

THE DIMENSION OF A COMPARABILITY GRAPH

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ABSTRACT. Dushnik and Miller defined the dimension of a partial order P as the minimum number of linear orders whose intersection is P . Ken Bogart asked if the dimension of a partial order is an invariant of the associated comparability graph. In this paper we answer Bogart's question in the affirmative. The proof involves a characterization of the class of comparability graphs defined by Aigner and Prins as uniquely partially orderable graphs. Our characterization of uniquely partially orderable graphs is another instance of the frequently encountered phenomenon where the obvious necessary condition is also sufficient.

1. Notation and definitions. In this paper we consider a partial order as an irreflexive, transitive binary relation. With a binary relation R on a set A we associate a graph $G(R)$ whose vertex set is A with distinct vertices x and y joined by an edge iff $x R y$ or $y R x$. A graph G is called a comparability graph if there exists a partial order P for which $G = G(P)$. Aigner and Prins [1] called a comparability graph G a uniquely partially orderable (UPO) graph if $G = G(P) = G(Q)$ implies $P = Q$ or $P = \hat{Q}$ where \hat{Q} denotes the dual of Q .

Let X be a graph and let $\{G_x | x \in V(X)\}$ be a family of graphs. Then the (Sabadussi) X -join [9] of this family is the graph with vertex set $\{(x, y) | x \in V(X), y \in V(G_x)\}$ with (x, y) adjacent to (z, w) iff x is adjacent to z in X or $x = z$ and y is adjacent to w in G_x . Every graph X is isomorphic to the X -join of a family of trivial graphs. If a graph G is isomorphic to the X -join of a family $\{G_x | x \in V(X)\}$ where X is nontrivial and at least one G_x is nontrivial, then G is said to be decomposable; otherwise G is said to be indecomposable.

Let G be a graph and let K be a subset of $V(G)$. K is said to be partitive iff for every vertex x with $x \notin K$, if there exists a vertex $y \in K$ such that x and y are adjacent, then x is adjacent to every vertex in K . A partitive subset K is said to be nontrivial when K is not the empty set, a singleton, or the entire vertex set. It is easy to see that a graph is indecomposable iff it has no nontrivial partitive sets.

Now let P be a partial order on a set A and let $\{Q_a | a \in A\}$ be a family of partial orders. If we denote the set on which each Q_a is defined by A_a , then the ordinal product [2] of this family over P is the partial order S on the set $\{(a, b) | a \in A, b \in A_a\}$ in which $(a_1, b_1) S (a_2, b_2)$ iff $a_1 P a_2$ or $a_1 = a_2$ and $b_1 Q_{a_1} b_2$. Clearly the comparability graph $G(S)$ is the $G(P)$ -join of the family $\{G(Q_a) | a \in A\}$.

Let e and f be edges of a graph G . Gilmore and Hoffman [6] defined a strong

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path from e to f as a sequence of (not necessarily distinct) vertices $x_0, x_1, x_2, \dots, x_n$ with $n \geq 2$ satisfying:

- (i) $\{x_0, x_1\} = e, \{x_{n-1}, x_n\} = f,$
- (ii) $\{x_i, x_{i+1}\} \in E(G)$ for $0 \leq i \leq n-1,$
- (iii) $\{x_i, x_{i+2}\} \notin E(G)$ for $0 \leq i \leq n-2.$

Edges e and f are said to be strongly connected, denoted $e \sim f$, when there exists a strong path from e to f . The binary relation \sim is an equivalence relation on $E(G)$ and is used in [4] and [6] to provide a transitive orientation for a comparability graph (see also [7]).

The following lemma is an immediate consequence of the results of Gilmore and Hoffman [6].

LEMMA 1. *Let e be an edge of a graph G and let K be the set of all vertices of G which are endpoints of an edge in the equivalence class of e under \sim . Then K is a partitive set of vertices in G .*

For a binary relation R on a set A and a subset $B \subseteq A$, let $R(B)$ denote the restriction of B to A . The following lemma is proved in [6].

LEMMA 2. *Let $e = \{x, y\}$ and $f = \{z, w\}$ be strongly connected edges of a comparability graph G . Let P and Q be partial orders so that $G = G(P) = G(Q)$. Then $P(\{x, y\}) = Q(\{x, y\})$ iff $P(\{z, w\}) = Q(\{z, w\})$.*

2. A characterization of UPO graphs. In this section we characterize UPO graphs. Our theorem will be another instance of the common phenomenon where the obvious necessary condition is also sufficient. We begin by repeating Aigner and Prins' observation [1] that a disconnected comparability graph is UPO iff it has at most one nontrivial component and that component is also UPO. Consequently we restrict our attention to connected graphs.

THEOREM 1. *A connected comparability graph is UPO iff every nontrivial partitive subset is an independent set of vertices.*

PROOF. Let G be a connected comparability graph with a nontrivial partitive subset K which contains an edge of G . Choose a vertex $x_0 \in K$, a partial order P_0 on $(G - K) \cup \{x_0\}$, and a partial order P_1 on K . Then define a partial order P on G by: If $x, y \in K$, then $x P y$ iff $x P_1 y$, if $x, y \in G - K$, then $x P y$ iff $x P_0 y$, if $x \in K, y \in G - K$, then $x P y$ iff $x_0 P_0 y$, and if $x \in G - K, y \in K$, then $x P y$ iff $x P_0 x_0$. Then define a partial order Q on G by $Q = (P - P(K)) \cup \hat{P}(K)$. It follows easily that $P \neq Q$ and $P \neq \hat{Q}$ and thus G is not UPO.

On the other hand, let G be a connected comparability graph in which every nontrivial partitive subset is an independent set of vertices. We prove that G is UPO. To accomplish this, we first prove that there is only one equivalence class in $E(G)$ under \sim . The fact that G is then UPO follows immediately from Lemma 2.

Choose an arbitrary edge e in G . It follows from Lemma 1 and the fact that G is connected that every vertex in G is the endpoint of an edge from the equivalence class of e . Suppose that there exists an edge $f = \{x, y\}$ which is not strongly connected to e . Choose an edge $e_1 = \{x, z\}$ with $e \sim e_1$. Then

$e_2 = \{y, z\} \in E(G)$ for if $\{y, z\} \notin E(G)$, then z, x, y is a strong path from e_1 to f .

Suppose first that $e_2 \sim e_1$. We now show that z is adjacent to every other vertex in G . Assume that this is not the case and choose an arbitrary vertex w with w not adjacent to z . Then choose an edge $e_3 = \{t, w\}$ with $f \sim e_3$ and a strong path $x_0, x_1, x_2, \dots, x_n$ from f to e_3 . We prove by induction that $\{z, x_i\} \in E(G)$ and $\{z, w_i\} \sim e_1$ for $0 \leq i \leq n$. First note that each edge $\{x_i, x_{i+1}\}$ is strongly connected to f for $0 \leq i \leq n - 1$. We then note that $\{z, x_0\}$ and $\{z, x_1\}$ are edges in G and each is strongly connected to e_1 . Now suppose that for some i with $1 \leq i \leq n - 1$ we have $\{z, x_i\} \in E(G)$ and $\{z, x_i\} \sim e_i$. Then if $\{z, x_{i+1}\} \notin E(G)$, it follows that z, x_i, x_{i+1} is a strong path from $\{z, x_i\}$ to $\{x_i, x_{i+1}\}$ and, hence, $f \sim \{x_i, x_{i+1}\} \sim \{z, x_i\} \sim e_1 \sim e$. We conclude that $\{z, x_{i+1}\}$ is an edge in G . Furthermore, we may conclude that $\{z, x_{i+1}\} \sim e_1$ since $\{z, x_{i+1}\} \sim \{z, x_{i-1}\}$. The inductive argument shows that z is adjacent to w and hence to every other vertex in G . It follows that the set $K = V(G) - \{z\}$ is a nontrivial partitive set containing the edge f . The contradiction shows that $e_2 \not\sim e_1$.

However it is straightforward to repeat the argument to show that the assumption that $e_2 \not\sim e_1$ leads to the conclusion that y is adjacent to every other vertex in G and thus the set $V(G) - \{y\}$ is a nontrivial partitive set containing e_1 . The contradiction completes the proof of our theorem.

3. The dimension of a comparability graph. Dushnik and Miller [3] defined the dimension of a partial order P , denoted $\text{Dim } P$, to be the minimum number of linear orders whose intersection is P . We note that if P is a partial order on a set X , then $\text{Dim } P(Y) \leq \text{Dim } P$ for every $Y \subseteq X$ and $\text{Dim } \hat{P} = \text{Dim } P$. We refer the reader to [3] and [10] for elementary properties of the dimension of partial orders.

If P and Q are partial orders for which $G(P) = G(Q)$, Bogart asked if it is always true that $\text{Dim } P = \text{Dim } Q$. The characterization of UPO graphs given in the preceding section will allow us to answer this question in the affirmative.

A partial order P on a set X is said to be irreducible if $\text{Dim } P(X - x) < \text{Dim } P$ for every $x \in X$. Hiraguchi [8] proved that if P is a partial order on a set $X = \{x_1, x_2, \dots, x_n\}$ and S is the ordinal product over P of the family $\{Q_{x_i} | 1 \leq i \leq n\}$ of partial orders then

$$\text{Dim } S = \max\{\text{Dim } P, \text{Dim } Q_{x_1}, \text{Dim } Q_{x_2}, \dots, \text{Dim } Q_{x_n}\}.$$

It is easy to show that the comparability graph of an irreducible partial order is indecomposable!

THEOREM 2. *If P and Q are partial orders so that $G(P) = G(Q)$, then $\text{Dim } P = \text{Dim } Q$.*

PROOF. Suppose the theorem is false and choose partial orders P and Q on a set X so that $\text{Dim } P < \text{Dim } Q$ but $|X|$ is minimum, i.e. $\text{Dim } P(Y) = \text{Dim } Q(Y)$ for every proper subset $Y \subseteq X$. It follows then that Q is irreducible since $\text{Dim } Q(X - x) = \text{Dim } P(X - x) \leq \text{Dim } P < \text{Dim } Q$ and thus $G(Q)$ is indecomposable. Since $G(Q)$ is indecomposable, it is UPO and thus either $P = Q$ or $P = \hat{Q}$. In either case $\text{Dim } P < \text{Dim } Q$ is not possible.

ADDED IN PROOF. The authors have learned that Theorem 1 was discovered previously by Shevrin and Filippov, *Partially ordered sets and their comparability graphs*, Siberian Math. J. **11** (1970), 497–509.

REFERENCES

1. M. Aigner and G. Prins, *Uniquely partially orderable graphs*, J. London Math. Soc. (2) **3** (1971), 260–266. MR **43** #1866.
2. G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R.I., 1967. MR **37** #2638.
3. B. Dushnik and E. Miller, *Partially ordered sets*, Amer. J. Math. **63** (1941), 600–610. MR **3**, 73.
4. S. Even, A. Pnueli and A. Lempel, *Permutation graphs and transitive graphs*, J. Assoc. Comput. Mach. **19** (1972), 400–410. MR **47** #1675.
5. A. Ghouila-Houri, *Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre*, C. R. Acad. Sci. Paris **254** (1962), 1370–1371. MR **30** #2495.
6. P. C. Gilmore and A. J. Hoffman, *A characterization of comparability graphs and of interval graphs*, Canad J. Math. **16** (1964), 539–548. MR **31** #87.
7. M. C. Golumbic, *Comparability graphs and a new matroid*, Ph.D Thesis, Columbia Univ., New York, 1975.
8. T. Hiraguchi, *On the dimension of partially ordered sets*, Sci. Rep. Kanazawa Univ. **1** (1951), 77–94. MR **17**, 19.
9. G. Sabidussi, *Graph derivatives*, Math. Z. **76** (1961), 385–401. MR **24** #A53.
10. W. T. Trotter, *Inequalities in dimension theory for posets*, Proc. Amer. Math. Soc. **47** (1975), 311–316.

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