BURLING GRAPHS, CHROMATIC NUMBER, AND ORTHOGONAL TREE-DECOMPOSITIONS

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ABSTRACT. A classic result of Asplund and Grünbaum states that intersection graphs of axis-aligned rectangles in the plane are χ-bounded. This theorem can be equivalently stated in terms of path-decompositions as follows: There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph that has two path-decompositions such that each bag of the first decomposition intersects each bag of the second in at most k vertices has chromatic number at most $f(k)$. Recently, Dujmović, Joret, Morin, Norin, and Wood asked whether this remains true more generally for two tree-decompositions. In this note we provide a negative answer: There are graphs with arbitrarily large chromatic number for which one can find two tree-decompositions such that each bag of the first decomposition intersects each bag of the second in at most two vertices. Furthermore, this remains true even if one of the two decompositions is restricted to be a path-decomposition. This is shown using a construction of triangle-free graphs with unbounded chromatic number due to Burling, which we believe should be more widely known.

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1. Burling graphs

For each $k \geqslant 1$, we define the *Burling graph* G_k and a collection $\mathcal{S}(G_k)$ of stable sets of G_k by induction on k as follows. First, let G_1 be the graph consisting of a single vertex and let $\mathcal{S}(G_1)$ contain just the single vertex stable set of G_1. Next, suppose $k \geqslant 2$ for the inductive case. First, take a copy H of G_{k-1}, which we think of as the 'master' copy. For each stable set $S \in \mathcal{S}(H)$, let H_S denote a new copy of G_{k-1}. Furthermore, for each stable set $X \in \mathcal{S}(H_S)$, introduce a new vertex $v_{S,X}$ adjacent to all vertices in X but no others. Let us denote by H'_S the graph obtained from H_S resulting from these vertex additions. The graph G_k is then defined as the union of H and H'_S over all $S \in \mathcal{S}(H)$. Its collection $\mathcal{S}(G_k)$ consists of two sets for each $S \in \mathcal{S}(H)$ and $X \in \mathcal{S}(H_S)$, namely: $S \cup X$ and $S \cup \{v_{S,X}\}$. Observe that $S \cup X$ and $S \cup \{v_{S,X}\}$ are both stable sets of G_k.

Burling defined the family $\{G_k\}$ in his PhD Thesis [2] in 1965 and proved that these graphs have unbounded chromatic number. However, this construction went mostly unnoticed until it was rediscovered in [10]. (One exception is a set of unpublished lecture notes of Gyárfás [7] from 2003, which has a section devoted to Burling graphs.)

Theorem 1 ([2])**.** *For every $k \geqslant 1$, the Burling graph G_k is triangle free and has chromatic number at least k.*

Proof. The fact that G_k is triangle free follows directly by observing that, when creating a vertex $v_{S,X}$ in the definition of G_k, its neighborhood is a stable set. To show that $\chi(G_k) \geqslant k$, we prove the following stronger statement by induction on k: For every proper coloring ϕ of G_k, there exists a stable set $S \in \mathcal{S}(G_k)$ such that ϕ uses at least k colors for vertices in S. This is obviously true for $k = 1$, so let us assume $k \geqslant 2$ and consider the inductive case. Let ϕ be a proper coloring of G_k. In what follows, the notations H, H_S, and H'_S refer to the graphs used in the definition of G_k. By induction, there is a stable set $S \in \mathcal{S}(H)$ such that ϕ uses at least $k-1$ colors on S. Similarly, there is a stable set $X \in \mathcal{S}(H_S)$ such that ϕ uses at least $k-1$ colors on X. If ϕ uses at least k colors on $S \cup X$, we are done since $S \cup X \in \mathcal{S}(G_k)$. If not, then ϕ uses exactly the same set C of $k-1$ colors on S and on X. This implies that the vertex $v_{S,X}$ is colored with a color not in C, and hence ϕ uses k colors on the stable set $S \cup \{v_{S,X}\} \in \mathcal{S}(G_k)$. □

Mycielski [9], and Erdős and Hajnal [6] each described easy constructions of triangle-free graphs with unbounded chromatic number that are classics nowadays. We believe that Burling graphs should be more widely known, for their definition is simple and yet they exhibit some unique properties. In particular, Burling graphs admit various geometric representations that are not known to exist for any other family of triangle-free graphs with unbounded chromatic number, which we briefly survey now.

First, recall that a class of graphs $\mathcal{C}$ is χ*-bounded* if there is a function f such that $\chi(G) \leqslant f(\omega(G))$ for all $G \in \mathcal{C}$, where $\omega(G)$ denotes the maximum size of a clique in G.

Burling [2] showed that each G_k can be obtained as the intersection graph of axis-aligned boxes in $\mathbb{R}^3$. Hence, this implies that intersection graphs of axis-aligned boxes in $\mathbb{R}^3$

are not χ-bounded. This is in contrast with the result of Asplund and Grünbaum [1] that $\chi(G) \in O(\omega^2(G))$ for intersection graphs G of axis-aligned rectangles. (We remark that Reed and Allwright [11] (see also [8]) described another interesting construction of axis-aligned boxes in $\mathbb{R}^3$ whose intersection graph has high chromatic number, with the extra property that the interiors of the boxes are pairwise disjoint, implying that the clique number is at most 4.)

In the 1970s, Erdős asked whether intersection graphs of line segments in the plane are χ-bounded. A negative answer was provided by Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [10]: The authors represented the Burling graphs as intersection graphs of segments in the plane. This result also disproves the conjecture of Scott [12] that, for every graph H, the class of graphs excluding every subdivision of H as an induced subgraph is χ-bounded. Indeed, segment intersection graphs—and thus in particular Burling graphs—do not contain any subdivision of H as an induced subgraph when H is the 1-subdivision of a non-planar graph. Later on, Chalopin, Esperet, Li, and Ossona de Mendez [3] showed that Burling graphs in fact even exclude all subdivisions of H as an induced subgraph when H is the 1-subdivision of K_4.

2. Application to orthogonal tree-decompositions

A *tree-decomposition* of a graph G is a pair $(T, \{B_t\}_{t\in V(T)})$ where T is a tree and the sets B_t $(t \in V(T))$ are subsets of $V(G)$ called *bags* satisfying the following properties:

(i) for each edge $uv \in E(G)$ there is a bag containing both u and v, and
(ii) for each vertex $v \in V(G)$, the set of vertices $t \in V(T)$ with $v \in B_t$ induces a non-empty subtree of T.

The *width* of the tree-decomposition is the maximum size of a bag minus 1. The *tree-width* of G is the minimum width of tree-decompositions of G. Path-decompositions and path-width are defined analogously, with the extra requirement that the tree T be a path. We refer the reader to Diestel [4] for background on tree-decompositions.

The following generalization of tree-decompositions was recently introduced by Stavropoulos [14, 13] and investigated by Dujmović, Joret, Morin, Norin, and Wood [5]. Suppose that $(T^1, \{B^1_t\}_{t\in V(T^1)}), \dots, (T^k, \{B^k_t\}_{t\in V(T^k)})$ are k tree-decompositions of a graph G. Let then the *k-width* of these decompositions be the maximum of $|B^1_{t_1} \cap \cdots \cap B^k_{t_k}|$ over all $(t_1, \dots, t_k) \in V(T_1) \times \cdots \times V(T_k)$. The *k-tree-width* of G, also called *k-medianwidth* of G in [14, 13], is the minimum k-width of all k-tuples of tree-decompositions of G. Replacing trees with paths, we obtain the corresponding notion of *k-path-width* of G, also known as *k-latticewidth* [13]. Intuitively, to show that the k-tree-width or k-path-width of G is small, we want to choose a k-tuple of tree/path-decompositions of G that are as 'orthogonal' as possible: For instance, to see that a grid has bounded 2-path-width, one can take a 'horizontal' path-decomposition where bags are unions of two consecutive columns, and a 'vertical' one where bags are unions of two consecutive rows.

The k-tree-width of G for $k = 1, 2, 3, \dots$ forms a non-increasing sequence of numbers that converges to the clique number $\omega(G)$ of G, and the same is true for the k-path-width of G [14, 13]. Thus these numbers can be seen as interpolating between the tree-width / path-width of G (plus one) and its clique number.

Some graph classes of interest already have bounded 2-tree-width. For instance, planar graphs, and more generally graphs excluding a fixed graph H as minor [5]. In fact, for planar graphs and some of their generalizations, one can even require one of the two tree-decompositions to be a path-decomposition such that each vertex appears in at most two bags, see [5] and the references therein. Note however that graphs with bounded 2-tree-width are not necessarily sparse: All bipartite graphs have 2-tree-width (and even 2-path-width) at most 2.

The k-path-width of a graph G can equivalently be defined as the minimum q such that G is a subgraph of an intersection graph H of axis-aligned boxes in $\mathbb{R}^k$ with $\omega(H) \leqslant q$. (To see this, recall that axis-aligned boxes in $\mathbb{R}^k$ satisfy the Helly property.) In particular, $\chi(G)$ is bounded from above by a function of the 2-path-width of G, since intersection graphs of axis-aligned rectangles in the plane are χ-bounded [1]. This prompted the authors of [5] to ask whether the same remains true for the 2-tree-width of G. We show that this is not the case, even if one the two decompositions is restricted to be a path-decomposition.

Theorem 2. *For every $k \geqslant 1$, the Burling graph G_k has a tree-decomposition $(T, \{B_t\}_{t \in V(T)})$ and a path-decomposition $(P, \{B_p\}_{p \in V(P)})$ such that $|B_t \cap B_p| \leqslant 2$ for every $t \in V(T)$ and every $p \in V(P)$.*

Proof. The proof is by induction on k. To facilitate the induction, we will prove that the tree-decomposition and the path-decomposition can be chosen such that

(i) $|B_t \cap B_p| \leqslant 2$ for every $t \in V(T)$ and every $p \in V(P)$;
(ii) for every $S \in \mathcal{S}(G_k)$, there exists $t \in V(T)$ such that $B_t = S$, and
(iii) $|S \cap B_p| \leqslant 1$ for every $S \in \mathcal{S}(G_k)$ and every $p \in V(P)$.

The claim is trivially true for $k = 1$, so let us consider the inductive case $k \geqslant 2$. As before, the notations H, H_S, and H'_S refer to the graphs used in the definition of G_k. Let $(T^H, \{B^H_t\}_{t \in V(T^H)})$ and $(P^H, \{B^H_p\}_{p \in V(P^H)})$ denote the tree-decomposition and path-decomposition of H given by the induction hypothesis. Similarly, for each stable set $S \in \mathcal{S}(H)$, let $(T^{H,S}, \{B^{H,S}_t\}_{t \in V(T^{H,S})})$ and $(P^{H,S}, \{B^{H,S}_p\}_{p \in V(P^{H,S})})$ denote the tree-decomposition and path-decomposition of H_S obtained from induction. (As expected, we assume that T^H, P^H, and all the $T^{H,S}$s and $P^{H,S}$s are pairwise vertex disjoint.)

Define the tree T as follows. Start with the union of T^H and $T^{H,S}$ for all $S \in \mathcal{S}(H)$. Then, for each $S \in \mathcal{S}(H)$, add an edge linking a vertex $t \in V(T^H)$ such that $B^H_t = S$ (which exists by induction) to an arbitrary vertex in $V(T^{H,S})$. Finally, for each $S \in \mathcal{S}(H)$ and $X \in \mathcal{S}(H_S)$, let $t_{S,X}$ denote a vertex in $V(T^{H,S})$ such that $B^{H,S}_{t_{S,X}} = X$. Add two leaves $t^1_{S,X}$, $t^2_{S,X}$ adjacent to $t_{S,X}$.

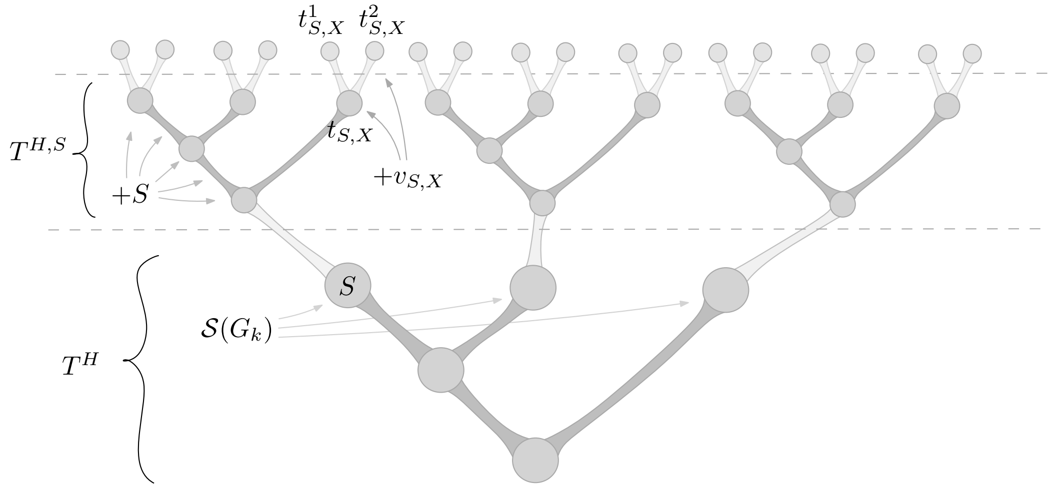

FIGURE 1. Tree-decomposition of G_k.

The bags B_t ($t \in V(T)$) of the tree-decomposition of G_k are defined as follows (see Figure 1 for an illustration) :

$$B_t := \begin{cases} B^H_t & \text{if } t \in V(T^H) \\ S \cup X \cup \{v_{S,X}\} & \text{if } t = t_{S,X} \text{ for some } S \in \mathcal{S}(H) \text{ and } X \in \mathcal{S}(H_S) \\ S \cup X & \text{if } t = t^1_{S,X} \text{ for some } S \in \mathcal{S}(H) \text{ and } X \in \mathcal{S}(H_S) \\ S \cup \{v_{S,X}\} & \text{if } t = t^2_{S,X} \text{ for some } S \in \mathcal{S}(H) \text{ and } X \in \mathcal{S}(H_S) \\ S \cup B^{H,S}_t & \text{if } t \in V(T^{H,S}) \text{ for some } S \in \mathcal{S}(H) \text{ and } t \neq t_{S,X} \text{ for all } X \in \mathcal{S}(H_S) \end{cases}$$

For each vertex $v \in V(G_k)$, the set of vertices $t \in V(T)$ such that $v \in B_t$ clearly induces a subtree of T. Moreover, the two endpoints of each new edge of the form $v_{S,X}x$ with $S \in \mathcal{S}(H)$, $X \in \mathcal{S}(H_S)$, and $x \in X$ lie in a common bag, namely B_t with $t = t_{S,X}$. It follows that $(T, \{B_t\}_{t \in V(T)})$ is a tree-decomposition of G_k.

We show that property (ii) holds. Recall that each set in $\mathcal{S}(G_k)$ is either of the form $S \cup X$ or of the form $S \cup \{v_{S,X}\}$ for some $S \in \mathcal{S}(H)$ and $X \in \mathcal{S}(H_S)$. In the former case, $S \cup X = B_t$ for $t = t^1_{S,X}$. In the latter case, $S \cup \{v_{S,X}\} = B_t$ for $t = t^2_{S,X}$. Hence, (ii) is satisfied.

Next, we define the path-decomposition of G_k. The path P indexing the decomposition is defined simply by taking the union of the paths P^H and $P^{H,S}$ for all $S \in \mathcal{S}(H)$, and connecting them in a path-like way (arbitrarily). The bags B_p ($p \in V(P)$) are defined as follows (see Figure 2 for an illustration):

$$B_p := \begin{cases} B^H_p & \text{if } p \in V(P^H) \\ B^{H,S}_p \cup \{v_{S,X} \mid X \in \mathcal{S}(H_S)\} & \text{if } p \in V(P^{H,S}) \text{ for some } S \in \mathcal{S}(H) \end{cases}$$

Observe that $(P, \{B_p\}_{p \in V(P)})$ is a path-decomposition of G_k. Indeed, for each vertex $v \in V(G_k)$ the set of vertices $p \in V(P)$ such that $v \in B_p$ clearly induces a subpath of

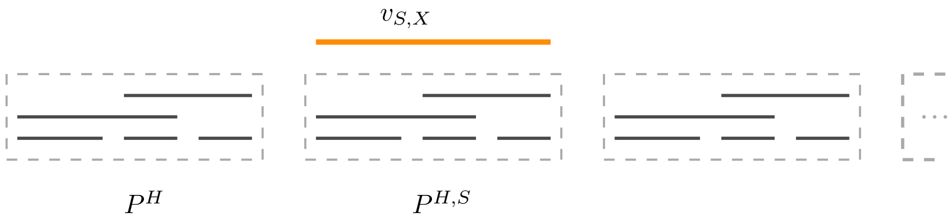

FIGURE 2. Path-decomposition of G_k.

P. Moreover, the two endpoints of each new edge of the form $v_{S,X}x$ with $S \in \mathcal{S}(H)$, $X \in \mathcal{S}(H_S)$, and $x \in X$ lie in a common bag since $v_{S,X} \in B_p^{H,S}$ for every $p \in V(P^{H,S})$.

Let us prove that property (iii) is satisfied. Consider sets $S \in \mathcal{S}(H)$ and $X \in \mathcal{S}(H_S)$, and a vertex $p \in V(P)$. First suppose $p \in V(P^H)$. Then $(S \cup X) \cap B_p = S \cap B_p^H$, and thus $|(S \cup X) \cap B_p| \leqslant 1$ holds by induction. Similarly, $(S \cup \{v_{S,X}\}) \cap B_p = S \cap B_p^H$ and again $|(S \cup \{v_{S,X}\}) \cap B_p| \leqslant 1$ follows from induction. Next assume $p \in V(P^{H,S})$ for some $S \in \mathcal{S}(H)$. Then $(S \cup X) \cap B_p = X \cap B_p^{H,S}$, and thus $|(S \cup X) \cap B_p| \leqslant 1$ by induction. Also, $(S \cup \{v_{S,X}\}) \cap B_p^{H,S} = \{v_{S,X}\}$ and hence $|(S \cup \{v_{S,X}\}) \cap B_p| = 1$. It follows that property (iii) holds.

It remains to show that our newly defined tree and path-decompositions together satisfy property (i). Let thus $t \in V(T)$ and $p \in V(P)$. First, suppose that $t \in V(T^H)$. If $p \in V(P^H)$, then $|B_t \cap B_p| \leqslant 2$ holds by induction. If $p \in V(P^{H,S})$ for some $S \in \mathcal{S}(H)$, then B_t and B_p are disjoint.

Next, suppose that $t = t_{S,X}$ for some $S \in \mathcal{S}(H)$ and $X \in \mathcal{S}(H_S)$. Thus, $B_t = S \cup X \cup \{v_{S,X}\}$. If $p \in V(P^H)$, then $B_t \cap B_p = S \cap B_p^H$, and we know that this set has size at most 1 by induction, since H satisfies property (iii). If $p \in V(P^{H,S'})$ for some $S' \in \mathcal{S}(H)$ distinct from S, then B_t and B_p are disjoint. If $p \in V(P^{H,S})$, then $B_t \cap B_p = (X \cap B_p^{H,S}) \cup \{v_{S,X}\}$. Since $|X \cap B_p^{H,S}| \leqslant 1$ holds by induction thanks to property (iii), we deduce that $|B_t \cap B_p| \leqslant 2$.

The above observations also imply that $|B_t \cap B_p| \leqslant 2$ if $t = t^1_{S,X}$ or $t = t^2_{S,X}$ for some $S \in \mathcal{S}(H)$ and $X \in \mathcal{S}(H_S)$, since $B_t \subseteq S \cup X \cup \{v_{S,X}\}$ in these cases. Finally, suppose that $t \in V(T^{H,S})$ for some $S \in \mathcal{S}(H)$ and $t \neq t_{S,X}$ for all $X \in \mathcal{S}(H_S)$. Then $B_t = S \cup B_t^{H,S}$. If $p \in V(P^H)$, then $B_t \cap B_p = S \cap B_p^H$, and (as in the above paragraph) that set has size at most 1 by induction, since H satisfies property (iii). If $p \in V(P^{H,S'})$ for some $S' \in \mathcal{S}(H)$ distinct from S, then B_t and B_p are disjoint. If $p \in V(P^{H,S})$, then $|B_t \cap B_p| = |B_t^{H,S} \cap B_p^{H,S}| \leqslant 2$ by induction.

Hence, $|B_t \cap B_p| \leqslant 2$ holds in all cases, and therefore property (i) is satisfied. $\square$

We conclude the paper with an open problem in the spirit of exploring how the Asplund-Grünbaum result [1] could be extended. A *spaghetti tree-decomposition* of a graph G is a tree-decomposition $(T, \{B_t\}_{t \in V(T)})$ of G such that T is rooted at some vertex

$r \in V(T)$ and, orienting all edges of T away from r, the subtree of T induced by $\{t \in V(T) : v \in B_t\}$ is a directed path for each vertex $v \in V(G)$.

Conjecture 3. *There exists a function* $f : \mathbb{N} \to \mathbb{N}$ *such that* $\chi(G) \leqslant f(k)$ *for every* $k \geqslant 1$ *and every graph* G *admitting a spaghetti tree-decomposition* $(T, \{B_t\}_{t \in V(T)})$ *and a path-decomposition* $(P, \{B_p\}_{p \in V(P)})$ *such that* $|B_t \cap B_p| \leqslant k$ *for every* $t \in V(T)$ *and* $p \in V(P)$.

We remark that, for all we know, the above conjecture could even be true with two spaghetti tree-decompositions.

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References

[1] E. Asplund and B. Grünbaum. On a coloring problem. *Math. Scand.*, 8:181–188, 1960.

[2] J. P. Burling. *On coloring problems of families of prototypes*. PhD thesis, University of Colorado, Boulder, 1965.

[3] J. Chalopin, L. Esperet, Z. Li, and P. Ossona de Mendez. Restricted frame graphs and a conjecture of Scott. *Electron. J. Combin.*, 23(1):Paper 1.30, 21, 2016. arXiv:1406.0338.

[4] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[5] V. Dujmović, G. Joret, P. Morin, S. Norin, and D. R. Wood. Orthogonal tree decompositions of graphs. arXiv:1701.05639.

[6] P. Erdős and A. Hajnal. On chromatic number of infinite graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 83–98. Academic Press, New York, 1968.

[7] A. Gyárfás. Combinatorics of intervals. Unpublished lecture notes (2003).

[8] C. Magnant and D. M. Martin. Coloring rectangular blocks in 3-space. *Discussiones Mathematicae Graph Theory*, 31:161–170, 2011.

[9] J. Mycielski. Sur le coloriage des graphs. *Colloq. Math.*, 3:161–162, 1955.

[10] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W. T. Trotter, and B. Walczak. Triangle-free intersection graphs of line segments with large chromatic number. *J. Combin. Theory Ser. B*, 105:6–10, 2014. arXiv:1209.1595.

[11] B. Reed and D. Allwright. Painting the office. *MICS Journal*, 1:1–8, 2008. Available online.

[12] A. D. Scott. Induced trees in graphs of large chromatic number. *J. Graph Theory*, 24(4):297–311, 1997.

[13] K. Stavropoulos. Cops, robber and medianwidth parameters. arXiv:1603.06871.

[14] K. Stavropoulos. On the medianwidth of graphs. arXiv:1512.01104.