A CHARACTERIZATION OF ROBERTS' INEQUALITY FOR BOXICITY

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F.S. Roberts defined the boxicity of a graph G as the smallest positive integer n for which there exists a function F assigning to each vertex x ∈ G a sequence F(x)(1), F(x)(2), ..., F(x)(n) of closed intervals of R so that distinct vertices x and y are adjacent in G if and only if F(x)(i) ∩ F(y)(i) ≠ ∅ for i = 1, 2, 3, ..., n. Roberts then proved that if G is a graph having 2n + 1 vertices, then the boxicity of G is at most n. In this paper, we provide an explicit characterization of this inequality by determining for each n ≥ 1 the minimum collection $\mathcal{G}_n$ of graphs so that a graph G having 2n + 1 vertices has boxicity n if and only if it contains a graph from $\mathcal{G}_n$ as an induced subgraph. We also discuss combinatorial connections with analogous characterization problems for rectangle graphs, circular arc graphs, and partially ordered sets.

1. Introduction

In this paper all graphs are finite and have no loops or multiple edges. For a graph G, we write x ⊥ y in G when x and y are adjacent vertices in G and x ≠ y in G when x and y are nonadjacent. We denote the number of vertices in G by |G|.

A graph H is called an induced subgraph of a graph G when the vertex set of H is a subset of the vertex set of G and distinct vertices of H are adjacent in H if and only if they are adjacent in G. When H is an induced subgraph of G, we will also say G contains H. We do not distinguish between isomorphic graphs.

A graph G is an interval graph when there is a function f which assigns to each vertex x ∈ G a closed interval f(x) of the real line R so that x ⊥ y in G if and only if f(x) ∩ f(y) ≠ ∅ and x ≠ y. Alternatively, an interval graph is the intersection graph of a family of closed intervals of the real line R.

The concept of an interval graph extends very naturally to higher dimensions by considering the intersection graph of a family of "boxes" in n-dimensional Euclidean space $\mathbb{R}^n$. Roberts [3] defined the boxicity of a graph G, denoted Box(G), as the smallest positive integer n for which G is the intersection graph of a family of boxes in $\mathbb{R}^n$. Formally, Box(G) is the smallest positive integer n for which there exists a function F which assigns to each vertex x ∈ G, a sequence F(x)(1), F(x)(2), ..., F(x)(n) of closed intervals of R so that x ⊥ y in G if and only if x ≠ y and F(x)(i) ∩ F(y)(i) ≠ ∅ for i = 1, 2, ..., n. The function F is called an interval coordinatization of length n for G. By convention, we define
Box \( (G) = 0 \) when \( G \) is a complete graph. Therefore, a graph \( G \) is an interval graph if and only if Box \( (G) \leq 1 \).

Roberts proved that if \( G \) is a graph having \( 2n + 1 \) vertices (where \( n \geq 1 \)), then Box \( (G) \leq n \). The principal result of this paper will be an explicit characterization of this inequality. For each \( n \geq 1 \), we will determine the minimum collection \( C_n \) of graphs so that if \( G \) is a graph and \( |G| = 2n + 1 \), then Box \( (G) = n \) if and only if \( G \) contains a graph from \( C_n \) as an induced subgraph.

2. Some inequalities for boxicity

If \( A \) is a subset of the vertex set \( V(G) \) of a graph \( G \), we denote by \( G - A \) the subgraph of \( G \) with vertex set \( V(G) - A \). It is obvious that if \( H \) is an induced subgraph of \( G \), then Box \( (H) \leq \) Box \( (G) \).

We now state without proof two elementary lemmas due to Roberts [3].

**Lemma 1.** If \( x \) is a vertex of \( G \), then Box \( (G) \leq 1 + \) Box \( (G - \{x\}) \).

**Lemma 2.** If \( x \neq y \) in \( G \), then Box \( (G) \leq 1 + \) Box \( (G - \{x, y\}) \).

It is easy to verify that every graph on three vertices is an interval graph and thus has boxicity at most one. The following inequality then follows from Lemma 2 by induction on \( n \).

**Theorem 1** (Roberts). If \( |G| = 2n + 1 \) where \( n \geq 1 \), then Box \( (G) \leq n \).

The join of two graphs \( G \) and \( H \), denoted \( G \oplus H \), is the graph formed by adding to disjoint copies of \( G \) and \( H \) all edges with one endpoint in \( G \) and the other in \( H \). We illustrate this definition with the graphs shown in Fig. 1.

![Fig. 1.](image)

The following lemma shows that boxicity is additive with respect to the join operation on graphs.

**Lemma 3.** Box \( (G \oplus H) = \) Box \( (G) + \) Box \( (H) \) for every pair of graphs \( G \) and \( H \).

**Proof.** Let \( t_1 = \) Box \( (G) \), \( t_2 = \) Box \( (H) \), and \( t_3 = \) Box \( (G \oplus H) \). We further assume
that $t_1 \geq 1$ and that $t_2 \geq 1$, i.e., neither $G$ nor $H$ is complete. The argument when $t_1 = 0$ or $t_2 = 0$ will follow with minor modifications.

We first show that $t_3 \leq t_1 + t_2$. Let $F_1$ be an interval coordinatization of length $t_1$ for $G$ and let $F_2$ be an interval coordinatization of length $t_2$ for $H$. Then choose an interval $[a, b]$ of $\mathbb{R}$ so that $F_1(x)(i) \cup F_2(y)(j) \subseteq [a, b]$ for every $x \in G$, $y \in H$, $i \leq t_1$ and $j \leq t_2$. For each vertex $z \in G \oplus H$ and each positive integer $k \leq t_1 + t_2$ we then define a closed interval $F_3(z)(k)$ of $\mathbb{R}$ by the following rule.

$$F_3(z)(k) = \begin{cases} F_1(z)(k) & \text{if } z \in G \text{ and } 1 \leq k \leq t_1, \\ [a, b] & \text{if } z \in G \text{ and } t_1 + 1 \leq k \leq t_1 + t_2, \\ [a, b] & \text{if } z \in H \text{ and } 1 \leq k \leq t_1, \\ F_2(z)(k - t_1) & \text{if } z \in H \text{ and } t_1 + 1 \leq k \leq t_1 + t_2. \end{cases}$$

It follows immediately that $F_3$ is an interval coordinatization of length $t_1 + t_2$ for $G \oplus H$, and thus $t_3 \leq t_1 + t_2$.

We now show that $t_3 \geq t_1 + t_2$. Let $F$ be an interval coordinatization of length $t_3$ of $G \oplus H$. Then let $S_1$ and $S_2$ be the subsets of $\{1, 2, 3, \ldots, t_3\}$ defined by

$$S_1 = \{i : \text{There exist nonadjacent vertices } x_1, x_2 \in G \\ \text{so that } F(x)(i) \cap F(x)(i) = \emptyset\}$$

and

$$S_2 = \{i : \text{There exist nonadjacent vertices } y_1, y_2 \in H \\ \text{so that } F(y)(i) \cap F(y)(i) = \emptyset\}.$$ 

We show that $S_1 \cap S_2 = \emptyset$, $|S_1| \geq t_1$ and $|S_2| \geq t_2$. This will allow us to conclude that $t_3 = |S| \geq |S_1| + |S_2| \geq t_1 + t_2$.

To see that $S_1 \cap S_2 = \emptyset$, we observe that if $i \in S_1$, $x_1, x_2 \in G$, and $F(x)(i) \cap F(x)(i) = \emptyset$, then

$$F(x)(i) \cap F(y)(i) \neq \emptyset \neq F(x)(i) \cap F(y)(i)$$

for every $y \in H$, i.e., the interval $F(y)(i)$ contains the open interval of $\mathbb{R}$ lying between the disjoint closed intervals $F(x)(i)$ and $F(x)(i)$. Hence $F(y)(i) \cap F(y)(i) \neq \emptyset$ for every $y_1, y_2 \in H$ and thus $i \not\in S_2$.

To see that $|S_1| \geq t_1$, let $|S_1| = m$ and $S_1 = \{i_1, i_2, \ldots, i_m\}$. Then the function $F'$ defined by $F'(x)(j) = F(x)(i_j)$ for every $x \in G$ and every $j \leq m$ is an interval coordinatization of length $m$ for $G$; hence, $m \geq t_1$. The same argument shows that $|S_2| \geq t_2$, so that our argument is complete in the case when $t_1 \geq 1$ and $t_2 \geq 1$.

We now consider the case when $t_1 = 0$ or $t_2 = 0$. First, if both $t_1$ and $t_2$ are zero, then $G, H,$ and $G \oplus H$ are complete graphs so that $\Box (G \oplus H) = 0 = \Box (G) + \Box (H)$. By symmetry, it remains only to consider the case where $t_1 = 0$ and $t_2 > 0$. Then $\Box (G \oplus H) \geq \Box (H)$ since $H$ is a subgraph of $G \oplus H$. To show that $\Box (G \oplus H) \leq \Box (H)$, we choose an arbitrary interval representation $F$ of length $t_2$ for $H$. We then select an interval $[a, b]$ of $\mathbb{R}$ so that $F(y)(i) \subseteq [a, b]$ for every $y \in H$ and $i \leq t_2$. Finally, we extend $F$ to $G \oplus H$ by defining $F(x)(i) = [a, b]$.
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for every $x \in G$ and $i \leq t_2$. It is clear that we have obtained an interval representation of $G \oplus H$ of length $t_2$ so that $\text{Box}(G \oplus H) \leq \text{Box}(H)$, and with this observation, the proof of the lemma is complete.

Let $G_1$ be the graph consisting of two nonadjacent vertices. For $n \geq 1$, we then define $G_n$ inductively by $G_{k+1} = G_k \oplus G_1$. It follows immediately that $|G_n| = 2n$ and $\text{Box}(G_n) = n$. Therefore, Roberts’ inequality (Theorem 1) is best possible. We now use the graph $G_n$ for $n \geq 1$ to examine the sharpness of the following inequalities which follow easily from Lemmas 1 and 2.

**Lemma 4.** If $K$ is a complete subgraph of $G$ with $|K| = k$, then $\text{Box}(G) \leq k + \text{Box}(G - K)$.

**Lemma 5.** If $I$ is an independent induced subgraph of $G$ with $|I| = i$, then $\text{Box}(G) \leq \lfloor i/2 \rfloor + \text{Box}(G - 1)$.

Label the vertices of $G_n$ with the symbols $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ so that the subgraphs $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are complete and $a_i$ and $b_j$ are adjacent if and only if $i \neq j$. Now suppose $1 \leq k \leq n$ and let $K$ be the $k$-element complete subgraph $\{a_1, a_2, \ldots, a_k\}$. It follows immediately from Lemma 3 that $\text{Box}(G) = k + \text{Box}(G - K)$ so that Lemma 4 is also best possible.

To test the accuracy of Lemma 5, it is first necessary to modify the graph $G_n$. For each $n \geq 1$, let $G_n^*$ be the graph obtained from $G_n$ by removing all edges between distinct vertices of $A$.

**Theorem 2.** $\text{Box}(G_n^*) = \{n/2\}$ for all $n \geq 1$.

**Proof.** Suppose first that $n = 2m$. For each $i \leq m$, let $F(a_{2i-1})(i) = [0, 1]$, $F(a_{2i})(i) = [4, 5]$, $F(b_{2i-1})(i) = [2, 4]$, $F(b_{2i})(i) = [1, 3]$, and if $j \neq 2i - 1$, $j \neq 2i$, then $F(b_j)(i) = [0, 5]$, and $F(a_j)(i) = [2, 3]$. Clearly the function $F$ is an interval coordinatization of length $m$ for $G_n^*$. Therefore, $\text{Box}(G_n^*) \leq m$ for all $m \geq 1$. The general result $\text{Box}(G_n^*) = \{n/2\}$ now follows from Lemma 2.

On the other hand, suppose $\text{Box}(G_n^*) = s$, and let $F$ be an interval coordinatization of length $s$ for $G_n^*$. For each $i \leq 2$, let

$$M(i) = \{j : F(a_j)(i) \cap F(b_j)(i) = \emptyset\}.$$  

We first observe that $|M(i)| \leq 2$ for each $i \leq s$, for if $j$, $j_2$, and $j_3$ are distinct elements of $M(i)$, we may assume by symmetry that $F(b_{j_1})(i)$ lies entirely to the left of $F(a_j)(i)$, $F(b_{j_2})(i)$ lies entirely to the left of $F(a_{j_2})(i)$, and that the right endpoint of $F(b_{j_2})(i)$ is at least as large as the right endpoint $F(b_{j_1})(i)$. But this would imply that $F(a_{j_2})(i) \cap F(b_{j_2})(i) = \emptyset$. Therefore, $|M(i)| \leq 2$.

Since we must clearly have $\sum_{i=1}^{s} |M(i)| \geq n$ it follows that $s \geq \{n/2\}$ and the argument is complete.
Now suppose \( 1 \leq k \leq n \) and \( I \) is the independent induced subgraph \( \{a_1, a_2, \ldots, a_k\} \) in \( G_n^k \). It follows from Theorem 2 and Lemma 3 that \( \text{Box}(G_n^k - 1) = \{(n-i)/2\} \). Therefore,

\[
\text{Box}(G_n^k) = \left\{ \frac{i}{2} \right\} + \text{Box}(G_n^k - 1),
\]

and the inequality in Lemma 5 is also best possible.

3. A characterization of Roberts' inequality

Let \( H_2 \) be the 5 element cycle \( \{c_1, c_2, c_3, c_4, c_5\} \) with \( c_i \perp c_{i+1} \) for \( i = 1, 2, 3, 4 \) and \( c_5 \perp c_1 \). For \( n \geq 2 \), we then define \( H_n \) inductively by \( H_{k+1} = H_k \oplus G_1 \). Since an interval graph does not contain a cycle of 4 or more vertices as an induced subgraph, we note that \( \text{Box}(H_2) = 2 \). By Lemma 3, we then conclude that \( \text{Box}(H_n) = n \) for every \( n \geq 2 \).

Consider the graph \( W_3 \) shown in Fig. 2.

![Fig. 2.](image)

We will now show that this graph has boxicity 3. First, note that \( \text{Box}(W_3) \leq 3 \) since \( |W_3| = 7 \). Now suppose that \( \text{Box}(W_3) < 3 \) and let \( F \) be an interval coordinatization of length two for \( W_3 \). For \( i = 1, 2 \), let

\[
E_i = \{ \{v_j, v_k\} : 1 \leq j < k \leq 7, F(v_j)(i) \cap F(v_k)(i) = \emptyset \}.
\]

It is easy to see that \( |E_1 \cup E_2| \geq 7 \) but that \( |E_1| \leq 3 \) and \( |E_2| \leq 3 \). The contradiction completes the argument.

We then define \( W_n \) for \( n \geq 3 \) by \( W_{k+1} = W_k \oplus G_1 \); by Lemma 3, we conclude that \( \text{Box}(W_n) = n \) for every \( n \geq 3 \).

The remainder of this section will be devoted to proving that the graphs \( G_n, H_n, \) and \( W_n \) provide an explicit characterization of Roberts' inequality for boxicity. In order to simplify the argument, we develop several preliminary lemmas. These lemmas will require the following result which follows from Lekkerkerker and Boland's characterization of interval graphs [2].

**Lemma 6.** If \( |G| \leq 5 \), then \( \text{Box}(G) = 2 \) if and only if \( G \) contains \( G_2 \) or \( H_2 \).

**Lemma 7.** If \( n \geq 1 \) and \( |G| = 2n \), then \( \text{Box}(G) = n \) if and only if \( G = G_n \).
Proof. For \( n = 1 \), we note that the complete graph on two vertices has boxicity zero while the independent graph on two vertices \( G_1 \) has boxicity one. The result follows from Lemma 6 when \( n = 2 \).

Now assume validity for all values of \( n \leq m \) where \( m \geq 2 \) and let \( G \) be a graph with \( |G| = 2m + 2 \) and \( \text{Box}(G) = m + 1 \). If \( G \) is complete, then \( \text{Box}(G) = 0 \), so \( G \) has vertices \( x, y \) with \( x \perp y \). Then \( G - \{x, y\} \) has \( 2m \) vertices and boxicity \( m \) and is therefore \( G_m \). Label \( G - \{x, y\} = G_m \) with the symbols \( a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \) so that \( a_i \perp a_j, b_i \perp b_j, \) and \( a_i \perp b_j \) if and only if \( i \neq j \) for \( i, j = 1, 2, \ldots, m \). Now the graphs \( G - \{a_1, b_1\} \) and \( G - \{a_2, b_2\} \) each have \( 2m \) points and boxicity \( m \) and must also be copies of \( G_m \). It follows that \( x \) and \( y \) are adjacent to every vertex of \( G - \{x, y\} \) and therefore \( G = \{x, y\} \oplus G_m = G_{m+1} \) and our proof is complete.

Suppose for some \( n \geq 1 \), \( G \) is a graph with \( 2n + 1 \) vertices. If \( G \) has a vertex \( x \) of degree \( 2n \), then \( G = \{x\} \oplus (G - \{x\}) \) so that \( \text{Box}(G) = \text{Box}(G - \{x\}) \) and thus \( \text{Box}(G) = n \) if and only if \( G \) contains \( G_n \).

Lemma 8. Let \( G \) be a graph with \( |G| = 7 \). If \( G \) has a vertex of degree 5, then \( \text{Box}(G) = 3 \) if and only if \( G \) contains \( G_3 \) or \( H_3 \).

Proof. Choose a vertex \( x \) of degree 5 and then choose \( y \) with \( x \perp y \). Now \( G - \{x, y\} \) has 5 vertices and boxicity 2 and therefore contains \( G_2 \) or \( H_2 \). If \( y \) is adjacent to each vertex of \( G - \{x, y\} \), then \( G = \{x, y\} \oplus (G - \{x, y\}) \) so that \( G \) contains \( G_2 \) or \( H_3 \).

Therefore, we may assume that there exists a vertex \( z \in G - \{x, y\} \) with \( z \perp y \). Therefore, \( G - \{z, y\} \) has boxicity 2 and

\[
G - \{x, y, z\} = \{x\} \oplus (G - \{x, y, z\})
\]

Thus, \( G - \{x, y, z\} = G_2 \). Label \( G - \{x, y, z\} = G_2 \) with the symbols \( a_1, a_2, b_1, b_2 \) so that \( a_1 \perp a_2, a_1 \perp b_2, b_1 \perp a_2, \) and \( b_1 \perp b_2 \). If \( y \) is adjacent to each vertex in \( G - \{x, y, z\} \), then \( G \) contains \( G_3 \) so we may assume without loss of generality that \( y \perp a_1 \). Then \( G - \{a_2, b_2\} \) has boxicity 2 and thus contains \( G_2 \) or \( H_2 \), but this is not possible since \( y \) has degree at most one and \( x \) has degree 3 in \( G - \{a_2, b_2\} \). The contradiction completes the proof.

Lemma 9. Let \( G \) be a graph with \( |G| = 7 \). Then \( \text{Box}(G) = 3 \) if and only if \( G \) contains \( G_3, H_3 \), or \( W_3 \).

Proof. Let \( G \) be a graph with \( |G| = 7 \) and \( \text{Box}(G) = 3 \). If \( G \) contains a vertex of degree 5 or 6, then \( G \) must contain \( G_3 \) or \( H_3 \), so we may assume without loss of generality that each vertex of \( G \) has degree at most 4.

Suppose that there exist a nonadjacent pair of vertices \( x, y \) so that \( G - \{x, y\} = H_2 \). Label the vertices of \( G - \{x, y\} \) with the symbols \( c_1, c_2, c_3, c_4, c_5 \) so that \( c_i \perp c_{i+1} \) for \( i = 1, 2, 3, 4 \) and \( c_5 \perp c_1 \). Since \( x \) has degree at most 4, we may assume
that \( x \pm c_2 \). Then \( G - \{c_1, c_2\} \) has boxicity 2 but does not contain \( G_2 \) or \( H_2 \). The contradiction allows us to conclude that for every nonadjacent pair of vertices \( x, y \) in \( G \), \( G - \{x, y\} \) contains \( G_2 \) but not \( H_2 \).

Now choose nonadjacent vertices \( x, y \) in \( G \) and a vertex \( z \) of \( G - \{x, y\} \) so that \( G - \{x, y, z\} = G_2 \). Label the vertices of \( G - \{x, y, z\} \) with the symbols \( a_1, a_2, b_1, b_2 \) as in the proof of Lemma 8. Suppose that \( x \neq z \). If

\[
G - y = \{x, y\} \oplus (G - \{x, y, z\})
\]

or

\[
G - z = \{x, y\} \oplus (G - \{x, y, z\}),
\]

then \( G \) contains \( G_3 \) so we may assume without loss of generality that \( y \pm a_1 \). Then

\[
G - \{a_2, b_2, y\} = G_2
\]

so that \( x \perp a_1, x \perp b_1, z \perp a_1 \), and \( z \perp b_1 \). And since \( b_1 \) has degree at most 4, we see that \( y \pm b_1 \). Therefore, \( G - \{y, b_1, a_1\} = G_2 \), \( x \perp a_2, x \perp b_2, z \perp a_2, z \perp b_2 \), and we conclude that

\[
G - y = \{x, z\} \oplus (G - \{x, y, z\}) = G_3.
\]

The contradiction allows us to conclude that \( x \perp z \) and \( y \perp z \).

Since \( y \) has degree at most 4, we may assume that \( y \pm a_1 \). Then

\[
G - \{a_2, b_2, a_1\} = G_2
\]

and thus \( x \perp b_1, y \perp b_1, \) and \( z \pm b_1 \). If \( y \pm a_2 \), then

\[
G - \{b_1, z, y\} = G_2
\]

and thus \( x \perp a_2, x \perp b_2, \) and \( x \pm a_1 \). Therefore, \( G - \{a_1, b_1, a_2\} = G_2 \) and thus \( y \perp b_2 \) and \( z \pm b_2 \). But \( G - \{z, b_2\} \) does not contain \( G_2 \). Therefore, we may assume that \( y \perp a_2 \). By symmetry, we may also assume \( y \perp b_2 \).

If \( x \pm a_1 \), then \( G - \{x, a_1, y\} = G_2 \) so \( z \perp a_2 \) and \( z \perp b_2 \). Therefore, \( x \pm a_2 \) and \( x \pm b_2 \). But \( G - \{x, b_2\} \) does not contain \( G_2 \). The contradiction allows us to conclude that \( x \perp a_1 \). Since \( x \) has degree at most 4, we may then assume that \( x \pm a_2 \).

It follows that \( G - \{x, a_2, b_1\} = G_2 \) and thus \( z \perp a_1, z \pm b_2 \). Also we see that \( G - \{y, a_1, b_2\} = G_2 \) and thus \( z \perp a_2 \). Finally we note that if \( x \pm b_2 \), then \( G - \{x, b_2\} \) does not contain \( G_2 \) so that we must have \( x \perp b_2 \). It follows that all adjacencies of \( G \) have been determined and that \( G = W_3 \).

We are now ready to establish our characterization of Roberts' inequality for boxicity. Theorem 3 will provide for each \( n \geq 1 \) the minimum collection \( \mathcal{C}_n \) of graphs so that if \( |G| = 2n + 1 \), then Box \( (G) = n \) if and only if \( G \) contains a graph from \( \mathcal{C}_n \) as an induced subgraph.

**Theorem 3.** Let \( n \geq 1 \) and let \( G \) be a graph with \( |G| = 2n + 1 \).

(i) If \( n = 1 \), then Box \( (G) = n \) if and only if \( G \) contains \( G_1 \).

(ii) If \( n = 2 \), then Box \( (G) = n \) if and only if \( G \) contains \( G_2 \) or \( H_2 \).

(iii) If \( n \geq 3 \), then Box \( (G) = n \) if and only if \( G \) contains \( G_n \), \( H_n \) or \( W_n \).

**Proof.** Part (i) is trivial since a complete graph has boxicity zero; part (ii) is
Lemma 6. We now proceed to prove part (iii) by induction on $n$. We first note that part (iii) is valid for $n = 3$ in view of Lemma 9. We then assume validity for all $n \leq m$ where $m$ is some integer with $m \geq 3$. Then let $G$ be a graph with $|G| = 2m + 3$ and $\text{Box}(G) = m + 1$. We will now show that $G$ contains $G_{m+1}$, $H_{m+1}$, or $W_{m+1}$.

Let $x, y$ be any pair of nonadjacent vertices in $G$. Then $G - \{x, y\}$ has $2m + 1$ vertices and boxicity $m + 1$ and therefore must contain $G_m$, $H_m$, or $W_m$. Suppose first that there exists a nonadjacent pair of $x, y$ of vertices of $G$ so that $G - \{x, y\} = H_m$. Label the vertices of $G - \{x, y\}$ with the symbols $a_1, a_2, \ldots, a_{m-2}, b_1, b_2, \ldots, b_{m-2}, c_1, c_2, c_3, c_4, c_5$ in the obvious fashion.

As in the proof of Lemma 9, if $x \neq c_2$, then $G - \{c_1, c_3\}$ has boxicity $m$ but does not contain $G_m$, $H_m$, or $W_m$ since $c_2$ has degree at most $2m - 3$ and $x$ has degree at most $2m - 2$ in $G - \{c_1, c_3\}$. We may therefore conclude that $x$ and $y$ are both adjacent to $c_1, c_2, c_3, c_4,$ and $c_5$.

Now consider the graph $G - \{c_1, c_3\}$ which has boxicity $m$. Since $c_4$ and $c_5$ have degree $2m - 1$ in $G - \{c_1, c_3\}$, $c_4 \neq c_2$, and $c_5 \neq c_2$, we see that $G - \{c_1, c_3\}$ is not $W_m$ or $H_m$. And therefore, $G - \{c_1, c_3\}$ must contain $G_m$. It is easy to see that we must have either $G - \{c_1, c_3, c_4\} = G_m$ or $G - \{c_1, c_3, c_5\} = G_m$. In either case, $x$ and $y$ are both adjacent to $a_1, a_2, \ldots, a_{m-2}, b_1, b_2, \ldots, b_{m-2}$ so that

$$G = \{x, y\} \oplus (G - \{x, y\}) = H_{m+1}.$$ 

Now suppose that $x$ and $y$ are nonadjacent vertices of $G$ and that $G - \{x, y\} = W_m$. Suppose first that $m = 3$ and label $G - \{x, y\}$ with the symbols $v_1, v_2, \ldots, v_7$ as shown in Fig. 2. Suppose further that $x \neq v_1$. Then $G - \{v_3, v_4\}$ has boxicity 3 but $v_1$ has degree at most 3 so $G - \{v_3, v_4\}$ is neither $W_3$ or $H_3$. But it is easy to see that $G - \{v_3, v_4\}$ does not contain $G$, either. We may therefore conclude that $x$ and $y$ are adjacent to $v_1, v_2, \ldots, v_7$ and therefore, $G = W_4$.

Now suppose that $m \geq 4$ and label the vertices of $G - \{x, y\}$ with the symbols $a_1, a_2, \ldots, a_{m-3}, b_1, b_2, \ldots, b_{m-3}, v_1, v_2, \ldots, v_7$ in the obvious fashion. As in the preceding paragraph, we may conclude that $x$ and $y$ are adjacent to $v_1, v_2, \ldots, v_7$. Now consider the graph $G - \{v_1, v_2\}$ which has boxicity $m$.

Now $v_1$ has degree $2m - 1$ and $v_3 \neq v_4$ and $v_5 \neq v_4$ in $G - \{v_1, v_2\}$ so $G - \{v_1, v_2\}$ is not $W_m$ or $H_m$. Therefore, $G - \{v_1, v_2\}$ must contain $G_m$. Clearly this requires $G - \{v_1, v_2, v_3\} = G_m$ and thus, $x$ and $y$ are adjacent to $a_1, a_2, \ldots, a_{m-3}, b_1, b_2, \ldots, b_{m-3}$. Therefore,

$$G = \{x, y\} \oplus (G - \{x, y\}) = W_{m+1}.$$ 

We may now assume that whenever $x$ and $y$ are nonadjacent vertices of $G$, the graph $G - \{x, y\}$ contains $G_m$, but not $H_m$ or $W_m$. Choose a nonadjacent pair of vertices $x, y$ and a vertex $z$ so that $G - \{x, y, z\} = G_m$. Label the vertices of $G - \{x, y, z\}$ with the symbols $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ in the usual fashion. Now suppose that $x \neq z$. If $y \neq a_1$, then $G - \{a_2, b_2\}$ does not contain $G_m$ so we may assume that $y$ is adjacent to $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$. Similarly, if $x \neq a_1,$
then \( G - \{a_2, b_2\} \) does not contain \( G_m \), so we may assume that \( x \) is also adjacent to \( a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \). But this implies that \( G - z = G_{m+1} \). We may therefore assume that \( x \perp z \). By symmetry, we may also assume \( y \perp z \).

Now suppose that \( x \pm a_1 \). Then we must have \( G - \{a_2, b_2, a_1\} = G - \{a_3, b_3, a_1\} = G_m \) and thus, \( G - a_1 = G_{m+1} \). We may therefore assume by symmetry that \( x \) and \( y \) are both adjacent to \( a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \) and, therefore, \( G - z = G_{m+1} \). With this case, the argument is complete.

4. The characterization of rectangle graphs

A graph \( G \) with \( \text{Box}(G) \leq 2 \) is the intersection graph of a family of rectangles (with sides parallel to the \( x \) and \( y \) axes) in the plane, so it is natural to refer to a graph with boxicity at most 2 as a rectangle graph. In this section, we discuss the problem of providing a forbidden subgraph characterization of rectangle graphs. While this is a very difficult unsolved combinatorial problem, we will solve the subproblem of determining a forbidden subgraph characterization for rectangle graphs with clique covering number two. We accomplish this by establishing combinatorial connections between this problem and characterization problems for partially ordered sets and circular arc graphs as discussed in [6]. In the interests of brevity, we provide only the key definitions here and refer the reader to [6] for details. If \( [a, b] \) and \( [c, d] \) are closed intervals of the real line \( \mathbb{R} \), we write \( [a, b] \triangleleft [c, d] \) when \( b < c \) in \( \mathbb{R} \). The interval dimension of a partially ordered set \( X \), denoted \( \text{I Dim}(X) \), is then the smallest positive integer \( n \) for which there exists a function \( F \) assigning to each point \( x \in X \) a sequence \( F(x)(1), F(x)(2), \ldots, F(x)(n) \) of closed intervals of \( \mathbb{R} \) so that \( x < y \) in \( X \) if and only if \( F(x)(i) \triangleleft F(y)(i) \) for \( i = 1, 2, 3, \ldots, n \).

A partially ordered set \( X \) is said to be \( t \)-interval irreducible when \( \text{I Dim}(X) = t \) and \( \text{I Dim}(X - x) = t - 1 \) for every \( x \in X \). Let \( \mathcal{P}_2 \) denote the collection of all 3-interval irreducible partially ordered sets of height 1.

A graph \( G \) is called a circular arc graph when it is the intersection graph of a family of arcs of a circle. Let \( \mathcal{A}_2 \) denote the collection of all graphs with clique covering number two which are not circular arc graphs but have the property that the removal of any vertex leaves a circular arc graph. Also, let \( \mathcal{B}_2 \) denote the collection of all graphs with clique covering number two which have boxicity 3, but have the property that the removal of any vertex leaves a subgraph with boxicity 2.

For a graph \( G \), we denote by \( \bar{G} \), the complement of \( G \), i.e., \( x \perp y \) in \( \bar{G} \) if and only if \( x \pm y \) in \( G \). Now let \( X \) be a partially ordered set of height one with maximal elements \( a_1, a_2, \ldots, a_m \) and minimal elements \( b_1, b_2, \ldots, b_n \). We associate with \( X \), graphs \( G_X \) and \( G_X^\perp \), each having

\[ \{a_1, a_2, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_n\} \]
as vertex sets. In $G_x$ and $G_x^\alpha$, the subgraphs induced by $\{a_1, a_2, \ldots, a_m\}$ and
$\{b_1, b_2, \ldots, b_n\}$ are complete. In $G_x$ we define $a_i \perp b_j$ if and only if $b_j < a_i$ while in
$G_x^\alpha$ we define $a_i \preceq b_j$ if and only if $b_j < a_i$.

Dually, for a graph $G$ with vertex set $\{a_1, a_2, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_n\}$ for which
the subgraphs induced by $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\}$ are complete, we
denote by $X_G$ the partially ordered set of height one for which $G = G_{x_G}$. Among
the results established in [6] is the following theorem relating circular arc graphs
to partially ordered sets.

**Theorem 4.** Let $X$ be a partially ordered set of height one. Then $X \in \mathcal{P}_2$ if and only
if $G_x^\alpha \in \mathcal{A}_2$.

Now let $G$ be a graph with vertex set $\{a_1, a_2, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_n\}$ for
which the subgraphs induced by $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\}$ are com-
plete. Suppose that $\text{Box}(G) = 2$ and let $F$ be an interval coordinatization of length
two for $G$. Since $\text{Box}(G) = 2$, assume by symmetry, that for $k = 1, 2$ there exist
$i_k, j_k$ with $1 \leq i_k \leq m$ and $1 \leq j_k \leq n$ so that $F(a_{i_k})(k) < F(b_{j_k})(k)$. Clearly, we may
further assume that $F(x)(k)$ is a subset of the open interval $(0, 1)$ for each vertex $x$
of $G$ and $k = 1, 2$.

Now consider the function $F'$ which assigns to each vertex $x$ of $G$ a pair
$F'(x)(1), F'(x)(2)$ of closed intervals of $\mathbb{R}$ defined as follows: For $i = 1, 2, \ldots, m$
and $k = 1, 2$, let $F'(a_i)(k) = [r, 1]$ where $r$ is the right end point of $F(a_i)(k)$; for
$j = 1, 2, \ldots, n$ and $k = 1, 2$, let $F'(b_j)(k) = [0, l]$ where $l$ is the left end point of
$F(b_j)(k)$. It follows that for $k = 1, 2, i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, we have
$F(a_i)(k) \cap F(b_j)(k) \neq \emptyset$ if and only if $F'(b_j)(k) < F'(a_i)(k)$ and therefore, $F'$
is an interval representation of length two for the partially ordered set $X_G$. This process
is easily seen to be reversible and we have thus established the following theorem relating
rectangle graphs to circular arc graphs and partially ordered sets.

**Theorem 5.** Let $X$ be a partially ordered set of height one. Then the following
statements are equivalent.

(i) $X \in \mathcal{P}_2$,
(ii) $G_x \in \mathcal{B}_2$,
(iii) $G_x^\alpha \in \mathcal{A}_2$.

The reader is referred to [6] where a complete determination of $\mathcal{P}_2$ is given.

5. Some related topics

We conclude this paper with some references to related papers. First, we note
that the author and K.P. Bogart [4] have proved a characterization theorem for
interval dimension which is analogous to Theorem 3. For an integer $n \geq 3$, let $S^n_n$
denote the partially ordered set $Y$ of height one for which $G_n = G_Y$. Then it follows that for each $n \geq 2$, if $X$ is a partially ordered set having $2n + 1$ points, then $\text{IDim}(X) = n$ if and only if $X$ contains $S^n_0$.

We also refer the reader to [1] where Feinberg has extended the concept of boxicity to arcs on a circle by defining the circular dimension of a graph, $D(G)$, as the smallest positive integer $n$ for which there exists a function $F$ assigning to each vertex $x$ of $G$ a sequence $F(x)(1), F(x)(2), \ldots, F(x)(n)$ of arcs on a circle so that $x \perp y$ in $G$ if and only if $x \neq y$ and $F(x)(i) \cap F(y)(i) \neq \emptyset$ for $i = 1, 2, \ldots, n$. Since $D(G) \leq \text{Box}(G)$, we have the analogous inequality $D(G) \leq [\lceil G \rceil / 2]$. However, Feinberg observed that $D(G_n) = 1$ for all $n \geq 1$. Feinberg constructed for each $n \geq 1$ a graph with $2^n + n - 1$ vertices and circular dimension $n$ and conjectured that this family characterized graphs with maximum circular dimension. This conjecture is incorrect, since it is straightforward to prove, using Erdos' probabilistic methods, that for large $n$, there exists a graph with $n$ vertices whose circular dimension exceeds $n/(4 \log n)$. However, the general question of the relative accuracy of $D(G) \leq [\lceil G \rceil / 2]$ is unanswered.

Finally, we mention the paper by Trotter and Harary [5], who defined the interval number of a graph $G$, denoted $i(G)$, as the smallest positive integer $n$ for which there exists a function $F$ assigning to each vertex $x$ of $G$ a sequence $F(x)(1), F(x)(2), \ldots, F(x)(n)$ of closed intervals of $\mathbb{R}$ so that distinct vertices $x, y$ of $G$ are adjacent in $G$ if and only if $F(x)(i) \cap F(y)(j) \neq \emptyset$ for some $i, j$ with $1 \leq i \leq n$ and $1 \leq j \leq n$. Alternately, $i(G)$ is the smallest $n$ for which $G$ is the intersection graph of a family of sets where each set is the union of $n$ intervals of $\mathbb{R}$. Trotter and Harary showed that the complete bipartite graph $K_{m,n}$ has interval number $((mn + 1)/(m + n))$.

References