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Planar Posets, Dimension, Breadth and the Number of Minimal Elements

William T. Trotter[1] **· Ruidong Wang**[1]

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Abstract In recent years, researchers have shown renewed interest in combinatorial properties of posets determined by geometric properties of its order diagram and topological properties of its cover graph. In most cases, the roots for the problems being studied today can be traced back to the 1970's, and sometimes even earlier. In this paper, we study the problem of bounding the dimension of a planar poset in terms of the number of minimal elements, where the starting point is the 1977 theorem of Trotter and Moore asserting that the dimension of a planar poset with a single minimal element is at most 3. By carefully analyzing and then refining the details of this argument, we are able to show that the dimension of a planar poset with t minimal elements is at most $2t + 1$. This bound is tight for $t = 1$ and $t = 2$. But for $t \geq 3$, we are only able to show that there exist planar posets with t minimal elements having dimension $t+3$. Our lower bound construction can be modified in ways that have immediate connections to the following challenging conjecture: For every $d \geq 2$, there is an integer $f(d)$ so that if P is a planar poset with $\dim(P) \geq f(d)$, then P contains a standard example of dimension d. To date, the best known examples only showed that the function f, if it exists, satisfies $f(d) \geq d + 2$. Here, we show that $\lim_{d\to\infty} f(d)/d \geq 2$.

Keywords Planar poset · Dimension

1 Introduction

We assume that the reader is familiar with basic notation and terminology for partially ordered sets (here we use the short term poset), including: minimal and maximal elements;

✉ William T. Trotter
trotter@math.gatech.edu

[1] School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

order diagrams; cover graphs and linear extensions. Extensive background information on the combinatorics of posets can be found in Trotter's survey article [28] and research monograph [27].

A family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of a poset P is called a *realizer* of P if $x \leq y$ in P if and only if $x \leq y$ in L_j for each $j = 1, 2, \ldots, t$. The *dimension* of P, denoted $\dim(P)$, is then the least positive integer d for which P has a realizer of size d.

A poset P is *planar* if its order diagram can be drawn without edge crossings in the plane. Of course, a poset can be non-planar even when its cover graph is planar, and we illustrate such a poset and its planar cover graph in Fig. 1.

For a positive integer t, let $M(t)$ be the maximum value of the dimension of a planar poset with t minimal elements. In this paper, our main result will be that the function $M(t)$ is well defined and satisfies the following inequality.

Theorem 1.1. *For every $t \geq 1$, $M(t) \leq 2t + 1$, i.e., if P is a planar poset and P has t minimal elements, then* $\dim(P) \leq 2t + 1$.

From below, we will show that the inequality in Theorem1.1 is tight for $t = 1$ and $t = 2$. But for $t \geq 3$, we are only able to produce the following lower bound.

Theorem 1.2. *For every $t \geq 3$, $M(t) \geq t + 3$, i.e., there is a planar poset P with t minimal elements such that* $\dim(P) = t + 3$.

The remainder of this paper is organized as follows. In the next section, we present a brief discussion of background material which serves to motivate this line of research and puts our results in historical perspective. Section 3 contains a compact summary of preliminary material necessary for our proofs. In Section 4, we present the proof of our main theorem, and in Section 5, we provide three constructions. The first establishes the lower bound in Theorem 1.2. The second and third are modifications of the first and are related to the conjecture that a planar poset with large dimension contains a large standard example. We close in Section 6 with some comments on some challenging problems which remain.

2 Background Discussion

Many combinatorial parameters for posets are *comparability invariants*, meaning that the parameter is the same for two posets with the same comparability graph. Among such parameters are height, width, dimension, and the number of linear extensions.

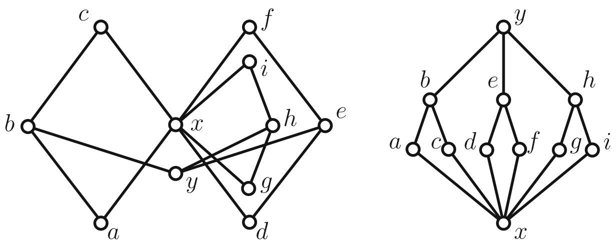

Fig. 1 A non-planar poset with a planar cover graph

On the other hand, very few parameters of a poset are determined by the cover graph. For example, a path of size n is the cover graph of a chain which has height n, width 1, dimension 1 and has only one linear extension. On the other hand, it is also the cover graph of a height 2 poset known as a *fence*, which has width $\lceil n/2 \rceil$, dimension 2 and exponentially many linear extensions. Later in this paper, we will show that for every $d \geq 2$, there are two posets P_1 and P_2, both having the same cover graph, with $\dim(P_1) = d$ and $\dim(P_2) = 2$.

For these reasons, it is understandable that comparability graphs have been studied much more extensively than cover graphs. However, in the last five years, there has been a surge of interest in cover graphs and order diagrams. To understand these recent developments, we first comment that comparability graphs are perfect graphs, and there is a polynomial time algorithm for recognizing them. So in modern terms, comparability graphs are well understood discrete structures.

By way of contrast, in [22] and [5], it is shown that it is NP-complete to determine whether a graph is a cover graph. Also, while there are fast algorithms [12] for testing whether a graph is planar, it is NP-complete to determine whether a poset is planar [11]. With only these remarks in mind, it is reasonable to conclude that cover graphs and diagrams are somewhat more subtle concepts and indeed are worthy of further study.

2.1 Standard Examples

For $d \geq 2$, the *standard example* S_d is the height 2 poset consisting of d minimal elements $a_1, a_2, \ldots, a_d$ and d maximal elements $b_1, b_2, \ldots, b_d$ with $a_i < b_j$ in S_d if and only if $i \neq j$. As is well known, $\dim(S_d) = d$ for all $d \geq 2$. Furthermore, for $d \geq 3$, S_d is *irreducible*, i.e., the removal of any point leaves a subposet whose dimension is $d - 1$.

Trivially, a poset containing a large standard example has large dimension, but the converse need not be true. First, we note that the class of interval orders is just the set of posets which do not contain S_2 as a subposet [9]. Interval orders can have arbitrarily large dimension, but to do so, they must have large height (see [10] and [19] for additional material on the dimension of interval orders). On the other hand, as noted in [6], for each pair (g, d) of positive integers, there is a height 2 poset so that the girth of the cover graph of P is at least g, while the dimension of P is at least d. Such posets arise as the adjacency posets of graphs with large girth and large chromatic number (see [7] for additional information on the dimension of adjacency posets). In general, these posets contain S_2, but they do not contain S_3.

2.2 Zeroes and Ones

If a poset has a unique minimal element, it is traditional to refer to this element as a *zero*. Similarly, a unique maximal element is called a *one*. If P has a zero, and $|P| \geq 2$, then the dimension of P is the same as the dimension of the subposet of P obtained when the zero is removed. So in some sense, the issue as to whether a poset has a zero is irrelevant when it comes to determining its dimension. An analogous remark applies to ones.

But there are exceptions to these blanket statements. The following 1971 result is due to Baker, Fishburn and Roberts [2].

Theorem 2.1. *If P is a planar poset and P has both a zero and a one, then* $\dim(P) \leq 2$.

In 1977, Trotter and Moore [30] extended this result as follows.

Theorem 2.2. *If P is a planar poset and P has a zero, then* $\dim(P) \leq 3$.

In Fig. 2, we show three posets which illustrate that the inequality in Theorem 2.2 is tight. There are many other such examples.

In Fig. 3, we show two drawings of the standard example S_4. The drawing on the left follows the natural convention for height two posets, resulting in a drawing with many crossings. On the other hand, the drawing on the right has no crossings and shows that S_4 is actually a planar poset. This elementary fact has been known for more than thirty years; and it has also been known that S_d is non-planar[1] for all $d \geq 5$.

In the 1970's, some researchers (including the first author on this paper) were hopeful that it might be true that $\dim(P) \leq 4$, whenever P is planar. This hope was dashed in 1981 by D. Kelly [16] who gave a construction which shows that for every $d \geq 5$, the non-planar poset S_d is a subposet of a planar poset. We illustrate Kelly's construction for the specific case $d = 6$ in Fig. 4. Modifying the drawing for larger values of d is straightforward.

Kelly's construction brought research on the dimension of planar posets to a halt. In retrospect, it should have prompted the following natural questions, all of which lay dormant for more than 30 years.

Question 2.3. If P is a planar poset and the dimension of P is large, which if any of the following three statements must be true?

1. P has large height.
2. P has many minimal elements.
3. P contains a large standard example.

Ironically, the first of these three questions came up fairly recently, but in a rather indirect manner. In 2009, Felsner, Li and Trotter [7] investigated the dimension of the adjacency posets of various classes of graphs. As a relatively minor by-product of this work, they noted that if P is a poset of height 2 and the cover graph of P is planar, then $\dim(P) \leq 4$. This led them to conjecture the following far more substantive result, which was subsequently proved in 2012 by Streib and Trotter [23].

Theorem 2.4. *For every $h \geq 1$, there is a least positive integer c_h so that if P is a poset of height h and the cover graph of P is planar, then* $\dim(P) \leq c_h$.

Trivially, $c_1 = 2$, while the results in [7] show that $c_2 = 4$. However, for $h \geq 3$, the existence of c_h is apparently very difficult to establish, and the bounds produced by the proof given in [23] are quite generous, since Ramsey theoretic techniques are used at several key steps. On the other hand, Kelly's example shows that $c_h = \Omega(h)$.

It should be noted that there is no analogue of Theorem 2.1 for cover graphs, as for every $d \geq 1$, there is a d-dimensional poset P with a zero and a one such that the cover graph of P is planar (see Trotter [25] and Streib and Trotter [23] for constructions).

Finally, we return to the issue of posets having the same cover graph but markedly different values of dimension. In Fig. 5, we show two posets with the same cover graph. The poset on the left is Kelly's construction with four additional points (they are darkened), so

[1] It is an easy exercise to show that the cover graph of S_d contains a homeomorph of $K_{3,3}$ when $d \geq 5$.

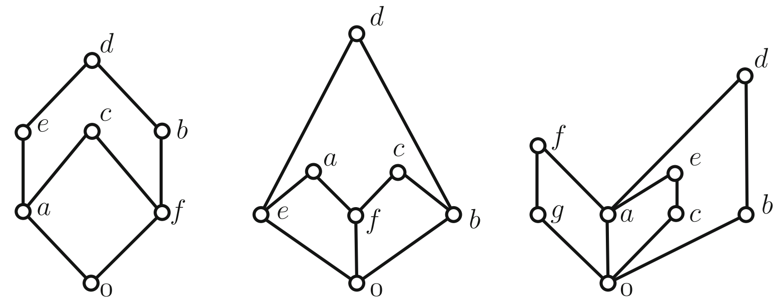

Fig. 2 Some 3-dimensional planar posets with a zero

this poset can have arbitrarily large dimension. On the other hand, the poset on the right is planar with a zero and a one, so it has dimension 2.

2.3 Tree-width and Topological Graph Theory

Quite recently, it was noted that there is an important connection between poset dimension, cover graphs, and topological graph theory. This connection has its roots in [30] where the following result is obtained as a relatively straightforward corollary to Theorem 2.2.

Theorem 2.5. *If P is a poset and the cover graph of P is a tree, then* $\dim(P) \leq 3$.

Of course, as presented in [30], Theorem 2.5 follows from the easily established fact that if P is a poset whose cover graph is a tree, then the poset obtained by adding a zero to P is planar. However, Theorem 2.5 has a strange history and was in fact proved before Theorem 2.2. The original motivation was to investigate what happens to the dimension of a poset when the Kimble split operation is applied repeatedly (see [26] for a full discussion). Moreover, Theorem 2.5 was first proved in the special case where the tree has height two. It was then extended to the case where the tree has arbitrary height. Finally, it was recognized that the argument actually proved the stronger inequality given in Theorem 2.2, with Theorem 2.5 then following as a corollary. Regardless, Theorem 2.5 is apparently the first instance

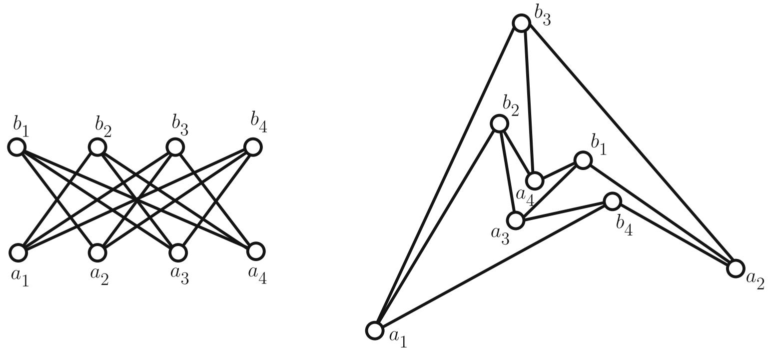

Fig. 3 A planar 4-dimensional poset

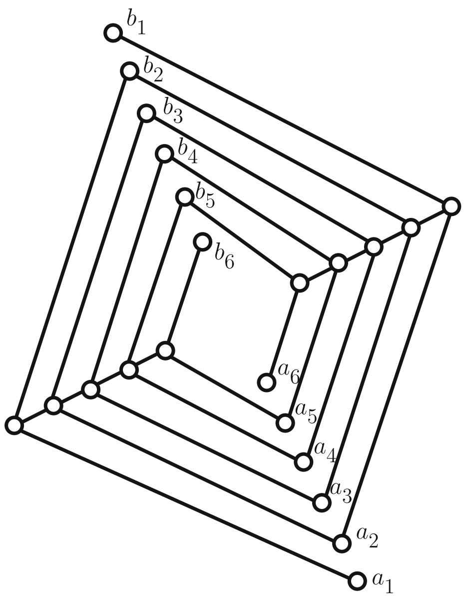

Fig. 4 Planar posets with arbitrarily large dimension

of a result where the dimension of a poset is bounded in terms of properties of the cover graph—not the order diagram.

In the language of tree-width, Theorem 2.5 would be restated as follows.

Theorem 2.6. *If P is a poset and the tree-width of the cover graph of P is 1, then* $\dim(P) \leq 3$.

G. Joret noted that the cover graphs of the posets given in Kelly's construction as illustrated in Fig. 4 have small tree-width (in fact, they have path-width at most 3). Joret also noted that Felsner, Trotter and Wiechert [8] had proved that if the cover graph of a poset P is outerplanar, then $\dim(P) \leq 4$. This result is significant because outerplanar graphs have tree-width at most 2. Finally, Joret recognized the importance of a key technical detail in

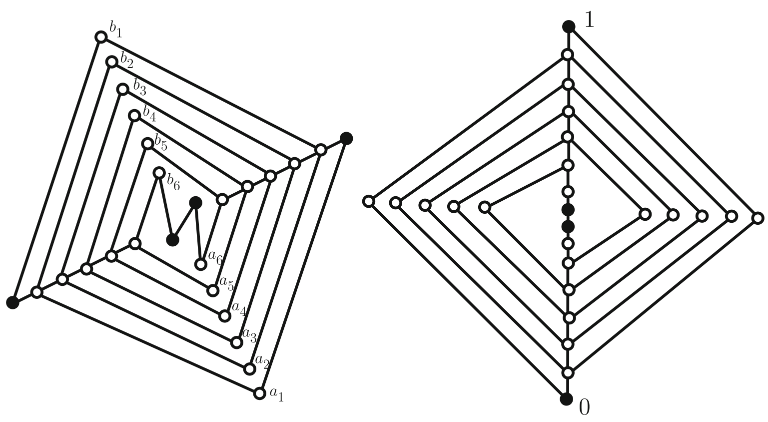

Fig. 5 Different Dimension - Same Cover Graph

the argument given by Streib and Trotter in [23], where it is shown that it is enough to prove the result for posets whose cover graphs are planar and have bounded diameter (in terms of the height). This detail is significant because planar graphs of bounded diameter have bounded tree-width. Together, these observations led Joret to conjecture the following result, subsequently proved in 2012 by Joret, Micek, Milans, Trotter, Walczak and Wang [13].

Theorem 2.7. *For every pair* (t, h) *of positive integers, there is a least positive integer* $d = d(t, h)$ *so that if* P *is a poset of height* h *and the cover graph of* P *has tree-width at most* t*, then* $\dim(P) \leq d$.

Evidently, in view of Theorem 2.5, we know that $d(1, h) \leq 3$ for all h, while Kelly's construction shows that $d(t, h)$ must go to infinity with h for all $t \geq 3$. It took a couple of years to settle whether $d(2, h)$ is bounded or goes to infinity with h. First, Biró, Keller and Young [4] settled the question for path-width with the following result.

Theorem 2.8. *If* P *is a poset whose cover graph has path-width at most 2, then* $\dim(P) \leq 17$.

Subsequently, the full question for tree-width was resolved by Joret, Micek, Trotter, Wang and Wiechert [14].

Theorem 2.9. *If* P *is a poset whose cover graph has tree-width at most 2, then* $\dim(P) \leq 1276$.

Moreover, quite recently, Walczak [31] has proved a sweeping extension of Theorem 2.7 with the following result.

Theorem 2.10. *For every pair* (G, h) *where* G *is a graph and* h *is a positive integer, there is an integer* $d = d(G, h)$ *so that if* P *is a poset of height* h *and the cover graph of* P *does not contain* G *as a minor, then* $\dim(P) \leq d$.

Walczak's proof makes use of advanced topics in topological graph theory, in particular the Robertson-Seymour and Grohe-Marx graph structure theorems. However, even more recently, Micek and Wiechert [21] have given a surprisingly short proof of Theorem 2.10, which is entirely combinatorial in spirit.

The reader may note that Theorems 2.5, 2.8, 2.9 and 2.10 are statements about cover graphs, while Theorem 2.1, 2.2 and our main theorem are statements about order diagrams, i.e., the gravity restrictions associated with a legal drawing of an order diagram play a key role.

3 Essential Preliminary Material

We include here a brief summary on the concepts of reversible sets of incomparable pairs and alternating cycles. Both have their origins in [30], but they have been refined and polished[2] in several recent papers, including [8], [23] and [13].

[2]Readers who are familiar with recent research in dimension theory may note that we do not need the concepts of strict alternating cycles or critical pairs, both of which have been used extensively in the literature.

Let P be a poset with ground set X. Then let $\text{Inc}(P)$ denote the set of all ordered pairs $(x, y) \in X \times X$ where x is incomparable to y in P. The binary relation $\text{Inc}(P)$ is of course symmetric, and it is empty when P is a total order.

A subset $\mathcal{S} \subset \text{Inc}(P)$ is *reversible* when there is a linear extension L of P so that $x > y$ in L for all $(x, y) \in \mathcal{S}$. When $\text{Inc}(P) \neq \emptyset$, the dimension of P is then the least positive integer d for which there is a covering:

$$\text{Inc}(P) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_d,$$

where $\mathcal{S}_j$ is reversible, for each $j = 1, 2, \ldots, d$.

When $k \geq 2$, an indexed subset $\{(x_i, y_i) : 1 \leq i \leq k\} \subseteq \text{Inc}(P)$ is called an *alternating cycle* of length k when $x_i \leq y_{i+1}$ in P, for all $i = 1, 2, \ldots, k$ (here, subscripts are interpreted cyclically so that $x_k \leq y_1$ in P). In [30], the following elementary result is proved.

Lemma 3.1. *Let P be a poset and let $\mathcal{S} \subseteq \text{Inc}(P)$. Then $\mathcal{S}$ is reversible if and only if $\mathcal{S}$ does not contain an alternating cycle.*

Recall that a subposet U of a poset P is called an *up set* in P when $y \in U$ whenever $x \in U$ and $y > x$ in P. Our proofs will use the following nearly self-evident proposition.

Proposition 3.2. *Let U be a non-empty up set in a poset P and let M be a linear extension of U. Then there is a linear extension L of P so that (1) the restriction of L to U is M and (2) $v < u$ in L if $u \in U$ and $v \in P - U$.*

Finally, we comment that in defining realizers, we allow repetition, so a poset of dimension d has a realizer of size t for every $t \geq d$.

4 Proofs of Upper Bounds

As the key ideas are essential for the proof of the improved upper bound in Theorem 1.1, we present a proof with updated notation and terminology for Theorem 2.2, i.e., we show that $\dim(P) \leq 3$ when P is planar and has a zero.

Let P be a planar poset with a zero, which we denote by 0. Fix a drawing with no edge crossings of the order diagram of P. Also, we impose a standard Euclidean coordinate system on the plane. Accordingly, for each $x \in P$, there is a pair $(\pi_1(x), \pi_2(x))$ of real numbers determined by the projections of the point in the diagram corresponding to x onto the two coordinate axes (we take the first coordinate axis to be horizontal while the second is vertical). Note that $\pi_2(x) < \pi_2(y)$ whenever $x < y$ in P.

For each element x of P, let $A(x)$ denote the set of all elements of P which cover x. We use the letter A to suggest "above". When $A(x) \neq \emptyset$, the drawing determines a natural "left-to-right" ordering on $A(x)$. Let $|P| = n$, and let $L_1 = \{x_1, x_2, \ldots, x_n\}$ be the unique linear extension of P obtained by carrying out the following depth-first search (depth-first linear extensions are also studied in [17], [18] and [20]).

(1) $x_1 = 0$, i.e., x_1 is the zero of P. Set $P_1 = \{x_1\}$.
(2) If $1 \leq i < n$ and the points in $P_i = \{x_1, x_2, \ldots, x_i\}$ have been determined, let S_i denote the set of minimal elements of $P - P_i$. Let j be the largest integer such that $1 \leq j \leq i$ and $A(x_j) \cap S_i \neq \emptyset$. Then x_{i+1} is the left most element in $A(x_j) \cap S_i$.

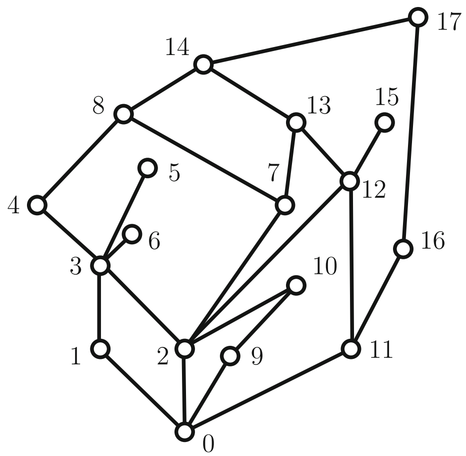

Fig. 6 A Linear Extension by Depth-First Search

We illustrate this definition with the poset shown in Fig. 6, where the points have been labelled with non-negative integers according to the linear extension L_1. Using symmetry, we may also define a second linear extension, denoted L_2, by taking the "right-to-left" ordering on each $A(x)$. For this poset, the resulting linear extension is:

$$L_2 = \{0, 11, 16, 9, 2, 10, 12, 15, 7, 13, 1, 3, 6, 5, 4, 8, 14, 17\}.$$

If x and y are incomparable points in P with $x < y$ in L_1 and $y < x$ in L_2, it is natural to say that x is *left of* y. Referring to the poset shown in Fig. 6, we see that 5 is left of 13. We will denote by $\mathcal{L}$ the set of all $(x, y) \in \mathrm{Inc}(P)$ with x left of y. Note that it is possible for x to be to the left of y when $\pi_1(x) > \pi_1(y)$. In Fig. 6, 7 is left of 9 even though $\pi_1(7) > \pi_1(9)$.

Dually, we say that x is *right of* y when y is left of x, and we let $\mathcal{R}$ denote the set of all $(x, y) \in \mathrm{Inc}(P)$ with x right of y. Analogously, we say that x is *inside* y when $x < y$ in L_1 and in L_2. For example, in Fig. 6, 6 is inside 14. Also, we let $\mathcal{I}$ denote the set of all $(x, y) \in \mathrm{Inc}(P)$ with x inside y. Note that when x is inside y, x is in the interior of a simple closed curve C consisting of two paths in the diagram, with all points on both paths less than or equal to y in P. Again referring to Fig. 6, one choice for the curve C is formed by the paths: $(2 < 3 < 4 < 8)$ and $(2 < 7 < 8)$ with $8 < 14$ in P (in general, there can be many choices for the curve C). Dually, we say that x is *outside* y when y is inside x, and we let $\mathcal{O}$ denote the set of all $(x, y) \in \mathrm{Inc}(P)$ with x outside y.

Here is an important and nearly self-evident property of pairs in $\mathcal{I}$.

Proposition 4.1. *Let* $(x, y) \in \mathcal{I}$. *If* $y' \geq x$ *and* $y' \parallel y$ *in* P, *then* $\pi_2(y') < \pi_2(y)$.

Proof When x is inside y, the point x is in the interior of a simple closed curve C whose boundary consists of two paths in the diagram, with all points on both paths less than or equal to y in P. So the point y' must also be in the interior of this curve. This forces $\pi_2(y') < \pi(y)$. □

We note that if $(x, y) \in \mathcal{R} \cup \mathcal{O}$, then (x, y) is reversed in L_1, and if $(x, y) \in \mathcal{L} \cup \mathcal{O}$, then (x, y) is reversed in L_2. So to complete the proof that $\dim(P) \leq 3$, we need only show that $\mathcal{I}$ is reversible.

Lemma 4.2. *The set $\mathcal{I}$ is reversible.*

Proof Suppose to the contrary that $\mathcal{I}$ is not reversible. Choose an integer $k \geq 2$ and an alternating cycle $\{(x_i, y_i) : 1 \leq i \leq k\}$ contained in $\mathcal{I}$. Since each $(x_i, y_i) \in \mathcal{I}$ and $x_i \leq y_{i+1}$, we know from Proposition 4.1 that $\pi_2(y_{i+1}) < \pi_2(y_i)$. But this statement cannot hold for all i. □

The reader may note that the conclusion that $\dim(P) \leq 2$ when P is planar and has both a zero and a one follows as an immediate corollary, since in this case, $\mathcal{I} = \mathcal{O} = \emptyset$, This implies that $\mathrm{Inc}(P) = \mathcal{R} \cup \mathcal{L}$ is a covering using two reversible sets.

4.1 Proof of the Main Theorem

We fix a positive integer $t \geq 2$ and a planar poset P with t minimal elements and show that $\dim(P) \leq 2t + 1$. The minimal elements of P are denoted by $m_1, m_2, \ldots, m_t$. As before, we choose a drawing without edge crossings of the order diagram of P. We will then show that $\dim(P) \leq 2t + 1$.

We start by letting $\mathcal{S}_0$ denote the set of all pairs $(x, y) \in \mathrm{Inc}(P)$ satisfying the following requirement: If $y' \geq x$ in P, then $\pi_2(y') < \pi_2(y)$. The proof given above showing that $\mathcal{I}$ is reversible actually yields the following modestly stronger result:

Lemma 4.3. *The set $\mathcal{S}_0$ is reversible.*

So there is a linear extension L_0 of P reversing all pairs in $\mathcal{S}_0$.

Next, for each $j = 1, 2, \ldots, t$, the subposet $U[m_j] = \{u \in X : u \geq m_j \text{ in } P\}$ is planar with a zero. Furthermore, $U[m_j]$ is an up set in P. For the subposet $U[m_j]$, the sets $\mathcal{L}_j$, $\mathcal{R}_j$, $\mathcal{I}_j$ and $\mathcal{O}_j$ of incomparable pairs in $U[m_j]$ are defined as in the argument for Theorem 2.2 which we have just presented. It follows that for each $j = 1, 2, \ldots, t$, there is a pair M_{2j-1}, M_{2j} of linear extensions of $U[m_j]$ so that M_{2j-1} reverses all pairs in $\mathcal{R}_j$ while M_{2j} reverses all pairs in $\mathcal{L}_j \cup \mathcal{O}_j$. Using Proposition 3.2, let L_{2j-1} and L_{2j} be any two linear extensions of P so that (1) for each $\alpha = 2j - 1, 2j$, the restriction of L_α to $U[m_j]$ is M_α and (2) $x > y$ in L_α if $x \in U[m_j]$ and $y \in P - U[m_j]$.

We claim that $\mathcal{R} = \{L_0, L_1, \ldots, L_{2t}\}$ is a realizer of P. To see this, let $(x, y) \in \mathrm{Inc}(P)$. Choose an integer j so that $x \in U[m_j]$ in P. If $y \notin U[m_j]$, then $x > y$ in both L_{2j-1} and in L_{2j}. So we may assume that $y \in U[m_j]$. If $(x, y) \in \mathcal{R}_j$, then $x > y$ in L_{2j-1}. If $(x, y) \in \mathcal{L}_j \cup \mathcal{O}_j$, then $x > y$ in L_{2j}. Finally, if $(x, y) \in \mathcal{I}_j$, then $(x, y) \in \mathcal{S}_0$ and $x > y$ in L_0. This completes the proof of Theorem 1.1.

The reader may also note that one can quickly establish the upper bound $M(t) \leq 3t$ via an inductive argument starting with Theorem 2.2 as the base case and using Proposition 3.2—without *any* information about how Theorem 2.2 is proved. But our improved bound required some knowledge of how the proof of Theorem 2.2 proceeds in order to determine where savings can be had.

5 Constructions for Lower Bounds

In Fig. 7, we show a planar poset with six minimal elements (the darkened points). On the other hand, the labelled points form the standard example S_9. It is easy to see that this construction can be modified to show that $M(t) \geq t + 3$ for all $t \geq 2$.

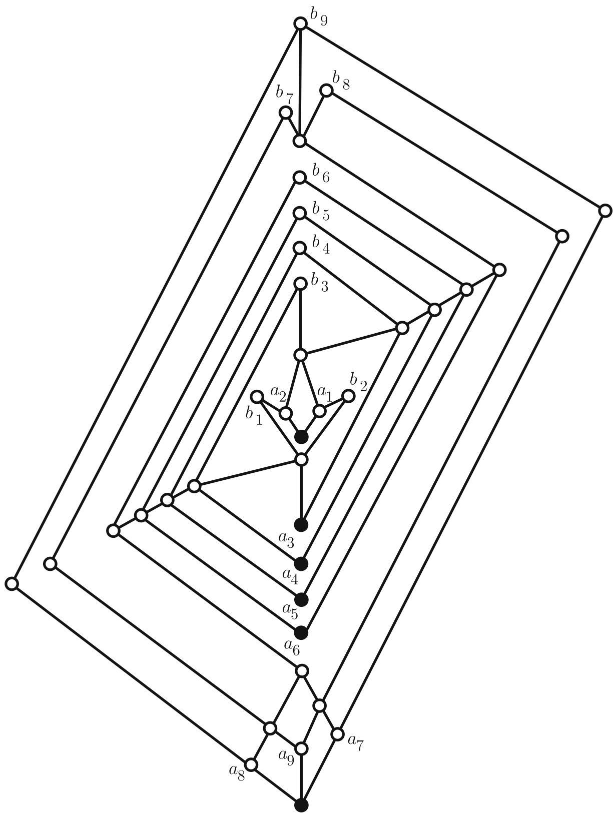

Fig. 7 Planar poset with six minimal elements and dimension 9

We are unable to settle the question as to whether the inequality in Theorem 1.1 is tight when $t \geq 3$.

5.1 Forcing Standard Examples

The question as to whether planar posets of large dimension must contain large standard examples was posed by Felsner, Trotter and Wiechert in [8], and researchers in the area tend to believe the answer is "yes." In order to put the problem in a formal framework, define the *breadth*[3] of a poset, denoted $\text{br}(P)$, as follows: First, set $\text{br}(P) = 1$ when P is an interval order. When P is not an interval order, set $\text{br}(P)$ to be the largest $d \geq 2$ for which P contains the standard example S_d.

With this notation in hand, the conjecture can now be stated formally as follows.

[3] To the best of our knowledge, the concept of breadth is due to K. Baker [1] who included it in an unpublished manuscript which was widely circulated some forty years ago and referenced in a number of published papers. Baker's manuscript also includes the proof of the fact that $\dim(P \times Q) = \dim(P) + \dim(Q)$ when P and Q have zeroes and ones.

Conjecture 5.1. For every $b \geq 1$, there is an integer $d = d_b$ so that if P is a planar poset and $\mathrm{br}(P) \leq b$, then $\dim(P) \leq d$.

Of course, if the value d_b is well defined, it must satisfy the inequality $d_b \geq b$, but to the best of our knowledge, no one has constructed examples of planar posets where $\dim(P) - \mathrm{br}(P)$ is large, and that will be our goal here. Specifically, we will show that for every d and every $\epsilon > 0$, there is a planar poset P with $\dim(P) \geq d$ while the ratio $\dim(P)/\mathrm{br}(P)$ is at least $2 - \epsilon$.

In [24], Trotter defined a family $\mathcal{C} = \{S_n^k : n \geq 3, k \geq 0\}$ of posets called *generalized crowns*. The poset S_n^k is a height 2 poset with $n + k$ minimal elements $a_1, a_2, \ldots, a_{n+k}$ and $n + k$ maximal elements $b_1, b_2, \ldots, b_{n+k}$. The partial order on S_n^k is defined (cyclically) as follows: for each i, a_i is incomparable with $b_i, b_{i+1}, \ldots, b_{i+k}$ and less than $b_{i+k+1}, b_{i+k+2}, \ldots, b_{i-1}$. The subfamily $\{S_3^k : k \geq 0\}$ is one of the infinite families of 3-irreducible posets (see [29], [15]), and posets in this family were called *crowns* by Baker, Fishburn and Roberts in [2].

In [24], the following formula is derived for the dimension of a generalized crown: For all $n \geq 3$, $k \geq 0$, $\dim(S_n^k) = \lceil 2(n + k)/(k + 2) \rceil$.

In Fig. 8, we show two posets. Consider first the poset on the left. It has $8 = 5 + 3$ minimal elements and 8 maximal elements. Furthermore, the subposet determined by the minimal and maximal elements together is the generalized crown S_5^2. Modifying the construction for other values of n and k is straightforward, so we may conclude that for all $n \geq 3$, $k \geq 0$, the generalized crown S_n^k is a subposet of a planar poset $P(n, k)$. Furthermore, we note that when $n \gg k \gg 1$, the breadth of the poset $P(n, k)$ is at most $(n + k)/k$ while $\dim(P(n, k)) \geq \dim(S_n^k) \geq 2(n + k)/(k + 2)$. So the ratio $\dim(P)/\mathrm{br}(P)$ is as close to 2 as desired.

The reader may note that when $n >> k >> 1$, the cover graph of the poset $P(n, k)$ has large tree-width, since it contains a large grid. On the other hand, on the right side of this

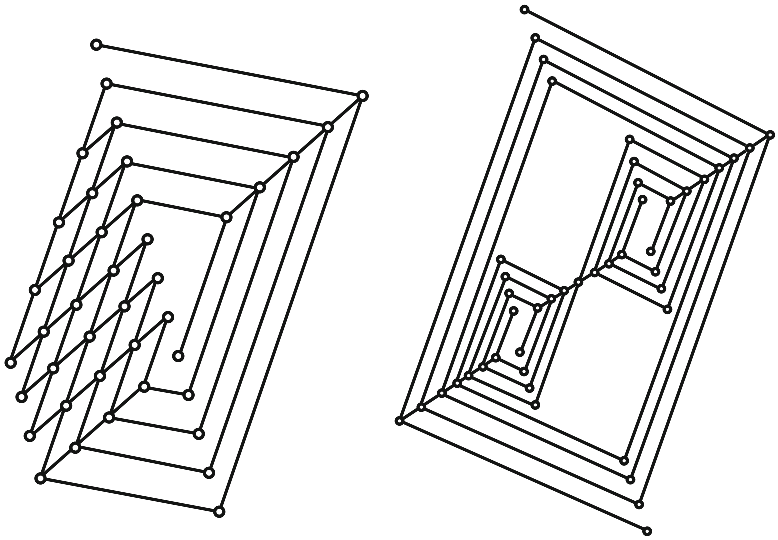

Fig. 8 Modifying the Kelly Construction

same figure, we show a modified form of the Kelly construction in which three standard examples are arranged in a cyclic manner. In this specific illustration, the standard examples are copies of S_4 but again, modifying the figure for other values of d is straightforward. Denote the resulting poset as P_d. Note that the three copies of S_d in P_d can be labelled as $S_d(1)$, $S_d(2)$, $S_d(3)$ so that for each $i = 1, 2, 3$, the minimal elements of $S_d(i)$ are less than the maximal elements of $S_d(i + 1)$. So for $i = 1, 2, 3$, there are d incomparable pairs of the form $(a_j(i), b_j(i))$ in $S_d(i)$, and there are $3d$ such pairs in P_d altogether. Any linear extension of P_d reverses only one such pair for the same value of i, and at most two such pairs altogether. It follows that the dimension of the poset is at least $3d/2$.

On the other hand, we leave it as an exercise to verify that the breadth of P_d is $d + 1$. Moreover, it is easy to see that the tree-width of the cover graph of P_d is bounded, independent of the value of d. It follows that there exist planar posets P with $\dim(P) \geq (3/2 - o(1))\,\mathrm{br}(P)$ such that the tree-width of the cover graph of P is bounded.

6 Open Problems

Of course, the obvious open problem is to tighten the upper and lower bounds on $M(t)$. Even after considerable effort, we are unable to settle whether $M(3)$ is 6 or 7. However, we tend to believe that $M(t) \leq (1 + o(1))t$, although we admit that we do not have much evidence to support this.

A second problem is to settle Conjecture 5.1. We tend to believe that it will be resolved in the affirmative and that the function d_b will be linear in b, but again this represents a belief without much evidence. Our construction suggests that there may also be refinements which take into consideration the tree-width of the cover graph.

Observing that the Kelly construction actually has two long incomparable chains, G. Gutowski and T. Krawczyk have made (personal communication) the following intriguing conjecture.

Conjecture 6.1. For every $k \geq 1$, there is an integer $d = d(k)$ so that if P is planar poset and P does not contain two chains of size k with all points in one chain incomparable with all points in the other, then $\dim(P) \leq d$.

Of course, the Gutowski-Krawczyk conjecture can be extended to minor closed classes, but even the version for planar graphs seems quite hard.

Recently, a much stronger and more general conjecture has begun to be circulated among researchers in this area, and we consider this a major challenge for the future.

Conjecture 6.2. For every pair (b, n) of positive integers, there is an integer $d = d(b, n)$ so that if P is a poset such that $\mathrm{br}(P) \leq b$ and the cover graph of P does not contain a K_n minor, then $\dim(P) \leq d$.

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