Q$ when $P$ is contained in $Q$. Of course, the $\mathcal{P}$-dimension of $P$ is at most $\dim(P)$, and to emphasize the distinction between the original definition of dimension and variants discussed in the remainder of this paper, the dimension is also called the ordinary dimension.

In [14], Bogart and Trotter defined the interval dimension of a poset $P = (X, \preceq)$ as the least $t$ for which $P$ is the intersection of $t$ interval orders on $X$. So a poset has interval dimension 1 if and only if it is an interval order. Posets with interval dimension at most 2 have also been studied extensively. In [114], Trotter gave a forbidden subposet characterization of height two posets having interval dimension at most 2. This characterization results in a complete listing of all minimal posets of height 2 having interval dimension 3. Polynomial time recognition algorithms for posets having interval dimension at most 2 have been provided by several authors, but the best to date is due to Ma and Spinrad [86].

One of the most appealing aspects of interval dimension is the positive solution of the removable pair conjecture. For ordinary dimension, Trotter conjectured (see [118], for example) that if $P$ is a poset having three or more points, then there is always a pair of points whose removal decreases the dimension by at most 1. In fact, he conjectured that the removal of a critical pair always decreases the dimension by at most 1. Although the removable pair conjecture remains open, this second conjecture was disproved by Reuter [95], and an infinite family of counterexamples was then constructed by Kierstead...
and Trotter [73].

However, for interval dimension, we have the following elementary result.

**Theorem 17.1** Let \( P = (X, P) \) be a poset and let \((x, y) \in \text{crit}(P)\). If \( Q = (Y, Q) \) is the subposet determined by \( Y = X - \{x, y\} \), then the interval dimension of \( P \) is at most one more than the interval dimension of \( Q \).

**Proof** Let \( Q_1, Q_2, \ldots, Q_l \) be interval orders on \( Y \) whose intersection is \( Q \). For each \( i \in [l] \), let \( P_i \) be an interval order on \( X \) so that \( P_i(Y) = Q_i \). Then let \( L \) be any linear extension of \( Y \) with \( D(x) = Y - D(x) \) and \( Y - U(y) < U(y) \) in \( L \). Define a partial order \( P_{t+1} \) on \( X \) by setting \( P_{t+1} = P \cup L \). It is easy to see that \( P_{t+1} \) is an interval order and that \( P = P_1 \cap P_2 \cap \cdots \cap P_{t+1} \).

Another appealing aspect of the concept of interval dimension is that there is a relatively simple characterization of posets having maximal dimension for a given number of points (see Bogart and Trotter [13]), while the corresponding problem for ordinary dimension is considerably more difficult. Several other inequalities relating interval dimension to other combinatorial parameters are simpler than the corresponding results for ordinary dimension, e.g., compare the forbidden subposet characterization of the inequality \( \dim(P, X) \leq \max\{2, |X - A|\} \), when \( A \) is an antichain, for ordinary dimension [111] with the result for interval dimension in [13].

Other aspects of the interplay between dimension and interval dimension are discussed in [30]. In [57], Habib, Kelly and Möhring show that the property of a poset having interval dimension at most 2 is a comparability invariant, i.e., it depends only on the underlying comparability graph and not on the specific order.

Bogart and Trotter also defined the semi-order dimension of a poset and noted that if the semi-order dimension of \( P \) is \( t \), then the ordinary dimension of \( P \) is at most \( 3t \). This result is tight when \( t = 1 \), but it is not known whether it is best possible when \( t \geq 2 \). In [31], Felsner and Möhring show that the property of a poset having semi-order dimension at most 2 is a comparability invariant.

In a somewhat different direction, more closely connected to the concepts discussed in the preceding section, Madej and West [87] define the interval inclusion number of a poset \( P = (X, P) \) as the least integer \( t \) for which there exist a function \( F \) assigning to each \( x \in X \) a set \( F(x) \subseteq \mathbb{R} \) so that:

1. For each \( x \in X \), \( F(x) \) is the union of at most \( t \) pairwise disjoint closed intervals of \( \mathbb{R} \), and
2. For each \( x, y \in X \), \( x \leq y \) in \( P \) if and only if \( F(x) \subseteq F(y) \).

In [88], Madej and West show that “almost all” posets on \( n \) points have interval number \( o(n) \), but it is still not known whether there exists a positive real number \( c \) so that for all \( n \), there exists a poset on \( n \) points with interval inclusion number exceeding \( cn \). It is easy to construct an interval order of dimension \( n \) and West note in [88], the set of all subposets of \( P \), shows that this last inequality is tight for the \( n \)-dimensional standard example \( S_n \), having all \( n \geq 2 \).

### 18 Intervals, angles and spheres

Over the past 10 years, there has been a large surge of problems which arise when posets are represented (geometrically defined objects) ordered by inclusion orders, and they are the natural extension graphs. For example, as is well known, if and only if it is isomorphic to the inclusion intervals of the real line. Space limitations prevent this range of research on inclusion orders, but to provide a taste of which related directly to interval orders.

Fishburn and Trotter [40] define a poset \( P \) is the inclusion order of a family of sets, each of which has a common point. They show that every interval order determined by a family of sets has a common point. Several authors showed that there exists a 5-dimensional poset which is a angle order, but the most elegant proof of the “degrees of freedom” developed by Alon and Fishburn.

A \( d \)-sphere with center \( x \) and radius \( r \) is the set of all \( x \) at distance to \( x \) is at most \( r \). A \( 1 \)-sphere is just \( P \) a sphere order if there is some \( d \) so that for \( P \) an order determined by a family of \( d \)-spheres on \( Q \), we may define the sphere dimension of a poset \( d \) which \( P \) is the inclusion order of a family of sets having dimension 1 if and only if it has ordinary dimension \( 1 \).

The problem of determining whether every interval order determined by a family of \( d \)-spheres on \( Q \), we may define the sphere dimension of a poset \( d \) which \( P \) is the inclusion order of a family of sets having dimension 1 if and only if it has ordinary dimension \( 1 \).

The problem of determining whether every interval order determined by a family of \( d \)-spheres on \( Q \), we may define the sphere dimension of a poset \( d \) which \( P \) is the inclusion order of a family of sets having dimension 1 if and only if it has ordinary dimension \( 1 \).
we have the following elementary result.

Let $(x, y) \in \text{crit}(P)$. If $Y = X - \{x, y\}$, then the interval order is larger than the interval dimension of $Q$.

- **Theorem 18.1**: Let $(x, y) \in \text{crit}(P)$. If $Y = X - \{x, y\}$, then the interval order is larger than the interval dimension of $Q$.

- **Theorem 18.2**: Let $(x, y) \in \text{crit}(P)$. If $Y = X - \{x, y\}$, then the interval order is larger than the interval dimension of $Q$.

- **Theorem 18.3**: Let $(x, y) \in \text{crit}(P)$. If $Y = X - \{x, y\}$, then the interval order is larger than the interval dimension of $Q$.

- **Theorem 18.4**: Let $(x, y) \in \text{crit}(P)$. If $Y = X - \{x, y\}$, then the interval order is larger than the interval dimension of $Q$.

- **Theorem 18.5**: Let $(x, y) \in \text{crit}(P)$. If $Y = X - \{x, y\}$, then the interval order is larger than the interval dimension of $Q$.

We have the following result.

- **Theorem 18.6**: Let $(x, y) \in \text{crit}(P)$. If $Y = X - \{x, y\}$, then the interval order is larger than the interval dimension of $Q$.

18. **Intervals, angles and spheres**

Over the past 10 years, there has been a flurry of work on geometric problems which arise when posets are represented by a family of sets (usually some geometrically defined objects) ordered by inclusion. These structures are called inclusion orders, and they are the natural order theoretic analogue of intersection graphs. For example, as is well known, a poset has dimension at most 2 if and only if it is isomorphic to the inclusion order determined by a family of intervals of the real line. Space limitations do not allow us to discuss the full range of research on inclusion orders, but we will attempt to highlight those which related directly to interval orders.

Fishburn and Trotter [40] define a poset $P = (X, P)$ to be an angle order when $P$ is the inclusion order of a family of subsets of the euclidean plane, with each set being an angular region determined by two rays emanating from a common point. They show that every interval order is an angle order and that every poset with dimension at most 4 is an angle order. They also showed that there exists a 7-dimensional poset which is not an angle order. Subsequently, several authors showed that there exists a 5-dimensional poset which is not an angle order, but the best elegant proof of this fact results from the theory of “degrees of freedom” developed by Alon and Scheinerman in [2].

A $d$-sphere with center $x$ and radius $r$ is the set of all points in $\mathbb{R}^d$ whose distance to $x$ is at most $r$. A 1-sphere is just a closed interval of $\mathbb{R}$. Call a poset $P$ a sphere order if there is some $d$ so that $P$ is isomorphic to the inclusion order determined by a family of $d$-spheres in $\mathbb{R}^d$. When $P$ is a sphere order, we may define the sphere dimension of a poset $P = (X, P)$ as the least $d$ for which $P$ is the inclusion order of a family of $d$-spheres. So a poset has sphere dimension 1 if and only if it has ordinary dimension at most 2.

The problem of determining whether every finite poset is a sphere order is posed by Brightwell and Winkler in [21], and it is widely believed that the answer is negative.

When $d = 2$, there are some interesting results and one especially vexing problem. For historical reasons, posets with sphere dimension at most 2 are called circle orders, although it might have been more accurate to call them disk orders. The recent article [101] contains a number of interesting perspectives on the problem of representing order by circles in the plane. The range and extent of the connections with other combinatorial problems is most surprising.

In [37], Fishburn shows that every interval order is a circle order. Trivially,
every poset with ordinary dimension at most 2 is a circle order—in fact, we can require that the circles all have centers on a fixed line in the plane. By the Alon/Scheinerman theory, there exist 4-dimensional posets which are not circle orders. However, it is not known whether every finite 3-dimensional poset is a circle order. Scheinerman and Weirman [102] showed that the countably infinite 3-dimensional poset $\mathbb{N}^3$ is not a circle order. Subsequently, a somewhat shorter proof of this result was given by Hurlbert [60]. The sharpest result to date is due to Fon-der-Flaass [43] who showed that $2 \times 3 \times \mathbb{N}$ is not a sphere order, but that $2 \times 2 \times \mathbb{N}$ is a circle order.

On the other hand, it is an easy exercise to show that if $P$ is a finite poset with ordinary dimension at most 3 and $n \geq 3$, then $P$ is the inclusion order of a family of regular $n$-gons in the euclidean plane, and it is easy to suspect that when $n$ is quite large relative to $|X|$, these polygons are extremely close to being circles. However, I would conjecture that there is a finite 3-dimensional poset which is not a sphere order.

In this discussion, the metric used to determine distance plays a critical role. Of course, if $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$, then the ordinary distance from $x$ to $y$ is $\sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$. But if we change this to $\max\{|x_i - y_i| : 1 \leq i \leq d\}$, then a $d$-sphere is just a cube. Furthermore, it is an easy exercise to show that every poset with dimension at most $d+1$ is the inclusion order of a family of cubes in $\mathbb{R}^d$. Again, by the Alon/Scheinerman theory, this is best possible, meaning that there are $(d+2)$-dimensional posets which cannot be represented by cubes in $\mathbb{R}^d$ ordered by inclusion.

19 Tolerances, thresholds and gaps

In the preceding two sections, we discussed higher dimensional analogues for interval graphs and interval orders. In this section, we discuss generalizations which arise when just one interval is assigned but more complex rules are used to determine edges and comparabilities. Here is the basic motivation. If we have an indexed family $F = \{I(x) : x \in X\}$ of closed intervals with distinct end points, then an interval graph results when we define an edge set $E$ by $(x, y) \in E$ if and only if $|I(x) \cap I(y)| > 0$. From an applications standpoint, the problem with this definition is that we take two vertices to be adjacent when their intervals intersect regardless of how small this intersection might be. Similarly, an interval order assigns $x$ to be less than $y$ only when $F(x)$ lies entirely to the left on $F(y)$. But there are many scheduling problems where we want to consider one job as preceding another even when there is some overlap in time.

We begin with generalizations of interval graphs. Golumbic and Monma [49] proposed the following definition. Given an indexed family $F = \{I(x) : x \in X\}$ of closed intervals of $\mathbb{R}$ and a subset $T = \{t_x : x \in X\}$ of the non-negative real numbers $\mathbb{R}_0$, define the tolerance graph $G = G(F, T) = (X, E)$ by setting

$$E = \{(x, y) : x, y \in X, x \neq y \text{ and } |I(x) \cap I(y)| > 0\}.$$ 

It is easy to see that an interval graph $G$ has a distinguishing representation and give each of the distance between any two end points $u, v$. The complement of an interval graph is a tree, and if $x \in X$, set $t_x = |I(x)|$. A tolerance graph is complete for all $x \in X$. Golumbic and Monma [49] showed that the complement of a comparability graph is the complement of a comparability graph. This argument does not work for tolerance graphs in [50], Golumbic, Monma, and Trotter show perfect. The proof in the general case follows a tolerance graph is perfectly orderable, and graph being perfectly orderable [24] is a way to show that interval graphs are perfect.

Here is an interesting way in which tolerance graphs. Recall that an interval graph is propagation using only intervals of length 1. This is in [6], Bogart, Fishburn, Isaak and Langley tolerance graphs are strictly larger than the classical interval graphs have unit length.

In the last several years, a number of new definitions of interval graphs have been introduced. Perhaps the best known is McMorris and Mulder [61] who proposed to set $y$ to subintervals $\{I_x : x \in X\}$, a subset $T = \{t_x : x \in X\}$ of $\mathbb{R}_0$ of non-negative reals, and a function $\phi : X \rightarrow \mathbb{R}_0$ set $E$ to consist of all 2-element sets $\{x, y\}$. The original definition of a tolerance graph $G = G(F, T) = (X, E)$.

Now here are some of the new ideas for $\phi$ (see [8]) propose to study a generalization of the extra conditions are imposed on the gap between comparable pairs of points. The definition is $\{I(x) = [a_x, b_x] : x \in X\}$ of closed intervals of the non-negative reals $\mathbb{R}_0$, and a function $\phi : X \rightarrow \mathbb{R}$ define a relation $P$ on $X$ by setting $(x, y) \in E$ if and only if $b_y - a_x > \phi(t_x, t_y)$. We call these posets as $P$-tolerance graphs. The relation $P$ is defined in terms of the gap between the non-negative reals $\mathbb{R}_0$.

In certain cases, $P$ will be a partial order. There are always the case if $\phi$ satisfies the triangle inequality $\phi(t_z, t_x) + \phi(t_y, t_z) \geq \phi(t_y, t_x)$ for all $t_x, t_y, t_z \in \mathbb{R}_0$. In particular, $\phi(t_x, t_y) = |t_x - t_y|$ on the other hand, $\phi(t_x, t_y) = \phi(t_y, t_x)$. The special case is called a max-gap order.

In another direction, Bogart and Trench
Interval Orders and Interval Graphs

It is easy to see that an interval graph is a tolerance graph. Just take a distinguishing representation and give each vertex a tolerance smaller than the distance between any two end points used in the representation. Also, the complement of an interval graph is a tolerance graph. In this case, for each $x \in X$, set $t_x = |I(x)|$. A tolerance graph is bounded if $0 \leq t_x \leq |I(x)|$, for all $x \in X$. Golumbic and Monma [49] showed that a bounded tolerance graph is the complement of a comparability graph and is therefore perfect. This argument does not work for tolerance graphs which are not bounded, but in [50], Golumbic, Monma and Trotter showed that all tolerance graphs are perfect. The proof in the general case follows by showing that the complement of a tolerance graph is perfectly orderable. Note that Chvátal’s concept of a graph being perfectly orderable [24] is a weakening of the key property used to show that interval graphs are perfect.

Here is an interesting way in which tolerance graphs differ from interval graphs. Recall that an interval graph is proper if and only if it has a representation using only intervals of length 1. This is not true for tolerance graphs. In [6], Bogart, Fishburn, Isaak and Langley show that the class of proper tolerance graphs is strictly larger than the class of tolerance graphs in which all intervals have unit length.

In the last several years, a number of new concepts for generalizing tolerance graphs have been introduced. Perhaps the most general is due to Jacobson, McMorris and Mulder [61] who proposed to study graphs defined by a family of intervals $\{I_x : x \in X\}$, a subset $T = \{t_x : x \in X\}$ of tolerances drawn from the set $R^0$ of non-negative reals, and a function $\phi : R^0 \times R^0 \to R^0$ by setting the edge set $E$ to consist of all 2-element sets $\{x, y\}$ for which $|I(x) \cap I(y)| > \phi(t_x, t_y)$. The original definition of a tolerance graph is just the function $\phi(t_x, t_y) = \min\{t_x, t_y\}$.

Now here are some of the new ideas for posets. McMorris and Jacobson (see [8]) propose to study a generalization of interval orders in which extra conditions are imposed on the gaps between intervals corresponding to comparable pairs of points. The definition requires an indexed family $\{I(x) = [a_x, b_x] : x \in X\}$ of closed intervals of $R$, a subset $T = \{t_x : x \in X\}$ of the non-negative reals $R^0$, and a function $\phi : R^0 \times R^0 \to R^0$. We then define a relation $P$ on $X$ by setting $(x, y) \in P$ if and only if (1) $x = y$ or (2) $b_y - a_x > \phi(t_x, t_y)$. We call these posets $\phi$-gap orders to reflect that the relation $P$ is defined in terms of the gap between the two intervals.

In certain cases, $P$ will be a partial order on $X$. For example, this is always the case if $\phi$ satisfies the triangle inequality: $\phi(t_x, t_y) + \phi(t_y, t_z) \geq \phi(t_x, t_z)$, for all $t_x, t_y, t_z \in R^0$. In particular, $P$ is always a partial order if $\phi(t_x, t_y) = \max\{t_x, t_y\}$. On the other hand, we may fail to get a partial order if $\phi(t_x, t_y) = \min\{t_x, t_y\}$. The special case where $\phi(t_x, t_y) = \max\{t_x, t_y\}$ is called a max-gap order.

In another direction, Bogart and Trenk [12] call a poset $P = (X, P)$ a
bi-tolerance order when there exists a triple $(I, F, G)$ where:

1. $I$ assigns to each $x \in X$ a closed interval $I(x) = [a_x, b_x]$ of $\mathbb{R}$;
2. $F = \{ f_x : x \in X \} \subset \mathbb{R}$, $G = \{ g_x : x \in X \} \subset \mathbb{R}$;
3. $a_x \leq f_x, g_x \leq b_x$ in $\mathbb{R}$, for every $x \in X$;
4. $x < y$ in $P$ if and only if $b_x < f_y$ and $g_x < a_y$ in $\mathbb{R}$.

For a vertex $x$, the value $f_x - a_x$ is called the left tolerance of $x$, and the value $b_x - g_x$ is called the right tolerance of $x$. When $f_x - a_x = b_x - g_x$, for all $x \in X$, we call the poset a tolerance order. It is an easy exercise to show that the tolerance orders are just the posets which arise from ordering the complement of a bounded tolerance graph. For this reason, bi-tolerance orders were originally called bounded bi-tolerance orders, but with time the adjective "bounded" seems to have been discarded.

Every interval order is a bi-tolerance order with $f_x = a_x$ and $g_x = b_x$, for every vertex $x$, but it is easy to see that there are bi-tolerance orders which are not interval orders. It is an easy exercise to show that if $P = (X, P)$ is a max-gap order, as evidenced by the intervals $\{ [a_x, b_x] : x \in X \}$ and the subset $T = \{ t_x : x \in X \} \subset \mathbb{R}$, then $P$ is also a bi-tolerance order, as evidenced by the intervals $\{ [a_x - t_x, b_x + t_x] : x \in X \}$ and the families $F = \{ a_x : x \in X \}$ and $G = \{ b_x : x \in X \}$.

In the definition of a bi-tolerance order, no restriction is placed on the order of $f_x$ and $g_x$. However, if desired, we can always assume that $g_x \leq f_x$. This follows from the observation that if $M$ is a positive number, then we may modify the representation by setting:

1. $a_x' = f_x - M$, $b_x' = g_x - M$, $b_x = b_x + M$ and $f_x' = f_x + M$.

When $M$ is sufficiently large, we always have $g_x' \leq f_x'$. We say the representation $(I, F, G)$ of a bi-tolerance order is separated if $g_x < f_y$, for every $x, y \in X$. Note that if $M$ is very large, then the new representation is separated. In fact, if desired, one can assume that $\{ a_x : x \in X \} = \{ 1, 2, \ldots, n \}$ and $\{ b_x : x \in X \} = \{ n + 1, n + 2, \ldots, 2n \}$, where $n = |X|$. This last observation makes use of the fact that it is only the order on the various values that matters. On the other hand, the question as to which bi-tolerance orders have totally bounded representations is not as well understood, at least not for posets of arbitrary height.

The notion of separation makes it clear that for every $x \in X$, we have two intervals $I_1(x) = [a_x, g_x]$, and $I_2(x) = [f_x, b_x]$. Furthermore, the intervals $\{ I_1(x) : x \in X \}$ form an interval order $P_1$, and the intervals in $\{ I_2(x) : x \in X \}$ form another interval order $P_2$. Since $x, y \in P$ if and only if $I_1(x) < I_1(y)$ and $I_2(x) < I_2(y)$, it follows that the bi-tolerance orders are just the posets with interval dimension at most two.

Interval Orders and Interval Graphs

Another interesting translation of the concept involves the geometric insight gained when selected from two parallel lines in the plane by Dagan, Columbic and Pinter [25]. The two intervals is a trapezoid, and the trapezoids is called a trapezoid order. So the posets with interval dimension at most the argument given by Habib, Kelly and Romano [27] dimension 2 is a comparability invariant. Ryan show that the family of unit area trapezoids is the family of proper trapezoid orders. It provide fast algorithms for finding optimal chains and antichains. Their results even the elements of the poset, and they are able (but fixed) interval dimension.

Returning to Bogart and Trenk's def, should comment that their formulation has of interesting families of posets which can be obtained on the triple $(I, P, G)$. As before, we may to describe bi-tolerance orders (also, tolerance orders are incomparable under inclusion and having Fishburn [38], we say a bi-tolerance order if

\[ f_x = g_x = \frac{a_x + b_x}{2}, \forall x \in X. \]

Finally, a totally bounded if $f_x \leq g_x$, for all $x \in X$.

The main theorem of [12] asserts that on posets of height two. Here are some of the

**Theorem 19.1** Let $P$ be a poset of height two.

1. $P$ is a proper bi-tolerance order.
2. $P$ is a unit bi-tolerance order.
3. $P$ is a tolerance order.
4. $P$ is a unit tolerance order.
5. $P$ is a 50%-tolerance order.
6. $P$ is a totally bounded bi-tolerance order.

For posets of arbitrary height, the various equivalencies become more surprising. Recognizing unit and proper interval graphs. This distinguishes proper tolerance orders. However, for bi-tolerance result of Bogart and Isaak [7].
a triple \((I, F, G)\) where:

1. A closed interval \(I(x) = [a_x, b_x] \) of \(\mathbb{R}\);
2. \(\{g_x : x \in X\} \subset \mathbb{R}\);
3. Every \(x \in X\);
   - \(x < f_y\) and \(g_x < a_y\) in \(\mathbb{R}\).

\(x\) is called the \textit{left tolerance} of \(x\), and the \textit{value} of \(x\). When \(f_x - a_x = b_x - g_x\), for all \(x \in X\), it is an easy exercise to show that the posets which arise from ordering the \(f\)-axis graph. For this reason, \(x\)\(x\), \(y\)\(y\), \(z\)\(z\), and \(w\)\(w\) are bi-tolerance orders, but with time the adjective discarded.

The tolerance order with \(f_x = a_x\) and \(g_x = b_x\), for all \(x \in X\), is a tolerance order, no restriction is placed on the \(x\)\(x\), \(y\)\(y\), \(z\)\(z\), or \(w\)\(w\) in \(\mathbb{R}\). If \(P = (X, P)\) is a tolerance order, then the intervals \(\{a_x, b_x : x \in X\}\) and the subsets also a tolerance order, as evidenced by \(\{c : c \in X\}\) and the families \(F = \{a_x : x \in X\}\).

Returning to Bogart and Trenk's definition of a bi-tolerance order, we should comment that their formulation has prompted the study of a number of interesting families of posets which can be described in terms of restrictions on the triple \((I, F, G)\). As before, we may use the adjectives \textit{proper} and \textit{unit} to describe bi-tolerance orders (also, tolerance orders) in which the intervals are incomparable under inclusion and have unit length, respectively. Following Fishburn [38], we say a bi-tolerance order is \textit{split} when \(f_x = g_x\), for all \(x \in X\), and we say a split bi-tolerance order is a \textit{50% tolerance order} when \(f_x = g_x = (a_x + b_x)/2\), for all \(x \in X\). Finally, we say the bi-tolerance order is \textit{totally bounded} if \(f_x \leq g_x\), for all \(x \in X\).

The main theorem of [12] asserts that almost all these definitions coincide on posets of height two. Here are some of the equivalencies proved in [12].

**Theorem 19.1** Let \(P\) be a poset of height 2. Then the following statements are equivalent.

1. \(P\) is a proper bi-tolerance order.
2. \(P\) is a unit bi-tolerance order.
3. \(P\) is a tolerance order.
4. \(P\) is a unit tolerance order.
5. \(P\) is a 50%-tolerance order.
6. \(P\) is a totally bounded bi-tolerance order. \(\blacksquare\)

For posets of arbitrary height, the various classes begin to separate, and equivalencies become more surprising. Recall that there is a distinction between unit and proper interval graphs. This distinction also holds between unit and proper tolerance orders. However, for bi-tolerance orders, we have the following result of Bogart and Isaak [7].
Theorem 19.2 Let \( P \) be a poset. Then the following statements are equivalent.

1. \( P \) is a unit bi-tolerance order.
2. \( P \) is a proper bi-tolerance order.

Proof A distinguishing representation of a unit bi-tolerance order shows that it is also a proper bi-tolerance order. Now let \( (I, F, G) \) be a distinguishing representation which evidences that a poset \( P = (X, P) \) is a proper bi-tolerance order. Without loss of generality, we may assume this representation is separated; we may also assume that \( \{a_x : x \in X\} = \{1, 2, \ldots, n\} \) and \( \{b_x : x \in X\} = \{n + 1, n + 2, \ldots, 2n\} \), where \( |X| = n \). This is now a representation in which each interval has length \( n \).

The next equivalence, due to Langley [82], is somewhat more surprising.

Theorem 19.3 Let \( P \) be a poset. Then the following statements are equivalent.

1. \( P \) is a unit bi-tolerance order.
2. \( P \) is a split interval order.

Proof Let \( (I, F, G) \) be a distinguishing unit representation of a poset \( P = (X, P) \). Modify the representation as follows.

\[
a'_x = f_x; \quad f'_x = g'_x = b_x \quad \text{and} \quad b'_x = g_x + 1. \tag{16}
\]

It follows easily that

\[
b_x < f_y \quad \text{and} \quad g_x < a_y \quad \text{if and only if} \quad b'_x < f'_y \quad \text{and} \quad g'_x < a'_y. \tag{17}
\]

This transformation is easily seen to be reversible.

Note that an interval order is also a split interval order. To see this, just consider a distinguishing representation of an interval order and set \( f_x = g_x = a_x \). On the other hand, there are split interval orders which are not interval orders, e.g. \( 2 + 2 \).

Bogart, Fishburn, Isaak and Langley prove the following equivalency in [6], which is now an immediate corollary to Theorem 19.3.

Corollary 19.4 Let \( P \) be a poset. Then the following statements are equivalent.

1. \( P \) is a unit tolerance order.
2. \( P \) is a 50% tolerance order.

Interval Orders and Interval Graphs

Although the family of tolerance orders therefore contains posets of arbitrarily large height, for many of the families involving restrictions, this is not the case. For example, here is a very recent result of Fishburn and Kierstead.

Theorem 19.5 If \( P \) is a split semi-order, then it has height \( h \).

Bi-tolerance orders must have large height, and it would be interesting to determine \( h(P) \) for a bi-tolerance order of height \( h \).

Acknowledgements

In the preparation of this article, the authors have benefited from conversations and communications with valued colleagues, Peter Fishburn and Hal Kierstead, and heartily appreciate their assistance.

This research is supported in part by the National Science Foundation.

References


Then the following statements are equivalent.

A splitting unit representation of a poset $P = (X, P)$ follows.

$$b'_x = g_x + 1.$$  

(16)

And only if $b'_x < f'_y$ and $g'_x < a'_y$,

(17)

to be reversible.  

so a split interval order. To see this, just

Then the following statements are equivalent.

Although the family of tolerance orders includes the interval orders and therefore contains posets of arbitrarily large dimension, this is not the case for many of the families involving restrictions on lengths and tolerances. For example, here is a very recent result of Fishburn and Trotter [42].

**Theorem 19.5** If $P$ is a split semi-order, then $\dim(P) \leq 6$.  

Bi-tolerance orders must have large height in order to have large dimension, and it would be interesting to determine (or estimate) the maximum value of a bi-tolerance order of height $h$.

**Acknowledgements**

In the preparation of this article, the author has benefited from many conversations and communications with valued colleagues, especially Ken Bogart, Peter Fishburn and Hal Kierstead, and he would like to express his deep appreciation for their assistance.

This research is supported in part by the Office of Naval Research.

**References**


[34] S. Felsner & W. T. Trotter, Coloring $\alpha$-sequences of sets, Discrete Mathematics.


[37] P. C. Fishburn, Interval orders and circuits, 234.

[38] P. C. Fishburn, Generalizations of semiorders, in press.

Interval Orders and Interval Graphs


[31] S. Felsner & R. Möhring, Note: Semi-order dimension two is a comparability invariant, in press.


[38] P. C. Fishburn, Generalizations of semiorders: A review note, in press.


[51] G. Hurlbert, A short proof that $\mathbb{N}^2$ is dense.


Interval Orders and Interval Graphs


[67] H. A. Kierstead, Coloring graphs on-line, in press.


Interval Orders and Interval Graphs


Interval Orders and Interval Graphs


Approximate Counting

Dominic Welsh

Summary I shall survey a range of very basic problems which have a unifying geometrical theme. For many of these problems, it is #P-hard but there is no obvious obstruction to finding an approximation algorithm. I shall outline the methods used to find such algorithms and describe what is currently known. In most cases, the methods work by approximating a part of the input so there remain many open questions.

1 Introduction

Consider the following problem.

How many different $4 \times 4$ matrices have row sums $1000, 9000, 3000$ and column sums $2000, 4000, 2000$?

The answer is approximately $10^{23}$, and it was found in 2 seconds real time on a SUN 3/50.

However had I posed a similar question in $d = 10$, it would be beyond the scope of everyday computer science.

A second problem for which I have thought about (and found too slow) is the following.

How does one generate a random planar graph? By “random” I mean uniformly at random, as in [15], where I shall discuss a range of problems. One common theme in geometric combinatorics is

---

1The integers were chosen by me on March 22, 1996.

Mount had running on the machines at MSRI, Berkeley.