THE DIMENSION OF POSETS WITH PLANAR COVER GRAPHS EXCLUDING TWO LONG INCOMPARABLE CHAINS

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ABSTRACT. It has been known for more than 40 years that there are posets with planar cover graphs and arbitrarily large dimension. Recently, Streib and Trotter proved that such posets must have large height. In fact, all known constructions of such posets have two large disjoint chains with all points in one chain incomparable with all points in the other. Gutowski and Krawczyk conjectured that this feature is necessary. More formally, they conjectured that for every $k \geqslant 1$, there is a constant d such that if P is a poset with planar cover graph and P excludes $\mathbf{k} + \mathbf{k}$, then $\dim(P) \leqslant d$. We settle their conjecture in the affirmative. The proof involves some intermediate results that we believe are of independent interest.

1. INTRODUCTION

We assume that the reader is familiar with basic notation and terminology for partially ordered sets (here we use the short term *posets*), including subposets, chains and antichains, minimal and maximal elements, linear extensions, order diagrams, and cover graphs. Extensive background information on the combinatorics of posets can be found in [18, 19]. We will also assume that the reader is familiar with basic concepts of graph theory, including subgraphs, induced subgraphs, paths and cycles, and planar graphs.

Here are some related concepts for posets. A subposet Q of P is *convex* if $y \in Q$ whenever $x, z \in Q$ and $x < y < z$ in P. When Q is a convex subposet of P, the cover graph of Q is an induced subgraph of the cover graph of P. Traditionally, the elements of a poset are called *points*, and this is what we do in this paper.

Dushnik and Miller [5] defined the *dimension* of a poset P, denoted by $\dim(P)$, as the least positive integer d for which there is a family $\mathcal{R} = \{L_1, \ldots, L_d\}$ of linear extensions of P such that $x \leqslant y$ in P if and only if $x \leqslant y$ in all $L_1, \ldots, L_d$. Clearly, if Q is a subposet of P, then $\dim(Q) \leqslant \dim(P)$. A poset has dimension 1 if and only it is a chain.

For $d \geqslant 2$, the *standard example* S_d is the poset of height 2 consisting of d minimal elements $a_1, \ldots, a_d$ and d maximal elements $b_1, \ldots, b_d$ with $a_i < b_j$ in S_d if and only if $i \neq j$. As noted in [5], $\dim(S_d) = d$ for every $d \geqslant 2$. So every poset that contains a large standard example has large dimension. On the other hand, it is well known that there are posets that have large dimension but do not contain the standard example S_2 (see the more comprehensive discussion in [2]).

In recent years, there have been a series of research papers exploring connections between the dimension of a poset P and graph-theoretic properties of the cover graph of P. This paper continues with that theme. A poset P is *planar* if it has a drawing with no edge crossings in its order diagram. A planar poset has planar cover graph, but it is well known that there are non-planar posets with planar cover graphs (see [18], page 67).

It is an easy exercise to show that the standard example S_d is a planar poset when $2 \leqslant d \leqslant 4$, while the cover graph of S_d is non-planar when $d \geqslant 5$. However, in [17], it is shown that for every $d \geqslant 5$, the non-planar poset S_d is a subposet of a poset with planar cover graph. Subsequently,
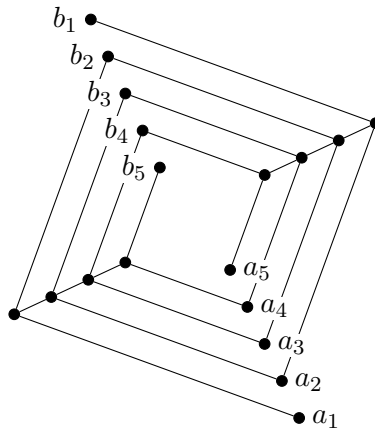
FIGURE 1. Kelly's example of a planar poset containing the standard example S_5 as a subposet

Kelly [14] proved the stronger result: for every $d \geqslant 5$, the non-planar poset S_d is a subposet of a planar poset P with $\dim(P) = d$ (see Figure 1).

In this paper, we do not distinguish between isomorphic posets, and we say that P *contains* Q when there is a subposet of P that is isomorphic to Q. Also, we say P *excludes* Q when P does not contain Q. For a positive integer k, let $\mathbf{k}$ denote a k-element chain. Also, let $\mathbf{k} + \mathbf{k}$ denote a poset consisting of two chains of size k with all points in one chain incomparable with all points in the other. The above-mentioned constructions of posets with planar cover graphs and arbitrarily large dimension raise the following questions.

Question 1.1. Which of the following statements are true for every poset P with planar cover graph and sufficiently large dimension?

(1) P has many minimal elements,
(2) P has large height, that is, P contains $\mathbf{k}$ for some large value of k,
(3) P contains $\mathbf{k} + \mathbf{k}$ for some large value of k,
(4) P contains S_k for some large value of k.

The construction in [17] shows that for every $d \geqslant 2$, there is a poset P satisfying the following conditions: (a) the dimension of P is d; (b) P has a unique minimal element and a unique maximal element; and (c) the cover graph of P is planar. On the other hand, in [22], the following result is proved for planar posets.

Theorem 1.2. *For every $t \geqslant 1$, if P is a planar poset with t minimal elements, then $\dim(P) \leqslant 2t + 1$.*

Furthermore, it is shown in [22] that this inequality is tight when $t = 1$ and $t = 2$. However, when $t \geqslant 3$, it is only known that there are planar posets with t minimal elements that have dimension $t + 3$. Since a poset and its dual have the same dimension, entirely analogous statements can be made about maximal elements.

The second question was answered in the affirmative in [16], where the following theorem (restated in a form consistent with the results of this paper) is proved.

Theorem 1.3. *For every $k \geqslant 2$, there is a constant d such that if P is a poset with planar cover graph and P excludes $\mathbf{k}$, then $\dim(P) \leqslant d$.*

The bound on d from [16] is very weak, due to extensive use of Ramsey theory in the proof; however, greatly improved bounds are available via [15].

Gutowski and Krawczyk [9] posed the third question and conjectured that it should also have affirmative answer. In this paper, we will settle their conjecture in the affirmative by proving the following theorem, which is the main result of this paper.

Theorem 1.4. *For every positive integer k, there is an integer d such that if P is a poset with planar cover graph and P excludes $\mathbf{k} + \mathbf{k}$, then $\dim(P) \leqslant d$.*

While the Gutowski-Krawczyk conjecture might seem entirely natural just from reflecting on the properties of the Kelly construction, it was also motivated by the results of [3, 4, 6, 13], where combinatorial properties of posets excluding $\mathbf{k} + \mathbf{k}$ played a central role.

The fourth question, which was apparently first raised in [18] (see the comment on page 119), remains open, and we consider it one of the most central challenges in this area of research. Most researchers feel that the answer is again "yes". Formally, we can state the following conjecture.

Conjecture 1.5. *For every $k \geqslant 2$, there exists an integer d such that if P is a poset with planar cover graph and P excludes the standard example S_k, then $\dim(P) \leqslant d$.*

The remainder of this paper is organized as follows. In the next section, we provide a brief summary of notation, terminology, and background material. This discussion would apply to any research problem involving dimension. Then, in Section 3, we develop some properties of the class of posets that exclude $\mathbf{k} + \mathbf{k}$. As these results may find application to other combinatorial problems for posets, the results of that section are presented for posets in general—with no assumption that the cover graph is planar. The proof of our main theorem is given in the next three sections. We conclude in Section 7 with a brief discussion of some open problems.

2. Notation, terminology, and background material

Let P be a poset. A family $\mathcal{R} = \{L_1, \ldots, L_d\}$ of linear extensions of P is called a *realizer* of P when the following holds: $x \leqslant y$ in P if and only if $x \leqslant y$ in all $L_1, \ldots, L_d$. Thus $\dim(P)$ is the least positive integer d such that P has a realizer of size d. Accordingly, to establish an upper bound of the form $\dim(P) \leqslant d$, the most natural approach is simply to construct a realizer of size d for P. However, in recent papers [7, 8, 10, 11, 15, 16, 21, 23], another approach has been taken. Clearly, a family $\mathcal{R}$ of linear extensions of P is a realizer of P if and only if for every $(x, y) \in \mathrm{Inc}(P)$, there is $L \in \mathcal{R}$ with $x > y$ in L. In this case, we say that L *reverses* the incomparable pair (x, y). More generally, when S is a set of incomparable pairs of P, a linear extension L *reverses* S when $x > y$ in L for every $(x, y) \in S$. A set $S \subseteq \mathrm{Inc}(P)$ is *reversible* when there is a linear extension L of P that reverses S, and a family $\mathcal{R}$ of linear extensions *reverses* S when for every $(x, y) \in S$, there is $L \in \mathcal{R}$ that reverses (x, y). With these ideas in hand, when $S \subseteq \mathrm{Inc}(P)$, we can define the *dimension* of S, denoted by $\dim(S)$, as the least positive integer d for which there is a family of d linear extensions of P that reverses S. Clearly, $\dim(P) = \dim(\mathrm{Inc}(P))$, so we can also say that $\dim(P)$ is the least positive integer d for which there is a partition of $\mathrm{Inc}(P)$ into d reversible sets.

An indexed family $\{(x_\alpha, y_\alpha)\}_{\alpha=1}^s$ of incomparable pairs of P with $s \geqslant 2$ is called an *alternating cycle* when $x_\alpha \leqslant y_{\alpha+1}$ in P for every index α considered cyclically in $\{1, \ldots, s\}$ (that is, $y_{s+1} = y_1$). An alternating cycle is *strict* when there are no other comparabilities, that is, $x_i \leqslant y_j$ in P if and only if $j \equiv i + 1 \pmod{s}$. The following elementary lemma, proved in [20], provides a convenient test to determine whether a subset of $\mathrm{Inc}(P)$ is reversible.

Lemma 2.1. *If P is a poset and $S \subseteq \mathrm{Inc}(P)$, then the following statements are equivalent:*

(1) *S is not reversible,*
(2) *S contains an alternating cycle,*

(3) *S contains a strict alternating cycle.*

A typical approach to show that the set $\mathrm{Inc}(P)$ can be partitioned into d reversible sets is by defining a *d-coloring* of the pairs in $\mathrm{Inc}(P)$ with the property that no (strict) alternating cycle is monochromatic. However, the rules for assigning colors can be quite complicated (see e.g. [10, 15, 16]), and that will certainly be the case here.

3. Posets that exclude two long incomparable chains

In this section, we present some general considerations on posets excluding two long incomparable chains. If a poset P excludes $\mathbf{1} + \mathbf{1}$, then P is a chain, so $\dim(P) = 1$. For the rest of this section, we fix an integer $k \geqslant 2$ and a poset P that excludes $\mathbf{k} + \mathbf{k}$. We make no assumption on the structure of the cover graph of P.

Let h denote the height of P, and let $C = \{c_1 < \cdots < c_h\}$ be a chain in P of size h. For each point $z \in P - C$, define integers $\mathrm{dn}(z)$ and $\mathrm{up}(z)$ as follows:

$$\mathrm{dn}(z) = \begin{cases} 0 & \text{if } z \text{ is incomparable with } c_1 \text{ in } P, \\ i & \text{otherwise, where } i \text{ is greatest in } \{1, \ldots, h\} \text{ such that } z > c_i \text{ in } P, \end{cases}$$

$$\mathrm{up}(z) = \begin{cases} h + 1 & \text{if } z \text{ is incomparable with } c_h \text{ in } P, \\ j & \text{otherwise, where } j \text{ is least in } \{1, \ldots, h\} \text{ such that } z < c_j \text{ in } P, \end{cases}$$

Note that $0 \leqslant \mathrm{dn}(z) \leqslant h - 1$ and $2 \leqslant \mathrm{up}(z) \leqslant h + 1$ for every point $z \in P - C$, by maximality of the chain C. Define

$$\mathrm{Dn}(i) = \{z \in P - C \colon \mathrm{dn}(z) = i\} \quad \text{for } 0 \leqslant i \leqslant h - 1,$$
$$\mathrm{Up}(j) = \{z \in P - C \colon \mathrm{up}(z) = j\} \quad \text{for } 2 \leqslant j \leqslant h + 1.$$

Lemma 3.1. *For $0 \leqslant i \leqslant h - 1$, the subposet $\mathrm{Dn}(i)$ of P is convex and has height at most $2k - 2$. More generally, for $0 \leqslant i \leqslant i + m \leqslant h - 1$, the subposet $\mathrm{Dn}(i, m)$ of P defined by*

$$\mathrm{Dn}(i, m) = \bigcup_{\alpha=i}^{i+m} \mathrm{Dn}(\alpha) \cup \{c_{i+1}, \ldots, c_{i+m}\}$$

is convex and has height at most $m + 2k - 2$.

Dually, for $2 \leqslant j \leqslant h + 1$, the subposet $\mathrm{Up}(j)$ of P is convex and has height at most $2k - 2$. More generally, for $2 \leqslant j \leqslant j + m \leqslant h + 1$, the subposet $\mathrm{Up}(j, m)$ of P defined by

$$\mathrm{Up}(j, m) = \bigcup_{\alpha=j}^{j+m} \mathrm{Up}(\alpha) \cup \{c_j, \ldots, c_{j+m-1}\}$$

is convex and has height at most $m + 2k - 2$.

Proof. We only show the proof of the first part, as the second is dual. It is clear that the subposet $\mathrm{Dn}(i, m)$ is convex. The fact that C is a maximum chain implies that the height of $\mathrm{Dn}(i, m)$ is at most $h - i$, so the desired inequality follows if $h - i \leqslant m + 2k - 2$. Suppose $h \geqslant i + m + 2k - 1 \geqslant i + m + k$. Let $Q = \{z \in \mathrm{Dn}(i, m) \colon \mathrm{up}(z) \leqslant c_{i+m+k}\}$. The fact that C is a maximum chain forces the height of Q to be at most $m + k - 1$. Furthermore, the height of the subposet $\mathrm{Dn}(i, m) - Q$ is at most $k - 1$, because all points of $\mathrm{Dn}(i, m) - Q$ are incomparable with the k-element chain $\{c_{i+m+1} < \cdots < c_{i+m+k}\}$. Hence the height of $\mathrm{Dn}(i, m)$ is at most $(m + k - 1) + (k - 1) = m + 2k - 2$. $\square$

For $0 \leqslant i \leqslant h$, define $\mathrm{Ar}(i, i + 1) = \{z \in P - C \colon \mathrm{dn}(z) \leqslant i \text{ and } \mathrm{up}(z) \geqslant i + 1\}$. Here, Ar stands for "around".

Lemma 3.2. *For $0 \leqslant i \leqslant h$, the subposet $\mathrm{Ar}(i, i+1)$ of P is convex and has height at most $4k-4$.*

Proof. It is clear that the subposet $\mathrm{Ar}(i, i+1)$ is convex. Let $Q = \{z \in \mathrm{Ar}(i, i+1) \colon \mathrm{up}(z) \leqslant i+k\}$. Thus $Q \subseteq \mathrm{Up}(i+1) \cup \cdots \cup \mathrm{Up}(i+k) \subseteq \mathrm{Up}(i+1, k-1)$. It follows from Lemma 3.1 that the height of Q is at most $(k-1) + 2k - 2 = 3k - 3$. Furthermore, the height of the subposet $\mathrm{Ar}(i, i+1) - Q$ is at most $k-1$, because all points of $\mathrm{Ar}(i, i+1) - Q$ are incomparable with the k-element chain $\{c_{i+1} < \cdots < c_{i+k}\}$. Hence the height of $\mathrm{Ar}(i, i+1)$ is at most $(3k-3) + (k-1) = 4k - 4$. $\square$

Lemma 3.3. *Let $z, w \in P - C$, $\mathrm{dn}(z) < \mathrm{up}(w)$, and $w \leqslant z$ in P. If C' is a chain in $P - C$ with least element w and greatest element z, then $|C'| \leqslant 4k - 4$.*

Proof. We have $z, w \in \mathrm{Ar}(\mathrm{dn}(z), \mathrm{dn}(z) + 1)$, which implies $C' \subseteq \mathrm{Ar}(\mathrm{dn}(z), \mathrm{dn}(z) + 1)$. We apply Lemma 3.2 to conclude that $|C'| \leqslant 4k - 4$. $\square$

Every incomparable pair (x, y) of P satisfies $\mathrm{dn}(y) < \mathrm{up}(x)$. We call an incomparable pair (x, y) of P *dangerous* if $\mathrm{dn}(x) < \mathrm{dn}(y) < \mathrm{up}(x) < \mathrm{up}(y)$, otherwise the pair (x, y) is *safe*.

Lemma 3.4. *If every convex subposet Q of P of height at most $2k-2$ satisfies $\dim(Q) \leqslant d_0$, then there is a set of at most $2d_0$ linear extensions of P that reverses all the safe incomparable pairs of P.*

Proof. It follows from Lemma 3.1 that $\dim(\mathrm{Dn}(i)) \leqslant d_0$ for $0 \leqslant i \leqslant h-1$ and $\dim(\mathrm{Up}(j)) \leqslant d_0$ for $2 \leqslant j \leqslant h+1$. First, consider d_0 linear extensions of P that

- have block form $\mathrm{Dn}(0) < c_1 < \mathrm{Dn}(1) < \cdots < c_{h-1} < \mathrm{Dn}(h-1) < c_h$,
- induce d_0 linear extensions of $\mathrm{Dn}(i)$ witnessing $\dim(\mathrm{Dn}(i)) \leqslant d_0$, for $0 \leqslant i \leqslant h-1$.

These linear extensions reverse all incomparable pairs (x, y) of P such that $x, y \in \mathrm{Dn}(i)$ for some i with $0 \leqslant i \leqslant h-1$, and all incomparable pairs (x, y) of P with $y \in C$. Then, consider d_0 more linear extensions of P that

- have block form $c_1 < \mathrm{Up}(2) < c_2 < \cdots < \mathrm{Up}(h) < c_h < \mathrm{Up}(h+1)$,
- induce d_0 linear extensions of $\mathrm{Up}(i)$ witnessing $\dim(\mathrm{Up}(i)) \leqslant d_0$, for $2 \leqslant i \leqslant h+1$.

These linear extensions reverse all incomparable pairs (x, y) of P such that $x, y \in \mathrm{Up}(i)$ for some i with $2 \leqslant i \leqslant h+1$, and all incomparable pairs (x, y) of P with $x \in C$. The block form of the $2d_0$ linear extensions implies that any other incomparable pair (x, y) of P is also reversed unless $\mathrm{dn}(x) < \mathrm{dn}(y) < \mathrm{up}(x) < \mathrm{up}(y)$, that is, the pair (x, y) is dangerous. $\square$

In view of Lemma 3.4, we can focus on reversing only the dangerous incomparable pairs when attempting for a bound on $\dim(P)$. This is the starting point of the proof of Theorem 1.4 in the next sections. We conclude this section with two results that will not be used further in the paper: one asserting that $\dim(P) = O(h)$ whenever the convex subposets of P with bounded height have bounded dimension, and the other asserting that linear dependence on h is necessary.

Proposition 3.5. *If every convex subposet Q of P of height at most $4k-4$ satisfies $\dim(Q) \leqslant d_1$, then $\dim(P) \leqslant (h+1)d_1$.*

Proof. We apply Lemma 3.4 to reverse all the safe incomparable pairs of P using at most $2d_1$ linear extensions. For every dangerous incomparable pair (x, y) of P, we have $x, y \in \mathrm{Ar}(\mathrm{dn}(y), \mathrm{dn}(y)+1)$, where $1 \leqslant \mathrm{dn}(y) \leqslant h-1$. For $1 \leqslant i \leqslant h-1$, it follows from Lemma 3.2 that $\dim(\mathrm{Ar}(i, i+1)) \leqslant d_1$, and we can extend any linear extension of $\mathrm{Ar}(i, i+1)$ witnessing the dimension to a linear extension of P. This way, we obtain a set of at most $(h-1)d_1$ linear extensions of P reversing all the dangerous incomparable pairs of P, and the proof is complete. $\square$

Proposition 3.6. *For every $n \geqslant 2$, there is a poset P excluding $\mathbf{3} + \mathbf{3}$ such that $\dim(P) \geqslant n$ and every convex subposet Q of P satisfies $\dim(Q) \leqslant h(Q) + 1$, where $h(Q)$ denotes the height of Q.*

Proof. The poset P consists of points $a_1, \ldots, a_n, b_1, \ldots, b_n, c_0, \ldots, c_n$ with the following cover relations: $c_0 < \cdots < c_n$, $a_i < c_i$ and $c_{i-1} < b_i$ for $1 \leqslant i \leqslant n$, and $a_i < b_j$ for $1 \leqslant j < i \leqslant n$. Every chain in P of size at least 3 contains a point of the form c_i, so P excludes $\mathbf{3} + \mathbf{3}$. The subposet of P induced on $a_1, \ldots, a_n, b_1, \ldots, b_n$ is isomorphic to the standard example S_n, so $\dim(P) \geqslant n$. Every convex subposet of P of height 1 is an antichain and therefore has dimension at most 2. Now, let $h \geqslant 2$, and let Q be a convex subposet of P of height h. To prove $\dim(Q) \leqslant h+1$, we can assume without loss of generality that Q is a maximal subposet of P of height h. It follows that Q is comprised of points $a_{i+1}, \ldots, a_n, b_1, \ldots, b_{i+h-1}, c_i, \ldots, c_{i+h-1}$ for some $i \in \{0, \ldots, n-h+1\}$. It is easy to check that the following $h+1$ linear extensions of Q form a realizer of Q:

$$a_{i+1} < \cdots < a_n < b_i < \cdots < b_1 < c_i < b_{i+1} < c_{i+1} < \cdots < b_{i+h-1} < c_{i+h-1},$$

$$c_i < a_{i+1} < c_{i+1} < \cdots < a_{i+j-1} < c_{i+j-1} < a_{i+j+1} < \cdots < a_n < b_{i+j} < a_{i+j}$$
$$< b_1 < \cdots < b_{i+j-1} < c_{i+j} < b_{i+j+1} < c_{i+j+1} < \cdots < b_{i+h-1} < c_{i+h-1} \quad \text{for } 1 \leqslant j \leqslant h-1,$$

$$c_i < a_{i+1} < c_{i+1} < \cdots < a_{i+h-1} < c_{i+h-1} < a_n < \cdots < a_{i+h} < b_1 < \cdots < b_{i+1}. \qquad \square$$

4. Proof of the main theorem

For the proof of Theorem 1.4, we fix a poset P that excludes $\mathbf{k} + \mathbf{k}$, where $k \geqslant 2$, and has planar cover graph, and we attempt to partition the set $\mathrm{Inc}(P)$ of incomparable pairs of P into a bounded number of reversible subsets, where the bound depends only on k. Following the notation and terminology of the preceding section, let h be the height of P, and let $C = \{c_1 < \cdots < c_h\}$ be a chain in P of size h. We use operators Dn, Up, and Ar as in the preceding section to denote appropriate convex subposets of $P - C$. We use operators dn and up in a different way than in the preceding section, namely, to refer to points of the chain C rather than integer numbers:

- for $z \in P$, we let $\mathrm{dn}(z) = c_i$ if $c_i \leqslant z$ and i is greatest in $\{1, \ldots, h\}$ with this property,
- for $z \in P$, we let $\mathrm{up}(z) = c_i$ if $z \leqslant c_i$ and i is least in $\{1, \ldots, h\}$ with this property.

We can write $\mathrm{dn}(z)$ or $\mathrm{up}(z)$ only for points $z \in P$ for which the respective point in C exists.

By Theorem 1.3, there is an integer d_1 such that every poset P' with planar cover graph and with height at most $4k - 4$ satisfies $\dim(P') \leqslant d_1$. It follows that every convex subposet of P of height at most $4k - 4$ has dimension at most d_1. For $1 \leqslant i \leqslant h-1$, in particular, Lemma 3.2 yields $\dim(\mathrm{Ar}(i, i+1)) \leqslant d_1$, so there is a coloring $\phi_i \colon \mathrm{Inc}(\mathrm{Ar}(i, i+1)) \to \{1, \ldots, d_1\}$ such that for each color $\gamma \in \{1, \ldots, d_1\}$, the set of incomparable pairs of $\mathrm{Ar}(i, i+1)$ that are assigned color γ by ϕ_i is reversible. We fix the integer d_1 and the colorings ϕ_i for $1 \leqslant i \leqslant h-1$ for this and the following sections.

By Lemma 3.4, we can reverse all the safe incomparable pairs of P using at most $2d_1$ linear extensions, and the remaining challenge is to reverse the dangerous incomparable pairs of P. For every dangerous incomparable pair (a, b) of P, we have $c_1 < b$ and $a < c_h$ in P, so the points $\mathrm{dn}(b)$ and $\mathrm{up}(a)$ of the chain C are defined. For any set S of dangerous incomparable pairs of P, let $A(S)$ denote the set of points $a \in P$ for which there is a point $b \in P$ with $(a, b) \in S$, and let $B(S)$ denote the set of points $b \in P$ for which there is a point $a \in P$ with $(a, b) \in S$.

Recall from the preceding section that for every dangerous incomparable pair (a, b) of P, we have $a, b \in \mathrm{Ar}(i, i+1) \cap \cdots \cap \mathrm{Ar}(j-1, j)$, where $c_i = \mathrm{dn}(b)$ and $c_j = \mathrm{up}(a)$. In particular, the dangerous incomparable pairs (a, b) of P with $\mathrm{dn}(b) = c_1$ belong to $\mathrm{Inc}(\mathrm{Ar}(1, 2))$ and thus can be reversed using d_1 linear extensions. Similarly, the dangerous incomparable pairs (a, b) of P with $\mathrm{up}(a) = c_h$ can be reversed using d_1 linear extensions.

Let S_0 be the set of dangerous incomparable pairs (a, b) of P with $\mathrm{dn}(b) > c_1$ and $\mathrm{up}(a) < c_h$ in P. It remains to partition the set S_0 into a bounded number of reversible subsets. By convention, we will write S only to denote some subset of the set S_0, we will write a, a', a_α, etc. only to denote a point from $A(S)$, and we will write b, b', b_α, etc. only to denote a point from $B(S)$.

For a set $S \subseteq S_0$, we call a pair $(a, b) \in S$ *left-safe* with respect to S if there is no point $b' \in B(S)$ with $a \leqslant b'$ and $\mathrm{dn}(b') < \mathrm{dn}(b)$ in P. The following lemma plays an important role in our argument.

Lemma 4.1. *For a set $S \subseteq S_0$, if S' is the set of pairs in S that are left-safe with respect to S, then $\dim(S') \leqslant d_1$.*

Proof. For a color $\gamma \in \{1, \ldots, d_1\}$, let $S'(\gamma)$ be the subset of S' consisting of all pairs $(a, b) \in S'$ such that $\phi_i(a, b) = \gamma$, where $c_i = \mathrm{dn}(b)$. It is enough to prove that the set $S'(\gamma)$ is reversible for every $\gamma \in \{1, \ldots, d_1\}$. Suppose to the contrary that the set $S'(\gamma)$ is not reversible for some $\gamma \in \{1, \ldots, d_1\}$. This means that $S'(\gamma)$ contains an alternating cycle $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^s$, where $s \geqslant 2$. For every $\alpha \in \{1, \ldots, s\}$, we have $a_\alpha \leqslant b_{\alpha+1}$ and therefore $\mathrm{dn}(b_\alpha) \leqslant \mathrm{dn}(b_{\alpha+1})$ in P, because the pair (a_α, b_α) is left-safe with respect to S. Since the latter inequality holds for every $\alpha \in \{1, \ldots, s\}$, there is a point $c_i \in C$ with $1 \leqslant i \leqslant h - 1$ such that $c_i = \mathrm{dn}(b_\alpha)$ for every $\alpha \in \{1, \ldots, s\}$. This implies that $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^s$ is a monochromatic alternating cycle in $\mathrm{Inc}(\mathrm{Ar}(i, i+1))$, which is a contradiction. $\square$

Let G denote the cover graph of P, which is a planar graph. We fix a plane straight-line drawing of G, that is, a drawing of G in the plane using non-crossing straight-line segments for edges. We assume, without loss of generality, that the least point c_1 of the maximum chain C lies on the outer face of the drawing.

A *witnessing path* for a pair (x, y) with $x \leqslant y$ in P is a path $u_0 \cdots u_r$ in G such that $x = u_0 < \cdots < u_r = y$ in P (in particular, u_i is covered by u_{i+1} in P for $0 \leqslant i \leqslant r - 1$). It is clear that every comparable pair (x, y) with $x \leqslant y$ in P has at least one witnessing path. For the purpose of our proof, it is convenient to fix one witnessing path, to be denoted by $W(x, y)$, for each pair (x, y) with $x \leqslant y$ in P. However, we need to choose the paths $W(x, y)$ in a consistent way, which is achieved in the following proposition.

Proposition 4.2. *There is a function $(x, y) \mapsto W(x, y)$ that maps each pair (x, y) with $x \leqslant y$ in P to a witnessing path $W(x, y)$ for (x, y) in such a way that the following holds:*

(1) *if $x \leqslant c \leqslant y$ in P and $c \in C$, then $W(x, y)$ passes through c; in particular, if $\mathrm{up}(x)$ and $\mathrm{dn}(y)$ are defined and $\mathrm{up}(x) \leqslant \mathrm{dn}(y)$ in P, then $W(x, y)$ passes through $\mathrm{up}(x)$ and $\mathrm{dn}(y)$;*

(2) *if $x_1 \leqslant x_2 \leqslant y_2 \leqslant y_1$ in P and $W(x_1, y_1)$ passes through x_2 and y_2, then $W(x_2, y_2)$ is the subpath of $W(x_1, y_1)$ from x_2 to y_2.*

Proof. Assume some (arbitrary) total order $\prec$ on the points of P. The order $\prec$ extends naturally to a lexicographic total order $\prec_{\mathrm{lex}}$ on finite sequences of points of P as follows:

- the empty sequence is the least element in $\prec_{\mathrm{lex}}$;
- for any two non-empty sequences $u_0 \cdots u_r$ and $v_0 \cdots v_s$ of points of P, we have $u_0 \cdots u_r \prec_{\mathrm{lex}} v_0 \cdots v_s$ if and only if $u_0 < v_0$ or $u_0 = v_0$ and $u_1 \cdots u_r \prec_{\mathrm{lex}} v_1 \cdots v_s$.

For every pair (x, y) with $x \leqslant y$ in P, let $W(x, y)$ be the $\prec_{\mathrm{lex}}$-minimum witnessing path among all witnessing paths for (x, y) passing through all points $c \in C$ with $x \leqslant c \leqslant y$ in P. It is clear that the paths $W(x, y)$ so defined satisfy both conditions of the proposition. $\square$

The *length* of a witnessing path $W(x, y)$, denoted by $\|W(x, y)\|$, is the number of edges in $W(x, y)$. It follows from Lemmas 3.1 and 3.2 that the following kinds of witnessing paths have bounded size:

- $\|W(\mathrm{dn}(z), z)\| \leqslant 2k - 2$ for any $z \in P$,
- $\|W(z, \mathrm{up}(z))\| \leqslant 2k - 2$ for any $z \in P$,
- $\|W(z, w)\| \leqslant 4k - 5$ for any $z, w \in P - C$ with $\mathrm{dn}(w) < \mathrm{up}(z)$.

For $i \in \{2, \ldots, h-1\}$, we classify every edge of the form $c_i z$ of G, where $z \notin C$, as a

- *left-edge* if the edges $c_i c_{i-1}$, $c_i z$, $c_i c_{i+1}$ occur in this order clockwise around c_i,
- *right-edge* if the edges $c_i c_{i-1}$, $c_i z$, $c_i c_{i+1}$ occur in this order counterclockwise around c_i.

Since $c_2 \leqslant \mathrm{dn}(b) < \mathrm{up}(a) \leqslant c_{h-1}$ in P for any $(a, b) \in S_0$, the above yields a partition of S_0 into four classes S_{LL}, S_{LR}, S_{RL}, and S_{RR} according to how the last edge of $W(a, \mathrm{up}(a))$ and the first edge of $W(\mathrm{dn}(b), b)$ are classified. That is, for every $(a, b) \in S_0$, we have

- $(a, b) \in S_{LL} \cup S_{LR}$ if the last edge of $W(a, \mathrm{up}(a))$ is a left-edge,
- $(a, b) \in S_{RL} \cup S_{RR}$ if the last edge of $W(a, \mathrm{up}(a))$ is a right-edge,
- $(a, b) \in S_{LL} \cup S_{RL}$ if the first edge of $W(\mathrm{dn}(b), b)$ is a left-edge,
- $(a, b) \in S_{LR} \cup S_{RR}$ if the first edge of $W(\mathrm{dn}(b), b)$ is a right-edge.

To complete the proof of Theorem 1.4, we will establish the following:

- $\dim(S_{LL}) = O(k^3 d_1)$ and $\dim(S_{RR}) = O(k^3 d_1)$, in the next section,
- $\dim(S_{LR}) = O(k^3 + k^2 d_1)$ and $\dim(S_{RL}) = O(k^3 + k^2 d_1)$, in Section 6.

This allows us to conclude that $\dim(S_0) = O(k^3 d_1)$ and thus $\dim(P) = O(k^3 d_1)$.

5. Same-side dangerous pairs

In this section, we show that $\dim(S_{LL}) = O(k^3 d_1)$ and $\dim(S_{RR}) = O(k^3 d_1)$. We present the argument only for S_{LL}, and the argument for S_{RR} is symmetric. To simplify the notation used in this portion of the proof, we (temporarily) set $S = S_{LL}$.

Recall that $c_2 < b$ in P for every $b \in B(S)$, as S contains only dangerous pairs with $\mathrm{dn}(b) > c_1$ in P. Furthermore, the choice of the witnessing paths guarantees that

- for any $b \in B(S)$, the common part of $W(c_1, b)$ with $W(c_1, c_h)$ is $W(c_1, \mathrm{dn}(b))$,
- for any $b, b' \in B(S)$, the common part of $W(c_1, b)$ and $W(c_1, b')$ is $W(c_1, z)$ for some $z \in P$.

Therefore, the union of $W(c_1, c_h)$ and the witnessing paths $W(c_1, b)$ over all $b \in B(S)$ forms a tree, which we call the *blue tree* and denote by BT. The points, edges, and paths in BT are called *blue points*, *blue edges*, and *blue paths*. For any two blue points u and v, let $\mathrm{BT}(u, v)$ denote the unique path in BT (made of blue points and blue edges) between u and v.

Let $\prec$ be the total order on the blue points determined by carrying out a clockwise-oriented depth-first search of BT from c_1 and setting $u \prec v$ when u is visited before v for the first time. In other words, for any two blue points u and v such that $v \notin W(c_1, u)$, if the common part of $W(c_1, u)$ and $W(c_1, v)$ is $W(c_1, z)$ (where $z \neq c_1, v$) and either $u \in W(c_1, v)$ or the blue paths $\mathrm{BT}(z, c_1)$ (which is opposite to $W(c_1, z)$), $W(z, u)$, and $W(z, v)$ go out of z in this order clockwise, then we declare $u \prec v$. It follows that $u \prec v$ for any two blue points u and v such that $\mathrm{dn}(u) < \mathrm{dn}(v)$ in P. The greatest point in this order is c_h. Figure 2 illustrates how the blue tree might appear.

Let $a \in A(S)$. Following [16], we call a blue point v *special* for a if the following holds:

- $a \leqslant v$ in P,
- $a \not\leqslant u$ in P whenever $u \in W(c_1, v)$ and $u \neq v$.

Let $\mathrm{Spec}(a)$ denote the set of all blue points v that are special for a (see Figure 2). Any two distinct points $v, v' \in \mathrm{Spec}(a)$ satisfy $v' \notin W(c_1, v)$ and $v \notin W(c_1, v')$, although they may be comparable in P. The set $\mathrm{Spec}(a)$ inherits the order $\prec$ from BT. Since $\mathrm{up}(a) \in \mathrm{Spec}(a)$, we have $\mathrm{Spec}(a) \neq \emptyset$. In fact, $\mathrm{up}(a)$ is the greatest point in the order $\prec$ on $\mathrm{Spec}(a)$.
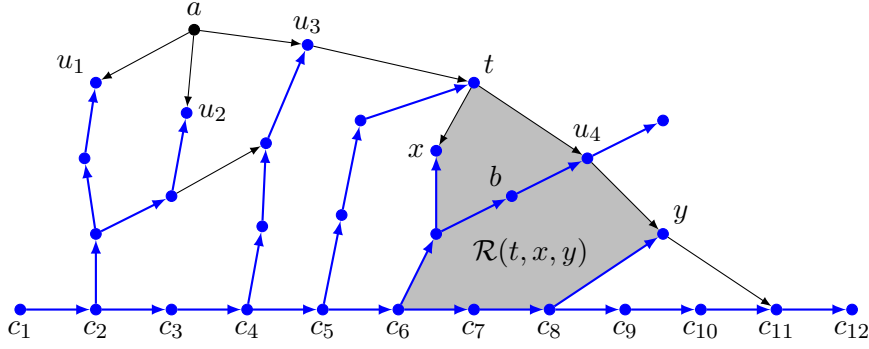
FIGURE 2. Illustration for the concepts introduced in Section 5; arrows on the cover graph edges point according to the increasing direction of P; $\mathrm{Spec}(a) = \{u_1 \prec u_2 \prec u_3 \prec t \prec x \prec u_4 \prec y \prec c_{11}\}$; $\mathrm{Spec}'(a) = \{u_1 \prec x \prec y \prec c_{11}\}$

The following straightforward property is stated for emphasis.

Proposition 5.1. *If $a \in A(S)$, $b \in B(S)$, and $a \leqslant b$ in P, then there is $v \in \mathrm{Spec}(a)$ with $v \in W(c_1, b)$ and $v \leqslant b$ in P.*

Now, suppose $a \in A(S)$, $x, y \in \mathrm{Spec}(a)$, and $x \prec y$ in BT. Every point $t \in P$ such that $a \leqslant t \leqslant x, y$ in P and t is the only common point of $W(t, x)$ and $W(t, y)$ gives rise to a region in the plane, denoted by $\mathcal{R}(t, x, y)$, whose boundary is the simple closed curve formed by the paths $\mathrm{BT}(x, y)$, $W(t, x)$, and $W(t, y)$. The boundary of $\mathcal{R}(t, x, y)$ traversed counterclockwise goes from t to x along $W(t, x)$, then to y along $\mathrm{BT}(x, y)$, and then back to t along $W(t, y)$ in the reverse direction (see Figure 2). It is possible that $t = x$ or $t = y$. If $a \neq t$, then a can lie either in the interior or in the exterior of $\mathcal{R}(t, x, y)$. Any region of the form $\mathcal{R}(t, x, y)$ where $x, y \in \mathrm{Spec}(a)$ and $a \leqslant t$ in P is called an *a-region*. (For such an a-region, it is *not* required that t lies on the paths $W(a, x)$ and $W(a, y)$.)

Let T_0 denote the subset of S consisting of all pairs that are left-safe with respect to S. By Lemma 4.1, we have $\dim(T_0) \leqslant d_1$. To complete the proof, we will show that $\dim(S - T_0) \leqslant \binom{2k-1}{2}(4k-4)d_1$ by partitioning $S - T_0$ into reversible subsets of the form $T(h_1, h_2, n, \gamma)$, where the four parameters are integers with $0 \leqslant h_1 < h_2 \leqslant 2k - 2$, $0 \leqslant n \leqslant 4k - 5$, and $1 \leqslant \gamma \leqslant d_1$. Membership in these sets will be determined in two stages. In the first stage, we will partition $S - T_0$ into subsets of the form $T(h_1, h_2)$ where $0 \leqslant h_1 < h_2 \leqslant 2k - 2$. In the second stage, for each pair (h_1, h_2) with $0 \leqslant h_1 < h_2 \leqslant 2k - 2$, we will further partition $T(h_1, h_2)$ into reversible subsets of the form $T(h_1, h_2, n, \gamma)$, where $0 \leqslant n \leqslant 4k - 5$ and $1 \leqslant \gamma \leqslant d_1$.

To describe the first partition, we need some more definitions. For a blue point v, let $h(v) = \|W(\mathrm{dn}(v), v)\|$. We have $0 \leqslant h(v) \leqslant 2k - 2$ for every blue point v, and we have $h(v) = 0$ if and only if $v \in C$. For a point $a \in A(S - T_0)$, let

$$\mathrm{Spec}'(a) = \{u \in \mathrm{Spec}(a) : \text{for every } v \in \mathrm{Spec}(a), \text{ if } v \prec u \text{ in BT, then } h(v) > h(u)\}$$

(see Figure 2). The set $\mathrm{Spec}'(a)$ inherits the order $\prec$ from $\mathrm{Spec}(a)$. The sequence of numbers $h(v)$ for the points $v \in \mathrm{Spec}'(a)$ considered in the order $\prec$ is strictly decreasing. The first point in $\mathrm{Spec}'(a)$ is the first point in $\mathrm{Spec}(a)$, and the last point in $\mathrm{Spec}'(a)$ is $\mathrm{up}(a)$.

Now, let $(a, b) \in S - T_0$. Since (a, b) is not left-safe with respect to S, there is a point $u \in \mathrm{Spec}(a)$ with $\mathrm{dn}(u) < \mathrm{dn}(b)$ in P. Consequently, there is a point $u \in \mathrm{Spec}'(a)$ with $\mathrm{dn}(u) < \mathrm{dn}(b)$ in P and thus $u \prec b$ in BT. We also have $b \prec \mathrm{up}(a) \in \mathrm{Spec}'(a)$ and $b \notin \mathrm{Spec}'(a)$. If follows that there are two points $x, y \in \mathrm{Spec}'(a)$ consecutive in the order $\prec$ on $\mathrm{Spec}'(a)$ such

that $x \prec b \prec y$ in BT. Let $h_1 = h(y)$ and $h_2 = h(x)$, so that $0 \leqslant h_1 < h_2 \leqslant 2k - 2$. We put the pair (a, b) to the set $T(h_1, h_2)$ of the first partition.

To complete the proof, it remains to show that $\dim(T(h_1, h_2)) \leqslant (4k - 4)d_1$ for each pair of integers (h_1, h_2) with $0 \leqslant h_1 < h_2 \leqslant 2k - 2$. To this end, we fix an arbitrary pair (h_1, h_2) of this form and show that $\dim(T(h_1, h_2)) \leqslant (4k - 4)d_1$ by explaining how $T(h_1, h_2)$ can be partitioned into reversible sets of the form $T(h_1, h_2, n, \gamma)$, where $0 \leqslant n \leqslant 4k - 5$ and $1 \leqslant \gamma \leqslant d_1$. This task will require some preliminary work.

In the reasoning used above to put a pair (a, b) in $T(h_1, h_2)$, the points x and y have been defined depending on both a and b. Now, however, for any point $a \in A(T(h_1, h_2))$, there are a unique point $y \in \mathrm{Spec}'(a)$ with $h(y) = h_1$ and a unique point $x \in \mathrm{Spec}'(a)$ with $h(x) = h_2$. Let $y(a)$ and $x(a)$ denote these points, respectively, for any $a \in A(T(h_1, h_2))$.

Lemma 5.2. *Let* $(a, b), (a', b') \in T(h_1, h_2)$ *be such that* $a' \leqslant b$ *and* $\mathrm{dn}(y(a)) < \mathrm{dn}(y(a'))$ *in* P. *Then, for every a-region of the form* $\mathcal{R}(x(a), y(a), t)$, *there is an a'-region of the form* $\mathcal{R}(x(a'), y(a'), t')$ *such that* $\|W(t', x(a'))\| < \|W(t, x(a))\|$.

Proof. Fix an a-region $\mathcal{R}(t, x(a), y(a))$, where $a \leqslant t$ in P. For simplicity of notation, let $x = x(a)$, $y = y(a)$, $\mathcal{R} = \mathcal{R}(t, x(a), y(a))$, $x' = x(a')$, and $y' = y(a')$. We prove the lemma in several steps, establishing the following claims:

(1) y' lies in the exterior of $\mathcal{R}$,
(2) b lies in the interior or on the boundary of $\mathcal{R}$,
(3) $a' \not\leqslant y$ in P,
(4) if $a' \leqslant u$ in P and $u \in W(c_1, x)$, then $u = x$,
(5) a' lies in the interior of $\mathcal{R}$,
(6) there is a point t' on $W(t, x)$ such that $a' \leqslant t \leqslant x, y'$ in P, $t' \neq t$, and t' is the only common point of $W(t', x)$ and $W(t', y')$,
(7) $x' = x$.

The conclusion of the lemma then follows directly from (6) and (7).

For the proof of (1), suppose y' does not lie in the exterior of $\mathcal{R}$. Since $\mathrm{dn}(y) < \mathrm{dn}(y')$ in P, the first edge of the path $W(\mathrm{dn}(y), y')$ is part of the chain C and lies in the exterior of $\mathcal{R}$. Therefore, the path $W(\mathrm{dn}(y), y')$ crosses the boundary of $\mathcal{R}$ at some point u other than $\mathrm{dn}(y)$. It follows that $u \leqslant x$ or $u \leqslant y$ in P, which contradicts the fact that $\mathrm{dn}(x) \leqslant \mathrm{dn}(y) < \mathrm{dn}(y') = \mathrm{dn}(u)$ in P.

For the proof of (2), suppose b lies in the exterior of $\mathcal{R}$. Since $x \prec b \prec y$ in BT, the blue path $W(\mathrm{dn}(b), b)$ enters the interior of $\mathcal{R}$ with the first edge that is not common with $W(\mathrm{dn}(x), x)$ nor $W(\mathrm{dn}(y), y)$. Therefore, it must exit the interior of $\mathcal{R}$ through a point u on $W(t, x)$ or $W(t, y)$. It follows that $a \leqslant u \leqslant b$ in P, which is a contradiction.

For the proof of (3), suppose $a' \leqslant y$ in P. It follows that a' has a special point u on the blue path $W(c_1, y)$. We have $h(u) \leqslant h(y) = h_1 = h(y')$. However, we also have $\mathrm{dn}(u) \leqslant \mathrm{dn}(y) < \mathrm{dn}(y')$ in P and thus $u \prec y'$ in BT. This is a contradiction with $y' \in \mathrm{Spec}'(a')$.

For the proof of (4), suppose $a' \leqslant u$ in P, $u \in W(c_1, x)$, and $u \neq x$. Assume without loss of generality that u is a special point for a'. We have $h(u) < h(x) = h_2 = h(x')$. We also have $\mathrm{dn}(u) \leqslant \mathrm{dn}(y) < \mathrm{dn}(y')$ in P and thus $u \prec y'$ in BT, which implies $h(u) > h(y')$, as $y' \in \mathrm{Spec}'(a')$. However, since x' and y' are consecutive in the order $\prec$ on $\mathrm{Spec}'(a')$, no point $u \in \mathrm{Spec}(a')$ can satisfy $h(x') > h(u) > h(y')$, which is a contradiction.

For the proof of (5), suppose a' does not lie in the interior of $\mathcal{R}$. By (2), b is not in the exterior of $\mathcal{R}$, so the path $W(a', b)$ intersects the boundary of $\mathcal{R}$ at some point u. Since $a' \leqslant u$ in P, it follows from (3) and (4) that u lies on $W(t, x)$. Thus $a \leqslant t \leqslant u \leqslant b$ in P, which is a contradiction.

By (5) and (1), a' is in the interior and y' is in the exterior of $\mathcal{R}$, so the path $W(a', y')$ crosses the boundary of $\mathcal{R}$. For the proof of (6), let t' be the last common point of $W(a', y')$ with the boundary of $\mathcal{R}$ in the order along $W(a', y')$. Since $a' \leqslant t'$ in P, it follows from (3) and (4) that t' lies on $W(t, x)$ and $t' \neq t$. Furthermore, it follows from the choice of t' that it is the only common point of $W(t', x)$ and $W(t', y')$. Therefore, t' satisfies all the conditions of (6).

It remains to prove (7). Suppose $x' \neq x$. The fact that $a' \leqslant t' \leqslant x$ in P and (4) imply that $x \in \mathrm{Spec}(a')$. Since $h(x) = h_2 = h(x')$ and $x' \in \mathrm{Spec}'(a')$, we have $x' \prec x$ in BT. Therefore, the first edge of the path $W(c_1, x')$ that is not common with $W(c_1, x)$ lies in the exterior of $\mathcal{R}$. If x' is not in the exterior of $\mathcal{R}$, then the path $W(c_1, x')$ crosses $W(t, x)$ or $W(t, y)$ and thus $a \leqslant t \leqslant x'$ in P. If x' is not in the interior of $\mathcal{R}$, then the path $W(a', x')$ crosses the boundary of $\mathcal{R}$ at some point u; again, since $a' \leqslant u$ in P, it follows from (3) and (4) that u lies on $W(t, x)$ and thus $a \leqslant t \leqslant u \leqslant x'$ in P. In either case, we have concluded that $a \leqslant x'$ in P. Consequently, there is a point $v \in \mathrm{Spec}(a)$ on the path $W(c_1, x')$. It follows that $v \prec x' \prec x$ in BT and $h(v) \leqslant h(x') = h(x)$, which is a contradiction to $x \in \mathrm{Spec}'(a)$. Thus $x' = x$. $\qquad\square$

For incomparable pairs $(a, b), (a', b') \in T(h_1, h_2)$, let $(a, b) \to (a', b')$ denote that $a' \leqslant b$ and $\mathrm{dn}(y(a)) < \mathrm{dn}(y(a'))$ in P. For a pair $(a, b) \in T(h_1, h_2)$, let $\ell(a, b)$ denote the greatest integer s for which there is a sequence $\{(a_\alpha, b_\alpha)\}_{\alpha=0}^s \subseteq T(h_1, h_2)$ such that $(a_0, b_0) \to \cdots \to (a_s, b_s) = (a, b)$.

Proposition 5.3. *For every incomparable pair $(a, b) \in T(h_1, h_2)$, the following holds:*

(1) $0 \leqslant \ell(a, b) \leqslant 4k - 5$,
(2) *if $(a, b) \to (a', b') \in T(h_1, h_2)$, then $\ell(a, b) < \ell(a', b')$.*

Proof. Fix a sequence $\{(a_\alpha, b_\alpha)\}_{\alpha=0}^s \subseteq T(h_1, h_2)$ such that $(a_0, b_0) \to \cdots \to (a_s, b_s) = (a, b)$, where $s = \ell(a, b)$. To see (1), choose any a_0-region of the form $\mathcal{R}(t_0, x(a_0), y(a_0))$, and apply Lemma 5.2 repeatedly to obtain a sequence of regions $\{\mathcal{R}(t_\alpha, x(a_\alpha), y(a_\alpha))\}_{\alpha=0}^s$ such that $\|W(t_0, x(a_0))\| > \cdots > \|W(t_s, x(a_s))\| \geqslant 0$; these inequalities are possible only when $s \leqslant \|W(t_0, x(a_0))\| \leqslant 4k - 5$, where the latter inequality follows from the fact that $\mathrm{dn}(x(a_0)) < \mathrm{up}(a_0) \leqslant \mathrm{up}(t_0)$. To see (2), set $(a_{s+1}, b_{s+1}) = (a', b')$ and observe that the sequence $\{(a_\alpha, b_\alpha)\}_{\alpha=0}^{s+1}$ witnesses $\ell(a', b') \geqslant s + 1$. $\qquad\square$

We partition the set $T(h_1, h_2)$ into subsets of the form $T(h_1, h_2, n, \gamma)$, putting every pair $(a, b) \in T(h_1, h_2)$ into the set $T(h_1, h_2, n, \gamma)$ such that $n = \ell(a, b)$ and γ is determined as follows:

- if $h_1 > 0$, then $\mathrm{dn}(b) \leqslant \mathrm{dn}(y(a)) < \mathrm{up}(a)$ in P, so (a, b) is an incomparable pair of $\mathrm{Ar}(i, i+1)$, where $c_i = \mathrm{dn}(y(a))$, and we let $\gamma = \phi_i(a, b)$;
- if $h_1 = 0$, then $y(a) = \mathrm{up}(a)$, so (a, b) is an incomparable pair of $\mathrm{Ar}(j-1, j)$, where $c_j = \mathrm{up}(a)$, and we let $\gamma = \phi_{j-1}(a, b)$.

It follows that $0 \leqslant n \leqslant 4k - 5$ (by Proposition 5.3 (1)) and $1 \leqslant \gamma \leqslant d_1$. To complete the proof, it remains to show the following.

Proposition 5.4. *Every set $T(h_1, h_2, n, \gamma)$ is reversible.*

Proof. Suppose not. Pick an alternating cycle $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^s$ contained in $T(h_1, h_2, n, \gamma)$, where $s \geqslant 2$. For $1 \leqslant \alpha \leqslant s$, since $\ell(a_\alpha, b_\alpha) = \ell(a_{\alpha+1}, b_{\alpha+1})$, it follows from Proposition 5.3 (2) that $(a_\alpha, b_\alpha) \not\to (a_{\alpha+1}, b_{\alpha+1})$. This and $a_{\alpha+1} < b_\alpha$ in P yield $\mathrm{dn}(y(a_\alpha)) \leqslant \mathrm{dn}(y(a_{\alpha+1}))$ in P, for $1 \leqslant \alpha \leqslant s$. This implies that there is $c \in C$ with $\mathrm{dn}(y(a_\alpha)) = c$ for $1 \leqslant \alpha \leqslant s$. If $h_1 > 0$ and $c_i = c$, then $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^s$ is an alternating cycle in $\mathrm{Ar}(i, i+1)$, and $\phi_i(a_\alpha, b_\alpha) = \gamma$ for $1 \leqslant \alpha \leqslant s$, which is a contradiction. If $h_1 = 0$ and $c_j = c$, then $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^s$ is an alternating cycle in $\mathrm{Ar}(j-1, j)$, and $\phi_{j-1}(a_\alpha, b_\alpha) = \gamma$ for $1 \leqslant \alpha \leqslant s$, which is again a contradiction. $\qquad\square$

We have partitioned S into $O(k^3 d_1)$ reversible subsets of the form $T(h_1, h_2, n, \gamma)$ with $0 \leqslant h_1 < h_2 \leqslant 2k - 2$, $0 \leqslant n \leqslant 4k - 5$, and $1 \leqslant \gamma \leqslant d_1$, thus proving that $\dim(S) = O(k^3 d_1)$.

6. Opposite-side dangerous pairs

In this section, we show that $\dim(S_{LR}) = O(k^2 d_1 + k^3)$ and $\dim(S_{RL}) = O(k^2 d_1 + k^3)$. We present the argument only for S_{RL}, and the argument for S_{LR} is symmetric. Recall that the set S_{RL} contains only dangerous incomparable pairs (a, b) of P with $\mathrm{dn}(b) > c_1$ and $\mathrm{up}(a) < c_h$ in P. We begin by setting $S = S_{RL}$. As the argument proceeds, the meaning of S changes, but the "new" set S is always a subset of the "old" set S. Each time the meaning of S changes, the target upper bound on $\dim(S)$ is adjusted accordingly.

In the argument given thus far, our main emphasis has been on classifying incomparable pairs. Now, we want to pay attention to comparable pairs. When $a \in A(S)$, $b \in B(S)$, and $a \leqslant b$ in P, we call (a, b) a *strong comparable pair* if $\mathrm{up}(a) \leqslant \mathrm{dn}(b)$ in P, and we call (a, b) a *weak comparable pair* if $\mathrm{up}(a) > \mathrm{dn}(b)$ in P. When a comparable pair (a, b) is weak, the witnessing path $W(a, b)$ has length at most $4k - 5$ and does not intersect the chain C.

Proposition 6.1. *For every strict alternating cycle $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^s$ in S, where $s \geqslant 2$, there is at most one index $\alpha \in \{1, \dots, s\}$ such that $(a_\alpha, b_{\alpha+1})$ is a strong comparable pair.*

Proof. Suppose there are two distinct indices $\alpha, \beta \in \{1, \dots, s\}$ such that the comparable pairs $(a_\alpha, b_{\alpha+1})$ and $(a_\beta, b_{\beta+1})$ are strong. Without loss of generality, we have $\mathrm{up}(a_\alpha) \leqslant \mathrm{up}(a_\beta)$ in P. It follows that $a_\alpha < \mathrm{up}(a_\alpha) \leqslant \mathrm{up}(a_\beta) \leqslant \mathrm{dn}(b_{\beta+1}) < b_{\beta+1}$ in P, which contradicts the assumption that the alternating cycle $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^s$ is strict. $\qquad\square$

When $T \subseteq S$, let $Q(T)$ denote the set of weak comparable pairs (a, b) such that $a \in A(T)$ and $b \in B(T)$. By Proposition 6.1, if $Q(T) = \emptyset$, then there is no strict alternating cycle in T and hence T is reversible. In particular, we can assume for the rest of this section that $Q(S) \neq \emptyset$.

For every weak comparable pair $(a, b) \in Q(S)$, the choice of the witnessing paths $W(a, \mathrm{up}(a))$ (which ends with a right-edge), $W(\mathrm{dn}(b), b)$ (which starts with a left-edge), and $W(a, b)$ guarantees the following:

- the common part of $W(a, \mathrm{up}(a))$ and $W(a, b)$ is $W(a, v)$ for some point v,
- the common part of $W(\mathrm{dn}(b), b)$ and $W(a, b)$ is $W(w, b)$ for some point w.

This yields a region $\mathcal{D}(a, b)$ in the plane whose boundary is a simple closed curve formed by the following four paths (see Figure 3):

- $W(\mathrm{dn}(b), \mathrm{up}(a))$, called the *middle* of $\mathcal{D}(a, b)$,
- $W(v, \mathrm{up}(a))$, called the *bottom* of $\mathcal{D}(a, b)$,
- $W(\mathrm{dn}(b), w)$, called the *top* of $\mathcal{D}(a, b)$,
- $W(v, w)$, called the *right side* of $\mathcal{D}(a, b)$.

Next, we determine a positive integer n and a sequence of weak comparable pairs $\{(a_i, b_i)\}_{i=1}^{n-1}$ using the following "greedy" procedure. We start by choosing $(a_1, b_1) \in Q(S)$ so as to maximize $\mathrm{dn}(b_1)$. Then, for $i \geqslant 2$, after (a_{i-1}, b_{i-1}) has been determined, we consider all pairs $(a, b) \in Q(S)$ such that the region $\mathcal{D}(a_{i-1}, b_{i-1})$ is contained in the interior of the region $\mathcal{D}(a, b)$, that is, $\mathrm{dn}(b) < \mathrm{up}(a) < \mathrm{dn}(b_{i-1})$ in P and the boundaries of $\mathcal{D}(a, b)$ and $\mathcal{D}(a_{i-1}, b_{i-1})$ are disjoint. If there is no such pair (a, b), then we set $n = i$ and the construction terminates. Otherwise, from all such pairs (a, b), we choose (a_i, b_i) so as to maximize $\mathrm{dn}(b_i)$ (see Figure 3).

For $1 \leqslant i \leqslant n - 1$, let $u_i = \mathrm{dn}(b_i)$ and $\mathcal{D}_i = \mathcal{D}(a_i, b_i)$. Furthermore, let $u_0 = c_h$ and $u_n = c_1$. It follows from the construction that $u_n < u_{n-1} < \cdots < u_1 < u_0$ in P. Define functions $M \colon A(S) \to \{1, \dots, n\}$ and $N \colon B(S) \to \{1, \dots, n\}$ as follows:
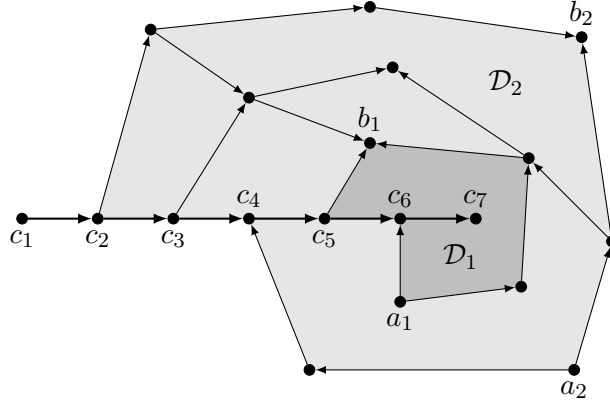
FIGURE 3. Illustration for the concepts used in Section 6; $\mathcal{D}_1 = \mathcal{D}(a_1, b_1)$ and $\mathcal{D}_2 = \mathcal{D}(a_2, b_2)$

- for $a \in A(S)$, $M(a)$ is the least positive integer j such that $u_j \leqslant \mathrm{up}(a)$;
- for $b \in B(S)$, $N(b)$ is the least positive integer j such that $u_j \leqslant \mathrm{dn}(b)$.

Let $q = 8k - 9$.

Proposition 6.2. *If $(a, b) \in Q(S)$, then $N(b) \leqslant M(a) + q$.*

Proof. Let $i = M(a)$ and $j = N(b)$. The graph G contains a path W_0 from $\mathrm{up}(a)$ to $\mathrm{dn}(b)$ which is formed by the bottom, the right side, and the top of $\mathcal{D}(a, b)$. The bottom and the top of $\mathcal{D}(a, b)$ have length at most $2k - 2$, while the right side has length at most $4k - 5$, so W_0 has length at most $(2k - 2) + (2k - 2) + (4k - 5) = 8k - 9$. In particular, there are at most $8k - 10$ points on W_0 distinct from $\mathrm{up}(a)$ and $\mathrm{dn}(b)$.

The path W_0 starts at the point $\mathrm{up}(a)$, which is in the interior of all $\mathcal{D}_{i+1}, \ldots, \mathcal{D}_{j-1}$, and ends at the point $\mathrm{dn}(b)$, which is in the exterior of all $\mathcal{D}_{i+1}, \ldots, \mathcal{D}_{j-1}$. Therefore, it must cross the boundaries of the regions $\mathcal{D}_{i+1}, \ldots, \mathcal{D}_{j-1}$ in $j - i - 1$ distinct points. This requires $j - i - 1 \leqslant 8k - 10$, so that $j \leqslant i + 8k - 9 = i + q$. □

For every $r \in \{0, \ldots, q-1\}$, let $S(r) = \{(a, b) \in S \colon M(a) \equiv r \pmod{q}\}$. Thus $S = \bigcup_{r=0}^{q-1} S(r)$. Since $q = O(k)$, it suffices to show that $\dim(S(r)) = O(kd_1 + k^2)$ for every $r \in \{0, \ldots, q-1\}$ to complete the proof. So we fix a value of r and update the meaning of S by setting $S = S(r)$. For every $m \in \{1, \ldots, n\}$ with $m \equiv r \pmod{q}$, let $S_m = \{(a, b) \in S \colon M(a) = N(b) = m\}$.

Proposition 6.3. *We have $\dim(S) \leqslant d_1 + \max\{\dim(S_m) \colon 1 \leqslant m \leqslant n, \ m \equiv r \pmod{M}\}$.*

Proof. Let $d_2 = \max\{\dim(S_m) \colon 1 \leqslant m \leqslant n, \ m \equiv r \pmod{q}\}$. For every $m \in \{1, \ldots, n\}$ with $m \equiv r \pmod{q}$, there is a coloring ψ_m that assigns a color $\psi_m(a, b) \in \{d_1 + 1, \ldots, d_1 + d_2\}$ to every incomparable pair $(a, b) \in S_m$ so that the set $\{(a, b) \in S_m \colon \psi_m(a, b) = \gamma\}$ is reversible for every color $\gamma \in \{d_1 + 1, \ldots, d_1 + d_2\}$.

Every pair $(a, b) \in S$ with $N(b) > M(a)$ is an incomparable pair of $\mathrm{Ar}(j - 1, j)$, where $c_j = u_{M(a)}$. Recall that ϕ_{j-1} is a coloring of $\mathrm{Inc}(\mathrm{Ar}(j - 1, j))$ that uses colors $\{1, \ldots, d_1\}$ and avoids monochromatic alternating cycles. For every $\gamma \in \{1, \ldots, d_1\}$, let $S(\gamma)$ be the subset of S consisting of all pairs (a, b) such that $N(b) > M(a)$ and $\phi_{j-1}(a, b) = \gamma$, where $c_j = u_{M(a)}$. For every $\gamma \in \{d_1 + 1, \ldots, d_1 + d_2\}$, let $S(\gamma)$ be the subset of S consisting of all pairs (a, b) such that $N(b) = M(a)$ and $\psi_{M(a)}(a, b) = \gamma$. Thus $S = \bigcup_{\gamma=1}^{d_1+d_2} S(\gamma)$.

We claim that the set $S(\gamma)$ is reversible for every $\gamma \in \{1, \ldots, d_1 + d_2\}$. Suppose not. Let $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^{s}$ be an alternating cycle contained in $S(\gamma)$, where $s \geqslant 2$. Suppose there

is $\alpha \in \{1, \ldots, s\}$ with $M(a_{\alpha+1}) > M(a_\alpha)$. Then $N(b_{\alpha+1}) \geqslant M(a_{\alpha+1}) \geqslant M(a_\alpha) + q$, as $M(a_{\alpha+1}) \equiv M(a_\alpha) \pmod{q}$. It follows that $(a_\alpha, b_{\alpha+1})$ is a weak comparable pair, but it violates Proposition 6.2. This shows that $M(a_1), \ldots, M(a_s)$ are all equal. Let $m = M(a_1) = \cdots = M(a_s)$. Consequently, $\{(a_\alpha, b_\alpha)\}_{\alpha=1}^s$ is an alternating cycle either in $\mathrm{Ar}(j-1, j)$, when $\gamma \in \{1, \ldots, d_1\}$ and $c_j = u_m$, or in S_m, when $\gamma \in \{d_1 + 1, \ldots, d_1 + d_2\}$. In both cases, this is a contradiction. $\qquad\square$

It remains to show that $\dim(S_m) = O(kd_1 + k^2)$ for $1 \leqslant m \leqslant n$. So we fix a value of m and update the meaning of S by setting $S = S_m$. It follows that

- $u_m < \mathrm{up}(a) < u_{m-1}$ in P for every $a \in A(S)$,
- $u_m \leqslant \mathrm{dn}(b) < u_{m-1}$ in P for every $b \in B(S)$.

If $m \geqslant 2$, then let D denote the set of points x on the boundary of $\mathcal{D}_{m-1}$ such that $u_{m-1} \not\leqslant x$ in P; that is, D is the union of the bottom and the right side of $\mathcal{D}_{m-1}$ except for their topmost points. If $m = 1$, then let $D = \emptyset$. It follows that $|D| \leqslant (2k-2) + (4k-5) - 1 = O(k)$.

Proposition 6.4. *Every pair* $(a, b) \in Q(S)$ *satisfies at least one of the following:*

(1) $\mathrm{dn}(b) = u_m$,
(2) *the top of* $\mathcal{D}(a, b)$ *intersects* D,
(3) *the right side of* $\mathcal{D}(a, b)$ *intersects* D.

Proof. Let $(a, b) \in Q(S)$. Suppose $\mathrm{dn}(b) \neq u_m$, so that $u_m < \mathrm{dn}(b) < \mathrm{up}(a) < u_{m-1}$ in P. The "greedy" construction of the sequence $\{(a_i, b_i)\}_{i=1}^{n-1}$ rejected the pair (a, b), so $m \geqslant 2$ and $\mathcal{D}_{m-1}$ is not contained in the interior of $\mathcal{D}(a, b)$. Since the middle of $\mathcal{D}_{m-1}$ lies in the interior of $\mathcal{D}(a, b)$, it follows that the top, the bottom, or the right side of $\mathcal{D}(a, b)$ intersects the boundary of $\mathcal{D}_{m-1}$.

If the bottom of $\mathcal{D}(a, b)$ intersects the top of $\mathcal{D}_{m-1}$ at a point x, then $x \leqslant \mathrm{up}(a) < u_{m-1} = \mathrm{dn}(b_{m-1}) \leqslant x$ in P, which is a contradiction. If the bottom of $\mathcal{D}(a, b)$ intersects the bottom or the right side of $\mathcal{D}_{m-1}$ at a point x, then $a_{m-1} \leqslant x \leqslant \mathrm{up}(a) < u_{m-1} < \mathrm{up}(a_{m-1})$ in P, which shows that $\mathrm{up}(a)$ is a better candidate for $\mathrm{up}(a_{m-1})$. If the top or the right side of $\mathcal{D}(a, b)$ intersects the boundary of $\mathcal{D}_{m-1}$ at a point x such that $u_{m-1} \leqslant x$ in P, then $\mathrm{dn}(b) < u_{m-1} \leqslant x \leqslant b$ in P, which shows that u_{m-1} is a better candidate for $\mathrm{dn}(b)$. We conclude that the top or the right side of $\mathcal{D}(a, b)$ intersects D. $\qquad\square$

Let R be the subset of the maximum chain C consisting of u_m and all points of the form $\mathrm{dn}(x)$ such that $x \in D$ and $u_m < \mathrm{dn}(x) < u_{m-1}$. It follows that $|R| \leqslant 1 + |D| = O(k)$. Let $S' = \{(a, b) \in S\colon \mathrm{dn}(b) \in R\}$. For every pair $(a, b) \in S$ with $\mathrm{dn}(b) = c_i$, we have $(a, b) \in \mathrm{Inc}(\mathrm{Ar}(i, i+1))$ and thus $\dim(\{(a, b) \in S\colon \mathrm{dn}(b) = c_i\}) \leqslant d_1$. It follows that $\dim(S') \leqslant |R|d_1 = O(kd_1)$.

Proposition 6.5. *For every pair* $(a, b) \in Q(S - S')$, *the right side of* $\mathcal{D}(a, b)$ *intersects* D.

Proof. Let $(a, b) \in Q(S - S')$. Since $b \in B(S - S')$, there is $a' \in A(S - S')$ such that $(a', b) \in S - S'$. If $\mathrm{dn}(b) = u_m$, then $(a', b) \in S'$ (as $u_m \in R$), which is a contradiction. Now, suppose the top of $\mathcal{D}(a, b)$ intersects D at a point x. Since $x \in W(\mathrm{dn}(b), b)$, we have $\mathrm{dn}(b) = \mathrm{dn}(x) \in R$, so $(a', b) \in S'$, which is a contradiction. We have excluded the cases (1) and (2) from Proposition 6.4, so the case (3) must hold. $\qquad\square$

We update the meaning of S once again by setting $S = S - S'$, and we prove that $\dim(S) = O(k^2)$. Let $\mathcal{X}$ denote the family of subsets X of D that are downward-closed in D, that is, such that $y \in X$ whenever $x \in X$, $y \in D$, and $y \leqslant x$ in P. Every nonempty set $X \in \mathcal{X}$ is characterized by the pair of points (x, y) such that x is the topmost point of X on the bottom of $\mathcal{D}_{m-1}$ and y is the topmost point of X on the right side of $\mathcal{D}_{m-1}$. It follows that there are at most $(2k-2)(4k-5)$ nonempty sets in $\mathcal{X}$, so $|\mathcal{X}| \leqslant (2k-2)(4k-5) + 1 = O(k^2)$.

For a point $b \in B(S)$, let $\downarrow b = \{x \in P \colon x \leqslant b \text{ in } P\}$. For every $b \in B(S)$, we have $\downarrow b \cap D \in \mathcal{X}$. For every $X \in \mathcal{X}$, let $S_X = \{(a,b) \in S \colon \downarrow b \cap D = X\}$.

Proposition 6.6. $Q(S_X) = \emptyset$ for every $X \in \mathcal{X}$.

Proof. Suppose there is a pair $(a,b) \in Q(S_X)$. By Proposition 6.5, the right side of $\mathcal{D}(a,b)$ intersects D, so there is $x \in D$ such that $a \leqslant x \leqslant b$ in P. Since $b \in B(S_X)$, we have $\downarrow b \cap D = X$, so $x \in X$. Since $a \in A(S_X)$, there is $b' \in B(S_X)$ such that $(a,b') \in S_X$. It follows that $\downarrow b' \cap D = X$ and thus $a \leqslant x \leqslant b'$ in P, which is a contradiction. $\qquad\square$

The last proposition and Proposition 6.1 imply that the set of incomparable pairs S_X is reversible for every $X \in \mathcal{X}$. Since $S = \bigcup_{X \in \mathcal{X}} S_X$, it follows that $\dim(S) \leqslant |\mathcal{X}| = O(k^2)$. This completes the proof that $\dim(S_{RL}) = O(k^2 d_1 + k^3)$.

7. Concluding remarks

Recently, a number of important results connecting dimension with structural graph theory have been proved [1, 10, 11, 12, 15, 23]. Based on these results, it is natural to make the following conjectures, the first of which is also made in [15].

Conjecture 7.1. *For every pair (k,n) of positive integers, there exists an integer d such that if P is a poset excluding $\mathbf{k} + \mathbf{k}$ and the cover graph of P does not contain a K_n-minor, then $\dim(P) \leqslant d$.*

Conjecture 7.2. *For every pair (k,n) of positive integers with $k \geqslant 2$, there exists an integer d such that if P is a poset excluding the standard example S_k and the cover graph of P does not contain a K_n-minor, then $\dim(P) \leqslant d$.*

One technical detail from Section 3 should be noted. In that section, we showed that if P excludes $\mathbf{k} + \mathbf{k}$ and every convex subposet of P of height at most $4k - 4$ has dimension at most d_1, then the dimension of P is bounded in terms of its height. In fact, we showed that $\dim(P) \leqslant d_1(h+1)$, where h is the height of P. A natural question is whether the $4k - 4$ is tight. We suspect not, and it might be enough just to assume that the convex subposets of P of height $k - 1$ have bounded dimension.

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