COMBINATORIAL PROBLEMS IN DIMENSION THEORY
FOR PARTIALLY ORDERED SETS

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Résumé. — Nous présentons quelques résultats récents en théorie de la dimension des ensembles partiellement ordonnés. Parmi les sujets abordés nous considérons des problèmes de construction d'ensembles partiellement ordonnés irréductibles, doublement irréductibles et des inégalités concernant des opérations telles que coupures, amalgamations et produits cartésiens. Nous donnons également une liste de quelques problèmes irrésolus dans ce domaine des mathématiques combinatoires.

Abstract. — We present some recent results in the dimension theory of partially ordered sets. Among the topics discussed are construction problems for irreducible posets, doubly irreducible posets, and inequalities involving splits, amalgamations, and cartesian products. We also give a list of some unsolved problems in this area of combinatorial mathematics.

1. Introduction and notation. — A partially ordered set (poset) is a pair \((X, P)\) where \(X\) is a set, always finite in this paper, and \(P\) is a reflexive, anti-symmetric, and transitive relation on \(X\). The relation \(P\) is called a partial order on \(X\). The notations \((x, y) \in P\), \(x \leq y\) in \(P\), and \(x \leq y\) in \(P\) are used interchangeably. We also write \(x < y\) in \(P\) when \((x, y) \in P\) and \(x \neq y\).

When \(x < y\) or \(y < x\) in \(P\), we say the distinct points \(x\) and \(y\) are comparable. When distinct points \(x\) and \(y\) are not comparable, we say \(x\) and \(y\) are incomparable and write \(x \nleq y\) in \(P\).

If \((X, P)\) and \((Y, Q)\) are posets, then a function \(f : X \to Y\) is called an embedding of \((X, P)\) in \((Y, Q)\) when \(x_1 \leq x_2\) in \(P\) if and only if \(f(x_1) \leq f(x_2)\) in \(Q\) for all \(x_1, x_2 \in X\). The posets are said to be isomorphic and the embedding \(f\) is called an isomorphism when \(f\) is also a surjection. In this paper we do not distinguish between isomorphic posets.

If \((X, P)\) is a poset and \(Y \subseteq X\), then the poset \((Y, Q)\) where \(Q = P \cap (Y \times Y)\) is called a subposet of \((X, P)\). Frequently we will find it convenient to use a single symbol to denote a poset. This notation is particularly useful in any discussion involving subposets. For example when \(Y\) is a subposet of \(X\), we will say \(Y\) is contained in \(X\) and write \(Y \subseteq X\).

A poset \(A\) is called an antichain if \(a_1, a_2, a_3 \in A\) for each distinct pair of points from \(A\). The width of a poset \(X\), denoted \(W(X)\), is the maximum number of points in an antichain contained in \(X\). A poset \(C\) is called a chain if each distinct pair of points from \(C\) is comparable. The length of a poset \(X\), denoted \(L(X)\), is the maximum number of points in a chain contained in \(X\). A chain \((X, P)\) is also called a totally ordered set or linearly ordered set. In this case the partial order \(P\) is called a total order or linear order.

If \(P\) and \(Q\) are partial orders on a set \(X\) and \(P \subseteq Q\), we say \(Q\) is an extension of \(P\). If \(Q\) is also a linear order, then we say \(Q\) is a linear extension of \(P\). A theorem of Szpilrajn [10] asserts that if \(P\) is a partial order on \(X\) and \(C\) is the collection of all linear extensions of \(P\), then \(C \neq \emptyset\) and \(\cap C = P\). Duchin and Miller [3] defined the dimension of a poset \((X, P)\), denoted \(\text{Dim}(X, P)\), as the smallest positive integer \(n\) for which there exist linear extensions \(L_1, L_2, \ldots, L_n\) of \(P\) so that \(P = L_1 \cap L_2 \cap \ldots \cap L_n\).

We consider \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) as a poset with ordering

\[(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)\]

if and only if \(a_i \leq b_i\) in \(R\) for \(i = 1, 2, \ldots, n\). We then have the following alternate definition due to Ore [8].

\(\text{Dim}(X)\) is the smallest positive integer \(n\) for which \(X \subseteq \mathbb{R}^n\).

If \(P\) is a partial order on \(X\), then the relation \(\bar{P} = \{(y, x) : (x, y) \in P\}\) is a partial order on \(X\) and is called the dual of \(P\). We will sometimes write \(\bar{X}\) to denote the dual of the poset \(X\). A poset and its dual have the same width, length and dimension.

A poset \(X\) is said to be irreducible if

\[\text{Dim}(X-x) < \text{Dim}(X)\]

for each \(x \in X\).

**Theorem 1.** — **Dim** \((X) \leq \lfloor X \rfloor/2\) when \(|X| \geq 4\).

Much simpler proofs of this theorem have been provided by Bogart [1], Rabinovitch [9], and Trotter [12]. Trotter’s proof is based on the following inequalities.

**Theorem 2** [4]. — **Dim** \((X) \leq W(Y)\).

**Theorem 3** [12]. — If \(A\) is an antichain of a poset \(X\) and \(|X - A| \geq 2\), then **Dim** \((X) \leq |X - A|\).

Rabinovitch’s proof of theorem 1 is one of many simple proofs that can be fashioned by combining appropriate removal theorems together with some case work to show that the result holds for posets of small order. However, it is not known whether or not every poset with at least three points contains a pair of points whose removal decreases the dimension at most one. If this does not hold, then there exists an irreducible poset \(X\) so that \(X - x\) is irreducible for every \(x \in X\). Such posets are called doubly irreducible posets.

**Problem 1.** — Determine whether doubly irreducible posets exist.

We note that forbidden subposet characterizations have been provided for theorems 1 and 3 ([2], [6], [14]).

**Problem 2.** — Give a forbidden subposet characterization of theorem 2.

3. Irreducible posets of large length. — Trotter [13] proved that irreducible posets with arbitrarily large length exist. In [5], Kelly and Rival constructed 3-dimensional irreducible posets of length \(n\) for every \(n \geq 2\).

**Theorem 4.** — For every \(m \geq 3\) and \(n \geq 2\), there exists an \(m\)-dimensional irreducible poset with length \(\geq n\).

**Proof.** — We proceed by induction on \(m\) using Kelly and Rival’s construction when \(m = 3\) together with the well known result that if \(X\) and \(Y\) have universal bounds, then \(**Dim** \((X \times Y) = **Dim** \((X) + **Dim** \((Y)\).\)

Let 2 denote the 2-element chain 0 < 1.

Now suppose that the theorem holds for \(m = k\). Then choose a \(k\)-dimensional irreducible poset \(X\) having length at least 2. Let \(X\) denote the poset obtained from \(X\) by attaching universal bounds to \(X\). Then let \(Y\) be a \(k + 1\)-dimensional irreducible subposet of the \(k + 1\)-dimensional poset \(X \times 2\). We now show that \(Y\) has length at least \(n\).

Let \(x_1 < x_2 < x_3 < \cdots < x_n\) be a \(n\)-element chain of \(X\). Now let \(i\) be chosen so that \(1 \leq i \leq 2n\). If neither \((x_i, 0)\) or \((x_i, 1)\) belongs to \(Y\), then \(Y\) is a subposet of \(%X - x_i \times 2\) and thus has dimension at most \(k\). The contradiction shows that \(Y\) contains at least one of the points \((x_i, 0)\) and \((x_i, 1)\). It follows that \(Y\) contains at least half of the points from \(\{ (x_i, 0) : 1 \leq i \leq 2n \}\) or at least half of the points \(\{ (x_i, 1) : 1 \leq i \leq 2n \}\). In each case \(Y\) contains an \(n\)-element chain.

**Problem 3.** — For each \(m \geq 3\) and \(n \geq 2\), construct an irreducible poset with dimension \(m\) and length \(n\).

4. Splits and rooted posets. — R. Kimble [7] defined the split of a poset \(X\), denoted \(S(X)\), as the poset of length two with maximal elements \(\{ x' : x \in X \}\) and minimal elements \(\{ x'' : x \in X \}\). The partial order on \(S(X)\) is given by the rule \(x' < y'\) in \(S(X)\) if and only if \(x \leq y\) in \(X\). Kimble proved the following result.

**Theorem 5.** —

\[ **Dim** \((X) \leq **Dim** \((S(X) \leq 1 + **Dim** \((X).\)\)

**Proof.** — Suppose \(L_1, L_2, \ldots, L_t\) generate the partial order on \(S(X)\). For each \(1 \leq i \leq t\), let

\[ S_i = \{ (x, y) \in X : (x', y') \in L_i \}. \]

It follows that there exists a linear order \(M_i\) on \(X\) so that \(P \cap S_i \subset M_i\) where \(P\) denotes the partial order on \(X\). Also it is easy to see that

\[ P = M_1 \cap M_2 \cap \ldots \cap M_t \]

and thus \(**Dim** \((X) \leq **Dim** \((S(X)\)).

Now let \(f : X \rightarrow R\) be an embedding of the chain \((X, M_i)\) in \(R\). Let \(x_0\) denote the maximum element of \((X, M_i)\) and let \(f(x_0) = r_0\). Then let \(x_i\) denote the minimum element of \((X, M_i)\) and let \(f(x_i) = r_i\).

Now consider the function \(g : S(X) \rightarrow X \times R\) defined by

\[ g(x') = (x, f(x)) \quad \text{and} \quad g(x'') = (x, f(x) + 1 + r_0 - r_i). \]

It is easy to see that \(g\) is an embedding and therefore

\[ **Dim** \((S(X) \leq **Dim** \((X \times R) \leq 1 + **Dim** \((X).\)\)

**Problem 4.** — Determine conditions on \(X\) which ensure that \(**Dim** \(S(X) = 1 + **Dim** \((X).\))

The question arose immediately as to whether the repeated splitting of a poset could increase the dimension without bound. Let us denote the \(n\)-th split of a poset by \(S^n(X)\), i.e.

\[ S^0(X) = X, \quad S^1(X) = S(X), \quad S^2(X) = S(S(X)), \quad \text{etc.} \]

The question then becomes whether there exists a constant \(k\) so that \(**Dim** \(S^n(X) \leq k + **Dim** \((X)\) for every \(X\) and every \(n \geq 0\).

If a one point poset is split four times, a 3-dimensional poset is obtained. Note that the Hasse diagram
of $S^r(X)$ is always a tree in the graph theoretic sense when $X$ is a one point poset. It was this observation and its application in the proof of theorem 7 that led Trotter and Moore [15] to investigate the relationship between dimension and planarity. One of the results of this investigation was the following theorem (see [16] for additional results).

**Theorem 6.** — If the Hasse diagram of a poset $X$ is a tree in the graph theoretic sense, then $\text{Dim} (X) \leq 3$.

We call a poset a rooted poset when a single point from the poset has been designated as a root. Now consider the following general situation. Suppose we have a poset $X$ and for each $x \in X$, suppose we also have a rooted poset $Y_x$ with root $r_x$. We form a poset $Z$ called the amalgamation of $\{ Y_x : x \in X \}$ by identifying the points $x$ and $r_x$ for each $x \in X$. For example, consider an arbitrary poset $X$. For each $x \in X$, let $Y_x$ be a 3-element chain $x' < r_x < x$. Then the amalgamation of $\{ Y_x : x \in X \}$ has $S(X)$ as a subposet.

**Theorem 7.** — Let $Z$ be the amalgamation of $\{ Y_x : x \in X \}$. If

$$t_1 = \text{Dim} (X), \quad t_2 = \max \{ \text{Dim} (Y_x) : x \in X \},$$

and $t = \max \{ t_1, t_2 \}$, then $t \leq \text{Dim} (Z) \leq t + 2$.

**Proof.** — The inequality $t \leq \text{Dim} (Z)$ is trivial since $X$ and each $Y_x$ are subposets of $Z$. On the other hand, let $L_1, L_2, \ldots, L_n$ be linear orders which generate the partial order on $X$ and for each $x \in X$, let $M_x^1, M_x^2, \ldots, M_x^n$ be linear orders which generate the partial order on $Y_x$. For each $i \leq t$, define a linear order $K_i$ on $Z$ by the following rules. If $w_1 \in Y_{x_1}$ and $w_2 \in Y_{x_2}$, with $x_1 \neq x_2$, then $w_1 < w_2$ in $K_i$ if and only if $x_1 < x_2$ in $L_i$. If $w_1, w_2 \in Y_x$ with $w_1 \neq w_2$, then $w_1 < w_2$ in $K_i$ if and only if $w_1 < w_2$ in $M_i$.

The remainder of our proof relies heavily of the concept of TM-cycles as introduced in [15]. We refer the reader to [17] and [18] for similar applications of this concept.

Let $S_1 = \{ (w_1, w_2) : w_1 \in Y_{x_1}, w_2 \in Y_{x_2}, w_1 < w_2 \}$ in $Z$, $x > y$ in $X$, $w_2 \leq y$ in $Y_y$ and $S_2 = \{ (w_1, w_2) : w_1 \in Y_{x_1}, w_2 \in Y_{x_2}, w_1 < w_2 \}$ in $Z$, $x > y$ in $X$, $w_2 \leq y$ in $Y_y$. We next show that neither $S_1$ nor $S_2$ contains any TM-cycles. Suppose first that $S_1$ contains a TM-cycle $\{ (a_i, b_i) : 1 \leq i \leq m \}$. We choose for each $i \leq m$ points $x_i, y_i \in X$ with $a_i \in Y_{x_i}, b_i \in Y_{y_i}, x_i > y_i$ in $X$, and $b_i \leq y_i$ in $Y_{y_i}$. Now choose an arbitrary integer $i \leq m$. Since $a_i < b_i$ in $Z$, we conclude that $a_i \not\geq x_i$ in $Y_{x_i}$, and $w \leq a_i$ in $Z$ implies that $w \in Y_{x_i} \setminus \{ x_i \}$ and that $w < a_i$ in $Y_{x_i}$. Hence $b_{i-1} \in Y_{x_i}$ and thus $x_i = y_{i-1}$. However, it is clear that this cannot hold for each $i$ since it would imply cyclically $y_{i-1} > y_i$ for $i = 2, 3, \ldots, m$ as well as $y_m > y_1$.

Now suppose that $S_2$ contains a TM-cycle

$$\{ (a_i, b_i) : 1 \leq i \leq m \}.$$

Choose the points $x_i$ and $y_i$ as in the preceding paragraph. Then $b_i \neq y_i$ and $b_i \leq a_{i+1}$ in $Z$ imply that $a_{i+1} \in Y_{y_i}$, i.e. $x_{i+1} = y_i$, and again a contradiction has been obtained.

Therefore there exist linear extensions $K_{i+1}$ and $K_{i+2}$ of $Z$ so that $S_1 \subset K_{i+1}$ and $S_2 \subset K_{i+2}$. It is straightforward to verify that $K_1 \cap K_2 \cap \cdots \cap K_{i+2}$ is the partial order on $Z$. Thus $\text{Dim} (Z) \leq t + 2$ and our proof is complete.

We note that if $X \leq Y$, then $S(X) \leq S(Y)$. Now consider an arbitrary poset $X$ to which we repeatedly apply the amalgamation process described earlier where we replace each point by a 3-element chain rooted at the middle point. The net effect of a finite number of such operations is the amalgamation of a family $\{ Y_x : x \in X \}$ of rooted trees. We therefore have the following theorem.

**Theorem 8.** — If $X$ is a poset of dimension three or more, then $\text{Dim} S^r(X) \leq 2 + \text{Dim} (X)$ for all $n \geq 2$.

We note that theorem 8 also holds for all posets although for reasons of brevity we do not include the details here.

5. Splits and cartesian products. — One of the best known inequalities in dimension theory is

**Theorem 9.**

$$\max \{ \text{Dim} (X), \text{Dim} (Y) \} \leq \text{Dim} (X \times Y) \leq \text{Dim} (X) + \text{Dim} (Y).$$

The problem is to determine just how accurate the lower bound on $\text{Dim} (X \times Y)$ really is. In particular we may ask the following question.

**Problem 5.** — For each $n \geq 1$, does there exist an $n$-dimensional poset $X$ for which $X \times X$ is also $n$-dimensional?

In general we seek posets for which $\text{Dim} (X \times X)$ is substantially less than $2 \text{Dim} (X)$, we may confine our attention to posets of length 2 for if $X$ is such a poset but the length of $X$ is large, then we may consider the split of $X$ instead. The author and J. L. Moore did prove as a starting point that for the standard example of an $n$-dimensional poset, the crown $S_n^0$ (see [11]), we have

$$\text{Dim} (S_n^r \times S_n^0) = 2n - 2.$$

References
