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Trees and circle orders

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Abstract This paper continues a recent resurgence of interest in combinatorial properties of a poset that are associated with graph properties of its cover graph and order diagram. The following two theorems appearing in a 1977 paper of Trotter and Moore have played important roles in motivating this more modern research: (1) The dimension of a poset is at most 3 when its cover graph is at tree; (2) The dimension of a poset is at most 3 when the poset has a zero and its order diagram is planar. Although the underlying ideas lay dormant for more than 30 years, the first of these two results has become the base case for recent results bounding the dimension of a poset in terms of (a) the tree-width of its cover graph, and (b) the maximum dimension of its blocks. The second result is the base case for bounding the dimension of a planar poset in terms of the number of minimal elements. Continuing with this line of research, we show that every poset whose cover graph is a tree is a circle order, i.e., it has a representation as a family of circular disks in the Euclidean plane partially ordered by inclusion.

Keywords Dimension · planar poset · circle order · width

Mathematics Subject Classification 06A07 · 05C35

1 Introduction, notation and terminology

Recall that a *partially ordered set* P consists of a ground set equipped with a binary relation $\leq$ which is reflexive, anti-symmetric and transitive, i.e.,

This paper is dedicated to the memory of Rudolf Halin, with deep appreciation for his many substantive contributions to graph theory and combinatorial mathematics.

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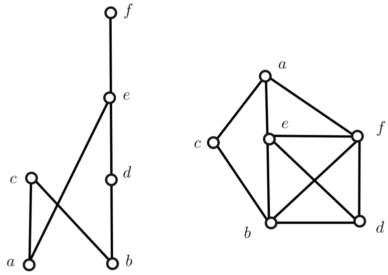

Fig. 1 A poset and its comparability grap

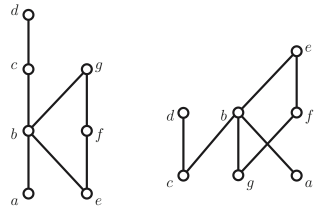

Fig. 2 Two posets with the same cover graph

(1) $x \leq x$ for all $x \in P$.
(2) If $x, y \in P$, $x \leq y$ and $y \leq x$, then $x = y$.
(3) If $x, y, z \in P$, $x \leq y$ and $y \leq z$, then $x \leq z$.

Following the now standard practice, we use the short form *poset* for a partially ordered set. We are concerned here with combinatorial problems for posets, and we assume some familiarity with basic concepts, including the notions of comparable and incomparable points; minimal and maximal elements; chains and antichains; and height and width. However, to assist readers who are relatively new to research topics in this area, we provide definitions for concepts which are just beyond these bare essentials. For readers who are completely new to combinatorics focusing on posets, we suggest consulting the research monograph [26] and survey article [27] for extensive background information.

The *size* of a poset P is just the cardinality of its ground set. Here we will focus exclusively on finite posets—although readers may note that there are cases in the literature where related results for infinite posets are investigated.

Associated with a poset P is a *comparability graph* and a *cover graph*. Both graphs have the ground set of P as their vertex set. In the comparability graph, distinct elements x and y of P are adjacent when either $x < y$ or $x > y$. In this case, we say that x and y are *comparable in* P. On the other hand, when $x < y$ and there is no point z with $x < z < y$ in P, then we say that x is *covered by* y in P. In the cover graph of P, distinct elements x and y are adjacent when one of them is covered by the other. Note that both the comparability graph and the cover graph of a poset are ordinary graphs without loops or multiple edges. Also, the edges are not directed. On the other hand, the *order diagram* (also called a *Hasse diagram*) of a poset P is a straight line drawing of the cover graph of P in a Euclidean plane, equipped with a standard horizontal-vertical coordinate system, with the additional restriction that the second coordinate (the vertical height) of x is lower than y in the drawing when x is covered by y in the poset. To make these concepts concrete, in Fig. 1, a poset P is shown on the left, and its comparability graph is shown on the right.

Also, in Fig. 2, we show two different posets with the same cover graph. Note that while both have size 7, the poset on the left has height 4 and width 2 while the poset on the right has height 3 and width 3.

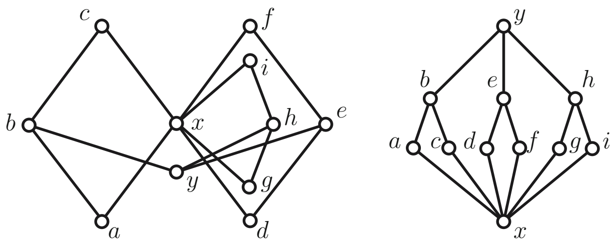

Fig. 3 A non-planar poset with a planar cover graph

Recall that a poset L is called a chain (also a linear order) when its width is 1, i.e., its comparability graph is a complete graph. Also, when P and L are posets with the same ground set, L is said to be a *linear extension of* P when L is a chain and $x < y$ in L whenever $x < y$ in P. The reader should note that we will follow the convention of writing $x < y$ in P in any setting where more than one partial order is being discussed on the same ground set.

A family $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of a poset P is called a *realizer* of P when $x < y$ in P if and only if $x < y$ in L_j for each $j = 1, 2, \ldots, t$. The least positive integer t for which P has a realizer of size t is called the *dimension of* P and is denoted $\dim(P)$. Of course, $\dim(P) = 1$ only when P is a chain. It is arguably true that dimension is the single most widely studied combinatorial parameter for posets, and many of the papers appearing on our bibliography are concerned primarily with properties of dimension. Our results here will work on the perimeter of this topic, as they are motivated by questions and conjectures involving dimension. However, our proofs are entirely self-contained and do not depend on dimension theoretic work.

A poset P is *planar* if its order diagram can be drawn without edge crossings. Although there are crossing edges in the drawing of the order diagram of the poset in Fig. 1, this poset is nevertheless planar, since it is clearly possible to find an alternative drawing of the order diagram in which there are no crossing edges. Of course, when a poset is planar, its cover graph is also planar. However, a poset can be non-planar even when its cover graph is planar, and we illustrate such a poset and its planar cover graph in Fig. 3.

As is well known, many familiar combinatorial parameters of a poset are *comparability invariants*, meaning that the parameter is the same for two posets with isomorphic comparability graphs. Elementary examples include size, height, and width. On the other hand, of these three parameters, only size is an invariant of the cover graph, as illustrated by the posets shown in Fig. 2. In fact, these other two parameters can vary wildly for posets with isomorphic cover graphs. To see this, just consider posets whose cover graphs are paths.

Although not entirely elementary, it is an direct consequence of the decomposition techniques developed by Gallai in [16] that the following parameters are also comparability invariants: (a) number of linear extensions, and (b) dimension. Again, posets whose cover graphs are paths show that the number of linear extensions can be markedly different, but any such poset has dimension at most 2. On the other hand, examples are constructed in [29] that show that for every $d \geq 2$, there are two posets P and Q with the same cover graph such that $\dim(P) = 2$ and $\dim(Q) = d$. So it is then natural to conclude that very little information about the combinatorial properties of a poset is revealed by graph theoretic properties of the cover graph.

Nevertheless, modern research has shown that in some cases, important combinatorial properties of a poset are indeed associated with the cover graph, and the results of this paper carry on with that theme (see [7,8,17,22,24,30] for related work).

In this paper, we will also be concerned with the concept of an *inclusion order*. It is easy to see that whenever P is a poset, it is possible to associate with each element x of P a subset $S(x)$ of some universal set U so that $x \leq y$ in P if and only if $S(x) \subseteq S(y)$. In this case, the family $\mathcal{F} = \{S(x) : x \in P\}$ is called an *inclusion representation* of P. Of course, researchers are primarily interested in posets which have inclusion representations where $S(x)$ has some special form, and there has been considerable attention paid to the case where $S(x)$ is a geometric figure, such as a box or sphere. We refer the reader to [13] for a comprehensive survey of work on geometric inclusion orders. Here, we will be concerned exclusively with the case of spheres.

When d is a positive integer, we let $\mathbb{R}^d$ denote d-dimensional euclidean space. For points $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_d)$ in $\mathbb{R}^d$, let $d(\mathbf{x}, \mathbf{y})$ denote the *distance* from $\mathbf{x}$ to $\mathbf{y}$, i.e.,

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_d - y_d)^2}$$

A d-*sphere* in $\mathbb{R}^d$ is determined by a *center* $\mathbf{c} \in \mathbb{R}^d$ and a non-negative real number r, called the *radius* of the sphere and consists of all points x such that $d(\mathbf{x}, \mathbf{c}) \leq r$.

A poset P is called a d-*sphere order* when it has an inclusion representation where $S(x)$ is a d-sphere for each $x \in P$. A 1-sphere in $\mathbb{R}$ is just an interval of the real line, and it is well known that a poset is a 1-sphere order if and only if it has dimension at most 2. Posets that are 2-sphere orders are called *circle orders*, although it can be argued that this terminology is misleading, since we are concerned with the entire disk for which the circle is just the boundary. Regardless, as there is an extensive literature in which the term circle order is used, we will also use that terminology here.

Our principal theorem will be the following result.

Theorem 1.1 *If P is a poset and the cover graph of P is a tree, then P is a circle order.*

Although our proof for Theorem 1.1 proceeds by induction and does not require advanced results in combinatorics or geometry, some work is required since, as we shall see, care must be taken with the induction. Also, we feel that our result suggests quite naturally some challenging open problems, which may turn out to be more important than Theorem 1.1, as we are fairly certain that they cannot be resolved without first developing some entirely new techniques.

The remainder of this paper is organized as follows. In the next section, we briefly discuss the historical background behind this line of research and the connections with quite recent research on the combinatorics of posets. The proof will then be given in Sect. 3. We close in Sect. 4 with the open problems referenced immediately above.

2 Why circle orders and trees are interesting

As we have noted, a poset P has dimension at most 2 if and only if it is a 1-sphere order. Interest in 2-sphere orders (circle orders) was spiked by Scheinerman [23] who proved that a graph G is planar if and only if its incidence poset (vertices and edges ordered by inclusion) is a circle order. One direction of their proof led to an interesting strengthening (see‘[4]) of the well-known [20] “kissing coin” theorem of Koebe: If V is the vertex set of a planar graph G, then there exists a family $\mathcal{F} = \{S(x) : x \in V\}$ of circles in the plane so that (1) the interiors of the circles in $\mathcal{F}$ are pairwise disjoint and (2) x and y are adjacent if G if and only if $S(x)$ and $S(y)$ are tangent.

We pause here to introduce three well known classes of posets. First, for an integer $d \geq 2$, the *standard example* S_d is the height 2 poset with d minimal elements labelled $a_1, a_2, \ldots, a_d$ and d maximal elements labelled $b_1, b_2, \ldots, b_d$. The partial order is defined by setting $a_i < b_j$ in S_d if and only if $i \neq j$. As is well known, $\dim(S_d) = d$ for all $d \geq 2$. Accordingly, any poset which contains a large standard example has large dimension. In [2], it is shown that when a poset has dimension which is relatively close to the maximum dimension a poset can have on a ground set of a given size, it must contain a large standard example.

At the other extreme, posets which do not contain the standard example S_2 are called *interval orders*. By Fishburn's theorem [9], a poset P is an interval order if and only if there is a family $\{[a_x, b_x] : x \in P\}$ of non-degenerate closed intervals of the real line $\mathbb{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. Important to our discussion here is the fact that interval orders can have arbitrarily large dimension (see [2,15] for details).

A poset P which has an inclusion representation where $S(x)$ is an angular region (infinite of course) determined by a point in the plane and two distinct rays emanating from the point are called *angle orders*. In [12], it is shown that each of the following is a subclass of the class of angle orders: (a) interval orders; (b) standard examples; and (c) all posets of dimension at most 4. In [10], Fishburn shows that every angle order is a circle order, and in [11], he shows that there are circle orders that are not angle orders.

Despite the exceptional classes we have just discussed, the Alon-Scheinerman [1] "degrees of freedom" implies that the following statements hold:

(1) Almost all posets P with $\dim(P) \geq 3$ are not interval orders.
(2) Almost all posets P with $\dim(P) \geq 5$ are not angle orders.
(3) For each $d \geq 2$, almost all posets P with $\dim(P) \geq d + 2$ are not d-sphere order.

For many years, researchers felt that it might be the case that all posets of dimension at most $d + 1$ are d-sphere orders, a statement that we have already noted is true when $d = 1$. But Felsner, Fishburn and Trotter [6] used techniques from lexicographic ramsey theory to prove that when n is sufficiently large[1], the finite 3-dimensional poset $P = \mathbf{n}^3$ is *not* a sphere order, i.e., there is no $d \geq 1$ for which the poset P can be represented as a family of d-spheres in $\mathbb{R}^d$ partially ordered by inclusion. As a consequence, when n is large, almost all posets which have dimension at least 3 are not sphere orders. To underscore the special role spheres are playing in these statements, we note that it is an easy exercise to show that every $n \geq 3$, a finite three dimensional poset has an inclusion representation using regular n-gons with a common orientation (homothetic copies). Also, it is shown in [14] that every finite 3-dimensional poset is an *ellipse order*. In fact, there is an inclusion representation using ellipses with major axes all on a common line.

2.1 Posets and trees

We mention here only a small sampling of recent results, all of which can be traced back to the following two theorems appearing in a 1977 paper of Trotter and Moore [25]. More extensive discussion is given in [17,24,29].

Theorem 2.1 *If P is a poset and the cover graph of P is a tree, then* $\dim(P) \leq 3$.

When a poset P has an element x_0 with $x_0 \leq x$ in P for every $x \in P$, it is traditional to refer to x as a *zero*. Trivially, when $|P| \geq 2$ and x_0 is a zero in P, then the dimension of the

[1] In view of the proof techniques, the value of n provided by the proof in [6] is *very* large. However, it might well be the case that $\mathbf{n}^3$ is not a sphere order when $n = 100$.

subposet $P - \{x_0\}$ is the same as the dimension of P. So in general, the dimension of a poset is unaffected by whether or not it has a zero. However, for planar posets, the presence of a zero can have considerable impact.

Theorem 2.2 *If P is a planar poset and P has a zero, then* $\dim(P) \leq 3$.

As detailed in [29], Theorems 2.1 and 2.2 have an interesting history. As presented in [25], Theorem 2.2 is proved first, with Theorem 2.1 deduced as an immediate corollary. This is accomplished by the easy exercise of showing that if P is a poset whose cover graph is a tree, then the super-poset Q obtained by adding a new element x_0 to P with $x_0 < x$ in Q for all $x \in P$ is planar. However, in fact, Theorem 2.1 was proved first for the special case of trees of height 2. The argument was then extended to trees of arbitrary height, and soon afterwards, it was realized that the more general result given in Theorem 2.2 could be proved.

As mentioned briefly in the abstract, the following powerful extension of Theorem 2.1 is proved by Joret, Micek, Milans, Trotter, Walczak and Wang [17].

Theorem 2.3 *For every pair (t, h) of positive integers, there is a constant $d = d(t, h)$ so that if P is a poset whose cover graph has tree-width t and the height of P is h, then* $\dim(P) \leq d$.

Evidently, $d(1, h) = 3$ for all $h \geq 2$. Examples constructed by Kelly [19] show that $d(t, h)$ must go to infinity with h for all $t \geq 3$. It took a couple of years to settle whether $d(2, h)$ is bounded or goes to infinity with h. First, Biró, Keller and Young [3] settled the question for pathwidth with the following result.

Theorem 2.4 *If P is a poset whose cover graph has path-width at most* 2*, then* $\dim(P) \leq 17$.

Subsequently, the full question for tree-width was resolved by Joret, Micek, Trotter, Wang and Wiechert [18].

Theorem 2.5 *If P is a poset whose cover graph has tree-width at most* 2*, then* $\dim(P) \leq 1276$.

Moreover, quite recently, Walczak [30] has proved a sweeping extension of Theorem 2.3 with the following result.

Theorem 2.6 *For every pair (n, h) of positive integers, there is an integer $d = d(n, h)$ so that if P is a poset of height h and the cover graph of P does not contain the complete graph K_n as a topological minor, then* $\dim(P) \leq d$.

Walczak's proof makes use of advanced topics in topological graph theory, in particular the Robertson–Seymour and Grohe–Marx graph structure theorems. However, even more recently, Micek and Wiechert [22] have given a surprisingly short proof of Theorem 2.6, which is entirely combinatorial in spirit.

Here is another interesting observation concerning posets and trees. A subposet Q of a poset P is said to be *convex* when it satisfies the following property: If x, y and z are distinct points of P with $x < y < z$ in P, then y is in Q whenever both x and z are in Q. Note that when Q is a convex subposet of P, the cover graph of Q is an induced subgraph of the cover graph of P. Also, a convex subposet of a poset P is called a *component* of P when its cover graph is a component of the cover graph of P.

Recall that an (induced) subgraph H of a graph G is called a *block of* G when H is a 2-connected subgraph of G and is maximal with respect to that property. Analogously, a convex subposet B of a poset P is called a *block of* P when the cover graph of B is a block in the cover graph of P. As is well known, the dimension of a poset P is equal to the maximum dimension among its components, except in the trivial case where P is disconnected and every component is a chain. In this exceptional case, P has dimension 2 while all components of P have dimension 1. On the other hand, we note that it is not immediately clear that there is any bound on the dimension of a poset in terms of the maximum dimension of its blocks. However, Trotter, Walczak and Wang [28] proved the following result.

Theorem 2.7 *For every $d \geq 1$, if $\dim(B) \leq d$ for every block B in a poset P, then $\dim(P) \leq d + 2$. Furthermore, this inequality is best possible.*

Theorem 2.1 is just the special case $d = 1$ in the preceding theorem, but the proof of the upper bound in Theorem 2.7 does not proceed by induction and handles all values of d at the same time. On the other hand, the Product Ramsey theorem is used to show that for $d \geq 2$, the inequality is best possible. Quite naturally, the examples produced are of enormous size.

In [29], Trotter and Wang extended Theorem 2.2 with the following result.

Theorem 2.8 *For every $t \geq 1$, if P is a planar poset with t minimal elements, then $\dim(P) \leq 2t + 1$.*

As noted previously, Theorem 2.2 is just the special case $t = 1$. Examples constructed in [29] show that for every $t \geq 2$, there is a planar poset P with t minimal elements satisfying $\dim(P) = t + 3$. So Theorem 2.8 is also tight when $t = 2$. However, for $t \geq 3$, there remains a gap between the best known upper and lower bounds.

3 Proof of the main theorem

We will call a poset P a *tree* if the cover graph of P is a tree[2], so in this section, we will prove that if P is a tree, then P is a circle order. Naturally our argument will proceed by induction on the size of the tree. However, we pause to explain why some care must be taken with how the induction proceeds. Consider the following tree: P is a poset of height 2 whose ground set is $\{x, y, z_1, z_2, \ldots, z_{n+1}\}$. The maximal elements in P are x and y while $z_1, z_2, \ldots, z_{n+1}$ are minimal elements in P. The point x covers z_i for each $i = 1, 2, \ldots, n+1$ while y covers only z_{n+1}.

Consider the representation of $P - \{y\}$ defined as follows. Start by setting $S(x)$ to be a circle of radius 2 centered at the origin. Then let S_1 be a circle of radius 1 centered at the origin, and consider a set $p_1, p_2, \ldots, p_n$ of points equally spaced around the boundary of the circle S_1. For each $i = 1, 2, \ldots, n$, take $S(z_i)$ to be the circle of radius $1/n$ which is in the interior of $S(x)$ and tangent to $S(x)$ at p_i. Then take $S(z_{n+1})$ to be a circle of radius $1/n$ centered at the origin. Clearly, when n is sufficiently large, there is no way to extend this representation by choosing an appropriate circle $S(y)$ for the maximal element y.

With these remarks in mind, the real challenge is to find the appropriate strengthening of the statement we are trying to prove. Once this is accomplished, the proof is not too difficult. However, the formulation of this stronger statement requires some additional notation and terminology.

[2] Readers who have a computer science background will sometimes use the term *tree* in a more restrictive sense. In particular, they require that the poset have a zero. In this paper, a poset which is a tree can have arbitrarily many minimal elements. Of course, it can also have arbitrarily many maximal elements.

If P is a finite poset with ground set X, the *split* of P (as defined[3] by R. Kimble), is a height 2 poset with minimal elements $\{x' : x \in X\}$ and maximal elements $\{x'' : x \in X\}$ and order relation $x' \le y''$ in the split of P if and only if $x \le y$ in P. Kimble noted that if Q is the split of P, then $\dim(P) \le \dim(Q) \le 1 + \dim(P)$.

Here, we will be concerned with a modest generalization of the notion of a split. Given a finite poset P with ground set X, we consider the *split-in-place* of P as the poset P^* which results from adding to P minimal elements $\{x' : x \in X\}$ and maximal elements $\{x'' : x \in X\}$. In P^* we have the comparabilities $x' \le y$, $x \le y''$ and $x' \le y''$ if and only if $x \le y$ in P. A straightforward modification of the argument for splits shows that then $\dim(P) \le \dim(P^*) \le 1 + \dim(P)$. Of course, P is a subposet of P^*. Furthermore, central to our approach is the fact that if P is a tree, then P^* is also a tree.

With these comments in mind, we will show that for every tree P, the split-in-place P^* of P is a circle order. In fact, we will prove the following more technical statement: For $i = 1$, let S_1 and S_2 be circles centered at the origin so that the radius of S_1 is 1 and the radius of S_2 is 2. Then P^* has a circle representation $\{S(u) : u \in P^*\}$ so that:

(1) The circles in $\{S(u) : u \in P^*\}$ are distinct and they form an inclusion representation of P^*, i.e. $u \le v \in P^*$ if and only if $S(u) \subseteq S(v)$.
(2) If $u < v$ in P^*, then $S(u)$ is contained in the interior of $S(v)$.
(3) For every u in P^*, $S_1 \subsetneq S(u) \subsetneq S_2$.
(4) If $u = x'$ for some $x \in X$, then $S(u)$ is tangent to S_1.
(5) If $u = x''$ for some $x \in X$, then $S(u)$ is tangent to S_2.
(6) For each $x \in X$, let $\arg(x')$ and $\arg(x'')$ denote respectively the angle from $[0, 2\pi)$, measured clockwise, to the point where $S(x')$ is tangent to S_1 and where $S(x'')$ is tangent to S_2. Then $\max\{\arg(x'), \arg(x'')\} = \pi + \min\{\arg(x'), \arg(x'')\}$.

We now proceed by induction on the number of points in P to show that there is a circle representation of the split-in-place P^* of a tree P satisfying these additional restrictions, noting that this statement holds trivially in the base case when P is a tree consisting of a single vertex. Now suppose that the theorem (in its strong form) holds for all trees with k vertices, for some positive integer k and consider a tree P on $k + 1$ vertices. Then let x be a leaf of P and let y be its unique neighbor in the cover graph. The basic idea behind the proof from this point on is that if x is a minimal element in P, then we use the circle $S(y')$ as a "template" for $S(x)$, $S(x')$ and $S(x'')$. A dual statement holds when x is a maximal element in P.

We first consider the case that x is a minimal element in P. Let Q denote the tree obtained by removing x from P and consider a circle representation of Q^* satisfying all the requirements listed above. Let $r(y)$, $r(y')$ and $r(y'')$ denote the radii of the circles $S(y)$, $S(y')$ and $S(y'')$, respectively.

Then let ϵ and δ be small positive numbers (the requirement as to how small ϵ and δ must be will be clear from the discussion to follow, but δ will always be smaller than ϵ). We will determine circles $S(x)$, $S(x')$ and $S(x'')$ to add to the circles for Q^* to form an appropriate representation for P^*. Furthermore $\arg(x') = \arg(y') + \epsilon$ and $\arg(x'') = \arg(y'') + \epsilon$. The circle $S(x')$ will have radius $r(x') = r(y')$. The circle $S(x)$ will have radius $r(x) = r(x') + \delta$ and will have the same center as $S(x')$. These requirements we have listed

[3] The notion of a split is due to Kimble who explained it to me in 1974. He also wrote a letter to me in which the concept is mentioned in conjunction with the construction of 3-irreducible posets. While splits have found a number of applications in the combinatorics of posets, and Kimble is always referenced as the inventor, there is apparently no published article to cite.

completely determine the circles $S(x)$ and $S(x')$—of course up to specifying values for ϵ and δ.

Now consider the point p_0 on the circle S_2 where our previous discussion will require $S(x'')$ to be tangent to S_2. Let S be the unique circle which (1) contains $S(x)$; (2) passes through the point p_0; (3) is contained in S_2; and (4) is tangent to both $S(x)$ and S_2. The circle $S(x'')$ is obtained from S by increasing its radius by δ and keeping it tangent to S_2 at p_0. Again, we note that this determines the circle $S(x'')$ (with the same remarks applying to the values of ϵ and δ).

Now we must verify that the resulting circles form an appropriate representation of P^*, provided ϵ is sufficiently small and in terms of ϵ, δ is sufficiently small. For starters, we observe that $S(x'')$ is tangent to S_2, $S(x')$ is tangent to S_1 and $S(x') \subsetneq S(x) \subsetneq S(x'')$. Furthermore, it is clear that $S(x) \subsetneq S(y)$. Also, the tangency requirements dictate that $S(x'')$ cannot be contained in in $S(u)$ for any $u \in P^*$ with $u \neq x''$. Similarly, $S(x')$ cannot contain $S(u)$ for any $u \in P^*$ with $u \neq x'$.

So it remains to show that (1) If $S(x') \subsetneq S(u)$ for some $u \in Q^*$, then $u \geq y$ in Q^*, and (2) If $S(u) \subsetneq S(x'')$ for some $u \in Q^*$, then $u \leq x$ in P^*. The first of these two statements follows from the fact that $S(x')$ is a small perturbation of the circle $S(y')$ so that if $S(u)$ contains $S(x')$ it also contains $S(y')$. This requires $u \geq y$ in Q^*.

We now prove the second statement. To accomplish this, we observe that the circle $S(x'')$ is closer to S_1 than any circle in the representation of Q^* except for those which are actually tangent to S_1. Also, $S(x'')$ cannot contain $S(u)$ for some $u = z'$ from Q because δ is smaller than ϵ, and the point on $S(x'')$ diametrically opposite p_0 is in the interior of all such circles.

This completes the proof in the case when x is a minimal element of P. Furthermore, it is easy to see that the argument for the case when x is a maximal element in P is symmetric. Accordingly, the proof is complete.

4 Open problems

In a footnote to his now classic paper [5], Dilworth pointed out that it follows as an immediate corollary to his chain covering theorem that the dimension of a poset is at most its width. On the one hand, this inequality is easily seen to be best possible, as evidenced by the family of standard examples. On the other hand, there is some subtlety behind this elementary inequality, as after many years, it is still not known whether the following question is NP-complete: Is the dimension of P less than the width of P?

For the problem discussed here, it would be challenging to decide whether every poset of width 3 is a circle order. Some small advance on this problem has been made by Lin [21] who showed that the answer is "yes" when P is ranked, i.e., every maximal chain has the same size. Note that the techniques used in [6] will not apply since $\mathbf{n}^3$ has enormous width. However, we still suspect that the answer will be negative.

In view of the historical connections between Theorems 2.1 and 2.2, it is natural to ask whether planar posets with a zero must be circle orders. Again, we conjecture that the answer is "no."

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