On Double and Multiple Interval Graphs

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ABSTRACT

In this paper we discuss a generalization of the familiar concept of an interval graph that arises naturally in scheduling and allocation problems. We define the interval number of a graph $G$ to be the smallest positive integer $t$ for which there exists a function $f$ which assigns to each vertex $u$ of $G$ a subset $f(u)$ of the real line so that $f(u)$ is the union of $t$ closed intervals of the real line, and distinct vertices $u$ and $v$ in $G$ are adjacent if and only if $f(u)$ and $f(v)$ meet. We show that (1) the interval number of a tree is at most two, and (2) the complete bipartite graph $K_{m,n}$ has interval number $[(mn + 1)/(m + n)]$.

1. INTRODUCTION

A graph $G$ is called an interval graph if there is a function $f$ that assigns to each vertex $u$ of $G$ a closed interval of the real line $R$ so that distinct vertices $u, v$ of $G$ are adjacent if and only if $f(u) \cap f(v) \neq \emptyset$. Structural characterizations of interval graphs have been provided by Lekkerkerker and Boland [7] who specified the forbidden subgraphs, Gilmore and Hoffman [2] in terms of cycles, and Fulkerson and Gross [1] in terms of matrices. Definitions not given here can be found in Ref. 5.

In this paper, we consider a generalization of the concept of an interval graph; we are motivated by scheduling and allocation problems that arise when a graph is used to model constraints on interactions between
components of a large scale system. For a graph $G$, we define* the interval number of $G$, denoted $i(G)$, as the smallest positive integer $t$ for which there exists a function $f$ which assigns to each vertex $u$ of $G$ a subset $f(u)$ of $\mathbb{R}$ which is the union of $t$ (not necessarily disjoint) closed intervals of $\mathbb{R}$ and distinct vertices $u, v$ of $G$ are adjacent if and only if $f(u) \cap f(v) \neq \emptyset$. The function $f$ is called a $t$-representation of $G$. Thus $G$ is an interval graph if and only if its interval number is one. Obviously every graph $G$ with $p$ vertices has an interval number $i(G) \leq p - 1$, and thus $i(G)$ is well defined.

A number $m$ is called an upper bound for a representation $f$ of a graph $G$ when $m > r$ for every number $r$ in $f(u)$ and every vertex $u$ of $G$.

We will frequently find it convenient to impose an additional restriction on a representation of a graph. A $t$-representation $f$ of a graph $G$ is said to be displayed if for every vertex $u$ of $G$, there exists an open interval $I_u$ contained in $f(u)$ so that $I_u \cap f(v) = \emptyset$ for every vertex $v$ in $G$ with $u \neq v$.

Recall that for any tree $T$, the tree $T'$ is obtained by removing all the endvertices of $T$. A caterpillar is a tree $T$ for which $T'$ is a path. It was noted in Harary and Schwenk [6] that $T$ is a caterpillar if and only if $T$ does not contain the subdivision graph of $K_{1,3}$ as a subtree.

**Theorem 1.** If $T$ is a tree, then $i(T) = 1$ if $T$ is a caterpillar and $i(T) = 2$ if it is not.

**Proof.** If $T$ is a tree and does not contain the subdivision graph of $K_{1,3}$ as a subtree, then it follows from the forbidden subgraph characterization of Ref. 7 that $T$ is an interval graph. On the other hand, if $T$ contains this subdivision graph, then $T$ is not an interval graph and $i(T) \geq 2$.

Now we proceed by induction on the number of vertices to show that every tree has a displayed 2-representation. If $T$ is the one point tree, the result is trivial. Next assume that for some $k \geq 1$, every tree on $k$ vertices has a displayed 2-representation and let $T$ be a tree with $k + 1$ vertices.

Choose an endvertex $u$ of $T$ and let $f$ be a displayed 2-representation of the tree $T - u$. Let $v$ be the unique vertex adjacent to $u$ in $T$ and let $I_v$ be an open interval contained in $f(v)$ so that $I_v \cap f(w) = \emptyset$ for every vertex $w$ in $T - u$ with $w \neq v$. Choose a closed interval $A$ contained in $I_v$.

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* Roberts [8] has studied another generalization of interval graphs. He defines the boxicity of a graph $G$ as the smallest positive integer $t$ for which there exists a function $f$ which assigns to each vertex $u$ of $G$ a sequence $f(u)(1), f(u)(2), \ldots, f(u)(t)$ of closed intervals of $\mathbb{R}$ so that distinct vertices $u, v$ of $G$ are adjacent if and only if $f(u)(i) \cap f(v)(i) \neq \emptyset$ for $i = 1, 2, 3, \ldots, t$.  

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Now choose an upper bound $m$ for $f$ and define $g(w) = f(w)$ for every vertex $w$ in $T - u$ and $g(u) = A U [m, m+1]$. It is clear that $g$ is a displayed 2-representation of $T$ and our proof is complete. 

2. COMPLETE BIPARTITE GRAPHS

We now derive our main result. We use the notation $[x]$ to represent the smallest integer among those which are at least as large as $x$.

Theorem 2. The interval number of the complete bipartite graph $K_{m,n}$ is given by

$$i(K_{m,n}) = [(mn + 1)/(m + n)].$$

Proof. We first show that $i(K_{m,n}) \leq [(mn + 1)/(m + n)]$. Suppose that $f$ is a $t$-representation of $K_{m,n}$. Without loss of generality, we may assume that for each vertex $u$ in $K_{m,n}$, $f(u)$ is the union $A_1(u) \cup A_2(u) \cup \cdots \cup A_t(u)$ of $t$ pairwise disjoint closed intervals.

We now use $f$ to determine a graph $G$. The vertices of $G$ are the ordered pairs of the form $(u, i)$ where $u$ is a vertex in $K_{m,n}$ and $1 \leq i \leq t$ with distinct vertices $(u, i)$ and $(v, j)$ adjacent in $G$ when $A_i(u) \cap A_j(v) \neq \emptyset$. The function $g$ defined by $g(u, i) = A_i(u)$ is a 1-representation of $G$ so $G$ is an interval graph. Since $G$ is bipartite, it is triangle-free. Since $G$ is an interval graph, it does not contain a cycle of four or more vertices as an induced subgraph. Therefore, $G$ is a forest. Note that $G$ has $(m + n)t$ vertices and at most $(m + n)t - 1$ edges.

Now suppose that $e = \{u, v, \}$ is an edge of $K_{m,n}$. Then there exist integers $i, j$ with $A_i(u) \cap A_j(v) \neq \emptyset$, and we may therefore define a function $h$ from the edge set of $K_{m,n}$ to the edge set of $G$ by setting $h(e) = h(\{u, v, \}) = \{(u, i), (v, j)\}$. Clearly, $h$ is a one-to-one function and since $K_{m,n}$ has $mn$ edges, we see that $mn \leq (m + n)t - 1$, i.e., $t \geq [(mn + 1)/(m + n)]$.

We will now show that $i(K_{m,n}) \geq [(mn + 1)/(m + n)]$. Let $t = [(mn + 1)/(m + n)]$. We will construct an interval graph $G$ with a 1-representation $g$. We will then construct a $t$-representation $f$ of $K_{m,n}$ by appropriately choosing, for each vertex $u$ of $K_{m,n}$, $t$ intervals from the range of $g$ as the intervals whose union is $f(u)$.

We begin by labeling the vertices of $K_{m,n}$ with the symbols $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n$ so that $a_i$ is adjacent to $b_j$ for all $i$ and $j$. Without loss of generality, we may assume $m \geq n$. Let $A = \{1, 2, 3, \ldots, m\}$ and $B = \{1, 2, 3, \ldots, n\}$. 


We next construct a graph $T$ whose vertex set is

$$\{u_k : 1 \leq k \leq nt\} \cup \{v_k : 1 \leq k \leq nt-1\} \cup \{w_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\},$$

where $T$ has the following adjacencies: $v_k$ is adjacent to $u_k$ and $u_{k+1}$ for $k = 1, 2, \ldots, nt-1$ and $w_{ij}$ is adjacent to $u_i$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The graph $T$ is a caterpillar and, by Theorem 1, is also an interval graph. Consequently any induced subgraph of $T$ is also an interval graph.

The next step in the construction is to color some, but not all, of the vertices of $T$ using the elements of $A$ as colors. We begin by assigning to $u_1, u_2, \ldots, u_{nt}$ the colors

$$1, 2, 3, \ldots, n, 1, 2, 3, \ldots, n, \ldots, 1, 2, 3, \ldots, n$$

in order. Note that each color from $B$ is used exactly $t$ times.

Now let $s = n - t$; then $2s \leq n - 1$. Suppose that $S$ is a set of either $2s$ or $2s - 1$ consecutive vertices from the sequence $v_1, v_2, \ldots, v_{nt-1}$. Consider a subset $S'$ of $S$ that contains $s$ vertices, no two of which are consecutive. Then let $B'$ be the subset of $B$ consisting of those integers $j$ for which there is a vertex $v$ from $S'$ and a vertex $u$ adjacent to $v$ with $u$ having color $j$. It is easy to verify that $B'$ must contain $2s$ elements, i.e., the $s$ vertices of $S'$ are adjacent to $2s$ distinctly colored vertices.

The next step is to assign colors to the first $ms$ vertices in the sequence $v_1, v_2, \ldots, v_{nt-1}$. Note that $t = [(mn + 1)/(m + n)]$ and $s = n - t$ imply that $ms \leq nt - 1$. At this point, we must consider two cases depending on the parity of $m$. If $m$ is even, then assign the vertices $v_1, v_2, \ldots, v_{ms}$ the colors

$$1, 2, 1, 2, \ldots, 1, 2, 3, 4, 3, 4, \ldots, 3, 4, \ldots, m - 1,$$

$$m, m - 1, m, \ldots, m - 1, m$$

in order. Note that each color in $A$ is to be used exactly $s$ times. If $m$ is odd, we modify this scheme as follows. We first assign color $m$ to $v_1, v_{n+3}, v_{2n+5}, \ldots, v_{(s-1)(n+2)+1}$. Note that for each $j = 1, 2, 3, \ldots, 2s$, there are integers $k, l$ for which $u_k$ is adjacent to $v_l$, where $v_l$ has color $m$ and $u_k$ has color $j$. Next assign to the $(m-1)s$ vertices in the sequence $v_1, v_2, \ldots, v_{nt}$, which were not assigned color $m$, the colors

$$1, 2, 1, 2, \ldots, 1, 2, 3, 4, 3, 4, \ldots, 3, 4, \ldots, m - 2,$$

$$m - 2, m - 1, \ldots, m - 2, m - 1$$

in order. Again we note that each color in $A$ is to be used exactly $s$ times.

When $m$ is even, observe that each color $i$ from $A$ is assigned to $s$ nonconsecutive vertices in a block of $2s - 1$ consecutive vertices from the sequence $v_1, v_2, \ldots, v_{nt-1}$. When $m$ is odd, we observe that distinct vertices that have been assigned color $m$ are at least $n + 2$ apart in the
sequence $v_1, v_2, \ldots, v_{nt-1}$. Therefore, we observe that each color $i$ from $A$ with $i \neq m$ is assigned to $s$ nonconsecutive vertices in a block of $2s$ or $2s-1$ consecutive vertices in the sequence $v_1, v_2, \ldots, v_{nt-1}$. For each color $i \in A$, define the set

$$B(i) = \{ j \in B: \text{There exist integers } k, l \text{ with } u_k \text{ adjacent to } v_l \text{ for which } u_k \text{ has been assigned color } j \text{ and } v_l \text{ has been assigned color } i \}.$$ 

We conclude that for all values of $m$ and for every color $i$ from $A$, the set $B(i)$ contains exactly $2s$ elements.

The next step in the construction is to assign colors to some, but not all, of the vertices in $\{w_{ij}: 1 \leq i \leq m, 1 \leq j \leq n\}$. The construction is the same for all values of $m$. Let $i$ be an element of $A$; assign color $i$ to vertex $w_{ij}$ if and only if $j$ is an element of $B - B(i)$. Now let

$$U_1 = \{u_k: 1 \leq k \leq nt - 1\} \cup \{w_{ij}: 1 \leq i \leq m, 1 \leq j \leq n\}$$

and let

$$U_2 = \{u_k: 1 \leq k \leq nt\}.$$ 

Observe that for each color $i$ from $A$, exactly $t$ vertices of $U_1$ have been assigned color $i$, and for each color $j$ from $B$, exactly $t$ vertices from $U_2$ have been assigned color $j$; furthermore, there exist adjacent vertices $u', u''$ with $u'$ from $U_1$, $u''$ from $U_2$, $u'$ having color $i$, and $u''$ having color $j$.

Now let $G$ be the subgraph of $T$ generated by the colored vertices and let $g$ be a $1$-representation of $G$. The final step in the construction is to use $g$ to define a $t$-representation $f$ of $K_{m,n}$. But this is accomplished simply by defining

$$f(a_i) = \bigcup \{g(u'): u' \text{ is a vertex from } U_1 \text{ and } u' \text{ has color } i\}$$

for $i = 1, 2, \ldots, m$

and

$$f(b_j) = \bigcup \{g(u''): u'' \text{ is a vertex from } U_2 \text{ and } u'' \text{ has color } j\}$$

for $j = 1, 2, \ldots, n$.

It is trivial to verify that $f$ is a $t$-representation of $K_{m,n}$. 

3. OTHER RESULTS

A preliminary version of this paper included a proof of the following result.
Theorem 3. If $G$ has $p$ vertices, then $i(G) \leq \lceil p/3 \rceil$.

This theorem may be established using a two-part argument in which it is proved inductively that a graph on $3n$ vertices has an $n$-representation and a triangle-free graph on $3n$ vertices has a displayed $n$-representation. The proof of the second part makes use of Turán's theorem for the maximum number of edges in a triangle-free graph.

However, the authors did not believe that the upper bound on the interval number of a graph provided by Theorem 3 was best possible. Motivated by the observation that the complete bipartite graph $K_{2n,2n}$ has $4n$ vertices and interval number $n + 1$, the authors conjectured that if $G$ is a graph with $p$ vertices, then $i(G) \leq \lceil (p + 1)/4 \rceil$.

The concept of interval number has been independently investigated by Griggs and West [4]. They obtained the formula given in Theorem 1 for the interval number of a tree as well as the upper bound given in Theorem 3. They also made the same conjecture concerning the maximum interval number of a graph with $p$ vertices. And they also provided an upper bound on the interval number of a graph in terms of the maximum degree of a vertex in the graph. Specifically, they showed that if the maximum degree of a vertex in a graph $G$ is $d$, then $i(G) \leq \lceil (d + 1)/2 \rceil$. This last result allowed them to determine that the interval number of the $n$-cube $Q_n$ is $\lceil (n + 1)/2 \rceil$, which answered a problem posed in the preliminary version of this paper.

The authors have recently learned that Griggs [3] has established the conjecture by proving that if $G$ has $4n - 1$ vertices, then $i(G) \leq n$.

4. AN OPEN PROBLEM

Lekkerkerker and Boland [7] gave a forbidden subgraph characterization of interval graphs by listing the collection $\mathcal{F}_2$ of graphs defined by

$$
\mathcal{F}_2 = \{ G : i(G) = 2 \text{ but } i(H) = 1 \text{ for every proper induced subgraph } H \text{ of } G \}.
$$

We propose the general problem of finding for $t \geq 3$, the collection

$$
\mathcal{F}_t = \{ G : i(G) = t \text{ but } i(H) \leq t - 1 \text{ for every proper subgraph } H \text{ of } G \}.
$$

The problem for $t = 3$ seems to both manageable and interesting since from applied viewpoint, graphs that are the intersection graphs of a family of sets each of which is the union of two intervals of the real line have practical significance, e.g., two work periods separated by a lunch break. By *double interval* graphs, we mean graphs with interval number
two. Theorem 2 shows that $K_{2n,2n}$ is in $\mathcal{I}_{n+1}$ for every $n \geq 1$ and that $K_{2n-1,2n+2}$ is in $\mathcal{I}_{n+1}$ for every $n \geq 2$. In particular, we note then that a forbidden subgraph characterization of double interval graphs will include $K_{4,4}$ and $K_{3,6}$.

References


