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Partially Ordered Sets with Equal Rank and Dimension

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Abstract

A family $F$ of linear extensions of a partial order $P$ is called a realizer when $\cap F = P$; a realizer $F$ is called irredundant when $\cap G \neq P$ for any proper subfamily $G \subset F$. The dimension of $P$ is the number of linear extensions in a smallest (hence irredundant) realizer; the rank is the number in a largest irredundant realizer. Rank is clearly equal or greater than dimension, and usually much greater. Here we determine those partial orders which have dimension equal to rank. Let $2^n$ be the partial order of subsets of an $n$-set. Basically, our posets are subposets of $2^n$ which include all the 1-sets and (n-1)-sets, as well as those posets obtained from these by stretching elements into chains.

1. Introduction

A partially ordered set (poset) is a pair $(X,P)$ where $X$ is a set (always finite in this paper) and $P$ is an irreflexive, transitive, binary relation on $X$. We write $x \geq y$ when either $(x,y) \in P$ or $x = y$. A family $F = \{L_1, L_2, \ldots, L_k\}$ of linear extensions of $P$ (all still defined on $X$) is called a realizer of $P$ when $\cap F = P$, i.e., $(x,y) \in P$ iff $(x,y) \in L_i$ for all $i$. A realizer is said to be irredundant when $\cap G \neq P$ for any proper nonempty subfamily $G$ of $F$.

The dimension $d(P)$ of $(X,P)$ is defined as the smallest number of linear extensions in any realizer of $P$. This old definition has both pure and applied interest. For instance,

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d(P) is easily seen to be the lowest number n such that the elements of X can be represented by points in \( \mathbb{R}^n \) so that P is precisely the partial order induced by the usual partial order on n-tuples. As for an application, posets arise as preference relations in measurement theory. One psychological explanation as to how people come to their preference, and in particular, why they are sometimes indifferent, is that they actually rate items on several linear scales simultaneously (perhaps unknowingly) and only sense a preference when all the scales agree as to the better choice. Then d(P) is just the smallest number of scales necessary to "explain" a person’s preference order P.

Unfortunately, dimension, though extensively studied, is extremely difficult to compute. See [7] for a good bibliography. It is easy enough to construct irredundant (i.e., minimal) realizers for any P, and every realizer containing d(P) linear extensions (i.e., a minimum realizer) is irredundant. However, not every irredundant realizer is minimum. In light of the difficulty of computing d(P) exactly, one might hope to get a good approximation by constructing some irredundant realizer of P.

With this approach in mind, the first two authors [1] made the following definition and asked the following question. The rank \( r(P) \) is the largest number of linear extensions in any irredundant realizer of P. How large can \( r(P) \) be relative to \( d(P) \)? Unfortunately, the answer is that, when \( |X| = n \), \( r(P) \) can be as large as \( n^2/4 \), while simultaneously \( d(P) \) can be 2. Therefore a different question is in order: for which types of posets is rank a good estimator of dimension? In particular, characterize those P for which \( r(P) = d(P) \). Let us call a poset satisfying this equality an RB poset. In this paper we characterize RB posets.

We will make use of a general theory of rank which we have presented elsewhere [2]. This theory makes the computation of rank much easier than the computation of dimension, but by no means trivial in general. Basically, the theory says that it suffices to look at all digraphs with vertex set X and which meet certain simple properties related to P; the largest number of edges in any such graph is \( r(P) \). Thus the actual construction of realizers of P is entirely bypassed!

The organization of this paper is as follows. In the next section we introduce notation and state our characterization theorem. In Section 3 we provide (and reference) those concepts and facts from the general theory of rank which we use in the proof of our theorem, which follows in Section 4.

For other results about rank, see [2,3,4].

2. Statement of the Characterization Theorem

Let \( n \) denote an n-element chain and \( Q_n \) an n-element anti-chain. If \( (X,P) \) and \( (Y,Q) \) are posets, define the join \( (X,P) \oplus (Y,Q) \) as the poset \( (X \cup Y, P \uplus Q \cup (X \times Y)) \), where \( \cup \) is disjoint union. Note that in \( (X,P) \oplus (Y,Q) \), every point of \( X \) is greater than every point of \( Y \). Let \( (X,P) \) be a poset and let \( \{(Y_\alpha, Q_\alpha)\}: x \in X \) be a family of posets. The ordinal sum \( \Sigma \sum (Y_\alpha, Q_\alpha) \) is the poset \( (Z,R) \) where \( x \in X \) and \( Z = \{(x,y) \in Y_\alpha \times X, x \in X \} \), and \( ((x,y),(x',y')) \in R \) iff either \( (x,x') \in P \) or \( x = x' \) and \( (y,y') \in Q_\alpha \). In this paper we are concerned only with ordinal sums where \( (X,P) \) is a collection of sets ordered by inclusion and each \( (Y_\alpha, Q_\alpha) \) is a chain.

Let \( \mathbb{R}^n \) denote the poset determined by the collection of all subsets of \( \{1, 2, \ldots, n\} \), ordered by inclusion. Let \( C_n \) denote the collection of all subposets of \( \mathbb{R}^n \) containing all the 1-element and (n-1)-element subsets. (We say \( (Y,Q) \) is a subposet of \( (X,P) \) if \( Y \subseteq X \) and \( Q \) consists of all pairs from \( P \) with elements from \( Y \).) For \( n \geq 3, \ 1 \neq n-1 \) and the subposet on
the collection of 1-element and (n-1)-element subsets is called the crown $C_n^*$. (See [5] for the more general $C_n^k$.) Finally, let $C_n$ be the collection of all posets which can be obtained by taking ordinal sums of families of chains over posets from $C_n$.

We can now state our Theorem. Note first that $d(F) = 1$ implies $r(F) = 1$ trivially, i.e., every linear order is an $F$-poset. So we may henceforth restrict ourselves to posets with $d(F) \geq 2$.

**Theorem 1.** Let $(X,P)$ be a poset and let $n$ be any integer with $n \geq 2$. Then $d(F) = r(F) = n$ if and only if $(X,P) \in C_n^*$; except if $n = 2$ then $(X,P)$ can also be any poset $n \ast 2 \ast p \ast q \ast g$, with $n,p,q \geq 0$.

3. Theory of Rank

For poset $(X,P)$, we say $x \not\leq y$ are incomparable, i.e., $(x,y) \in P^* = I$, iff $(x,y), (y,x) \not\in P$. There is a partial order called forcing induced on $I$ by $P$, namely, $(x,y)$ forces $(x',y')$ iff the transitive closure $Tr(P \cup \{(x,y)\})$ contains $(x',y')$. The set of nonforcing pairs, $N(P) = N$, is the set of $(x,y) \in I$ which force no other pairs. Equivalently, $(x,y) \in N$ iff for all $z \in X$, $(x,z) \in P \Rightarrow (z,y) \in P$, and $(z,x) \in P \Rightarrow (z,x) \in P$. $N$ is a family of ordered pairs on $X$, will be thought of as a digraph on $X$. The size of a digraph is the number of edges. A "path" will always mean "directed path".

It is easy to show that if $(x,y)$ is not forcing, then $(y,x)$ is unforced. Moreover, since $X$ is finite, every $(x,y) \in I$ is forced by some unforced pair. It follows easily (or see the proof of the similar Lemma 4.4, in [2]) that we have

**Lemma 2.** A family $F$ of linear extensions of $P$ is a realizer of $F$ iff $(x,y) \in F$ for every $(x,y) \in N$.

Thus we say that $P$ realizes $F$ if it "turns over" every $(x,y) \in N$.

A subgraph $G \subseteq N$ is unipathic relative to $F$, and is henceforth called a $U_p$ subgraph, if whenever there are two edge-disjoint directed paths from $x$ to $y$ in $F$, then $(x,y) \in F$. (If $F$ is an antichain, a $U_p$ graph is unipathic as previously defined in the literature.)

**Theorem 3** (Main Theorem on Rank, Form 2, in [2]). Let $F$ be nonlinear. Let $r$ be the size of a largest subgraph of $N$ which is either a cycle or acyclic $U_p$. Then $r = r(F)$.

Note: to show $r \geq r(F)$ is relatively easy; $r \leq r(F)$ is hard.

In rank theory, cycles are a nuisance, for $a)$ $N$ can have lots of cycles, but $b)$ rarely does the size of some cycle equal $r(F)$. To elaborate on claim a), $Y \subseteq X$ is said to have duplicate holdings in $F$ iff for all $x \in Y$, either $(x,z) \in F$ for all $z \in Y$ or none, and $(z,y) \in F$ for all $z \in Y$ or none. Also, a complete two-way digraph on $X$ has both edges $(y,y')$ and $(y',y)$ for all $y,y' \in Y$. Then we have

**Lemma 4** (3.9 in [2]). There is a cycle in $N$ with vertex set $Y$ since the restriction of $N$ to $Y$ is a complete two-way digraph $\Rightarrow Y$ has duplicate holdings in $F$.

As for claim b), call $F$ rank degenerate if every largest subgraph described in Theorem 3 is a cycle. Every rank degenerate poset must have duplicate holdings, but much more is true:

**Theorem 5** (see proof of Main Theorem, Form 1, in [2]). $P$ is rank degenerate iff $P$ is a subposet of $n \ast 3 \ast g$.

Once the small class of rank degenerate posets is removed, Theorem 3 requires that we avoid cycles. Happily, it is possible to restrict attention to a subset of $N$ which is already acyclic. Namely, for any linear order $L$ on $X$ (L need not even extend $P$) define

$$N^* = N \cup \{(x,y) \in N: (x,y) \not\in N \text{ or } (y,x) \not\in N\}.$$

It might seem that $N^*$ depends on $L$, but in fact, up to isomorphism, it does not. This follows from Lemma 4. Thus Theorem 3 readily implies:
Theorem 6 (Main Theorem, Form 3, from [2]). Suppose $P$ is not rank degenerate. Then $r(P)$ is the size of any largest $U_P$-graph in $N$.

As we show in Section 4, with two exceptions, RD posets do not have duplicate holdings, so for them Theorem 6 applies and $N = H$.

To prove Theorem 1, we need not so much the main Theorem(a) on rank as certain more preliminary facts about $N$. In addition to Lemma 4 these are:

**Lemma 7** (3.5 in [2]). If $x_1, x_2, \ldots, x_n$ are the vertices in order of a directed path in $P \uplus N$, and at least one edge is in $P$, then $(x_i, x_{i+1}) \in P$.

**Lemma 8** (3.8 in [2]). $\text{Tr}(P \uplus N) = P \uplus N \cup \{(x,x) : x \text{ on a cycle in } N\}$.

By a transitive tournament in $N$, we mean a subgraph which is a linear order. From Lemmas 7 and 8 we immediately get:

**Lemma 9.** If $(x_i, x_{i+1}) \in N$, $i = 1, 2, \ldots, n-1$, and $(x_1, x_n) \in N$, then for all $1 \leq i \leq n$, $(x_i, x_j) \in N$. That is, $x_1, \ldots, x_n$ is the ordered vertex set of a transitive tournament in $N$.

(Note: basically the same argument proves the first implication in Lemma 4.)

Finally, we need one older result often used in studying dimension. A set $\{(x_i, y_i) : 1 \leq i \leq n\}$ is called a TM-cycle for $P$ when $(x_i, y_j) \in P$ for all $i$, and $y_j \geq x_i$ in $P$ if $j = 2i \mod n$. Then it is easy to verify (see Theorem 1 of [6] and commentary afterwards):

**Theorem 10.** Let $(x, P)$ be a poset and let $S \subseteq I$. Then $\text{Tr}(P \uplus S)$ is a partial order iff $S$ contains no TM-cycle.

We will use Theorem 10 as follows. Suppose we find $N \subseteq H$ such that in the reverse set $\overline{H} = \{(y, x) : (x, y) \in H\}$, every pair of edges is in a TM-cycle. Then no two edges in $N$ can be turned over in the same linear extension of $P$. Thus, by the definition of dimension, $d(P) \leq |N|$.

4. Proof of Theorem 1.

The outline of our proof is:

1) All posets listed in the Theorem are RD posets.
2) With two exceptions, RD posets have no duplicate holdings.
3) For any $P$, let $t$ be the number of maximal transitive tournaments in $N$. Then $d(P) \leq t$. If for some distinct maximal transitive tournaments $T_1, T_2$, there is a linear extension of $P \uplus T_1 \uplus T_2$, then $d(P) < t$. Now assume $P$ has no duplicate holdings. Then $t = r(P)$. If in addition some maximal transitive tournament has more than one edge, then $t > r(P)$. Thus, if any RD poset not excepted in 2), each edge of $N$ is a maximal tournament; no two edges of $N$ can be turned over simultaneously; and $|N| = d(P)$.
4) Let $P$ be an RD poset with $d(P) = n$ and no duplicate holdings. Label $N$ as $(a_1, b_1) : 1 \leq i \leq n$ so that we may set $A = (a_i : 1 \leq i \leq n)$, $B = (b_i : 1 \leq i \leq n)$. Then with one exception, the subposet of $P$ on $A \cup B$ is the crown $S_n$.
5) For all $P$ described in 4), $P$ as a whole is as described in the Theorem.

We now proceed in turn with these parts.

1). Recall that elements in posets in $C_3$ correspond to subsets of $N = \{1, \ldots, n\}$. Set $i = n - 1$. For any fixed $P$ in $C_n^*$, let $c_i$ be the element at the top of the chain corresponding to $i$. Let $d_i$ be the element at the bottom of the chain corresponding to $i$. Let $C = (c_1, \ldots, c_n)$, $D = (d_1, \ldots, d_n)$. It is straightforward to show that $N = (c_i, d_i) : 1 \leq i \leq n$. See Theorem 6.4 in [2] for the main details. When $n = 3$, the edges of $N$ are independent; for $n \geq 2$ they may be a path or a cycle. In any event, $r(P) \leq n$ by Theorem 3. On the other hand, every pair of edges in $N$ are on a TM-cycle. So $d(P) \leq n$ by Theorem 10.
and the remarks after it. So P is an RD poset.

As for P = n = r = 2, it is easy to see that N consists of 2 disjoint 2-cycles and, by Theorem A, r(P) = 2. Clearly, d(P) = 2 also.

2. Suppose (X,P) has duplicate holdings. Let x_1, ..., x_k be the disjoint maximal duplicate holdings sets. Thus each |x_i| ≥ 2. Pick some x_1 ∈ X_1 for each i, and set Y = X - x(X_1-x_2).

Let Q be the subposet of P on Y. It is easy to see that if d(Q) > 1, then d(Q) = d(P); and if d(Q) = 1, then d(P) = 2. It is equally easy to see that in all cases, r(P) > r(Q): for let H be a largest subgraph of N, as in Theorem 3. H may be empty (if d(Q) = 1), but since Q has no duplicate holdings, H is acyclic in any event (Lemma 4). Now append to H any one edge between vertices of, say, x_1. The new graph is in N, and is acyclic in N, if any edge between vertices of, say, x_1. Thus r(P) > r(Q).

Therefore, if P is an RD poset with duplicate holdings, d(P) = r(P) = 2 and P = n_1 = n_2 = n_3 = ... = n_k = n_k+1, with all n_i ≥ 0 and some n_i > 1. i.e., P is a so-called weak order. However, it is easy to check (use Theorem 3 and Lemma 4) that a nonlinear weak order has r(P) = 2 if and only if a) k = 1 and n_1 = 2, or b) k = 2 and n_1 = n_2 = 2. (For the general formulas for rank of a weak order, see Cor. 6.6 in [2].) Case a) is included in C_1; case b) is the exceptional case of Theorem A.

3. For any transitive tournament T in N, P ∪ T always has a linear extension: it is easy to see that T does not contain any TM-cycles. Since every edge in N is in some maximal T, d(P) ≤ t by Lemma 2. Likewise, if some P ∪ T_1 ∪ T_2 has a linear extension, Lemma 2 gives d(P) ≤ t-1.

Now suppose P has no duplicate holdings. Let T_1, ..., T_k be the maximal transitive tournaments. For each i, let a_i be the longest edge of T_i, that is, the edge from the first element a_i to the last element b_i. We claim that the graph G = (e_1, ..., e_k) is U, hence r(G) ≥ t by Theorem 6. We show H is U by showing something much stronger for every path in H with 2 or more edges, if x and y are its initial and final vertices, then (x,y) ∈ P. By Lemma 7, it suffices to prove this property for paths of length exactly 2.

So suppose (a_1, b_1) (a_2, b_2) is a path, i.e., b_1 = a_2. Then (a_1, b_2) ∈ P ∪ N by Lemma 8. We show (a_2, b_2) ∈ N. For let x_1 = a_2, x_2, ..., x_k = b_1 be the vertices in order of T_1, and let x_1 = a_2, x_2, ..., x_k = b_1 be the vertices in order of T_2. If (a_1, b_1) ∈ N, then by Lemma 9, every (x_i, x_j) ∈ E for i < j. But then x_1, ..., x_k is a bigger tournament than either T_1 or T_2, contradicting maximality. (There cannot be any repeats among the x_i's, for then N would contain a cycle.)

Now suppose some T, say T_1, has more than one edge, i.e., at least 3 vertices. Pick any e_i = (a_i, x) ≠ e_j from T_1. We claim H ∈ T satisifies the same condition on paths of length 2 as N, hence in U. Thus r(P) ≥ t + 1. To prove this claim, we need only show the condition for paths e_i, e_j; and e_i, e_j, i ≠ j ≥ 2; for e_i, e_j do not form a directed path. The proof is essentially the same as the proof for H: if any such path had its end pair in N, then T_1 would not be maximal.

4. If i ≠ j, then a_i ≠ a_j and b_i ≠ b_j. For otherwise e_i, e_j can be turned over together, violating d(P) = n by Lemma 2. So N consists of vertex disjoint paths. Such a path can contain 3 edges. For if x_1, x_2, x_3, x_4 were consecutive vertices in N, then (x_1, x_2) and (x_3, x_4) could be turned over together; a TM-cycle would imply (x_1, x_2) ∈ P ∪ I = 0. Moreover, if one path in N contains 2 edges, then there are no other paths: for suppose N contained (x_1, x_2), (x_3, x_4), and (x_1, x_3). Since d(P) = n, Lemma 2 and Theorem 10, applied to (x_2, x_3), (x_1, x_2)), would imply (x_1, x_3) ∈ P, and applied to (x_1, x_2), (x_3, x_4) would imply (x_1, x_2) ∈ P. Thus (x_2, x_3) ∈ P ∪ N = 0.
Thus either \( n = 2 \) and \( N \) is a 2-path (the exception), or else \( N \) consists of \( n \) independent edges (here \( n \geq 2 \), since \( P \) is not linear). In the latter case, since no \( a_i, b_i \) may be turned over together, Theorem 10 implies \((a_i,b_j) \in P \). Moreover \((a_i,b_j) \in P \) else either \( a_i \) or \( b_j \) is in \( P \). Likewise, \((b_j,a_i) \in P \). Thus the subposet of \( P \) on \( A \cup B \) is the crown \( S^2_n \). Previously we defined the crown for \( n \geq 3 \) only, but when \( n = 2 \) the poset we now get on \( A \cup B \) is a natural choice to call \( S^2_2 \); only the interpretation of vertices in terms of distinct 1-sets and (n-1)-sets is missing, and as we will see, distinctness of set labels is about to be lost anyway.

5). For all \( x \in X \), define \( A(x) = \{i : a_i \geq x \text{ in } P\} \), \( B(x) = \{i : x \geq b_i \text{ in } P\} \). We will show:

i) For each \( x \), \( A(x) \), \( B(x) \) partition \( n = \{1, \ldots, n\} \).

ii) \((x,y) \in I \iff \) either \( B(x) \) or \( B(y) \) is contained in the other.

iii) If \( B(x) \subseteq B(y) \), then \((y,x) \in P\).

It follows from ii) that for each \( S \subseteq \mathbb{N} \), the restriction of \( P \) to \( \{x : B(x) = S\} \) is a linear order. It then follows from iii) that \( P \) is an ordinal sum of chains, where the sum is over a subposet \((Z, \mathcal{O})\) of \( 2^n \). It follows from the analysis in 4) of the subposet on \( A \cup B \) that \( Z \) contains all 1-element and (n-1)-element subsets.

When \( n = 2 \), then \( n = 1 \) and \( A \cup B \) gives us each I-set at least once, whether \( A \cup B \) is a crown or the exception. Thus, when we prove i) - iii), we have finished the proof of our Theorem.

We need two observations. First, for any \( P \) whatever, if \((x,y) \in I \), then \((x,y) \) forces some \((z,w) \in N \), where \( z \geq x \) and \( y \geq w \) in \( P \). Second, for all \( P \) as in 4), including the exception, \((a_i,b_j), (b_j,a_i) \in I \). Now to prove i), clearly no \( i \in A(x) \cap B(y), \) for then \((a_i,b_j) \in P \). Suppose some \( i \in A(x) \cup B(y) \). It cannot be that both \((x,a_i), (b_j,x) \in P \), for then \((b_j,a_i) \in P \). So either \((a_i,x) \in I \) or \((x,b_j) \in I \). \((a_i,x) \) would force some \((a_i,b_j), \) and since then \( a_i \geq x \) and \( x \geq b_j \) in \( P \), we would have \( i = 1 \) and \( (a_1,b_1) \in I \).

\( i \in B(x) \). Dually, \((x,b_j) \) would force the contradiction \( i \in A(x) \).

To prove ii), if \((x,y) \in I \), then \((x,y) \) forces some \((a_i,b_j) \).

By the definition of forcing, \( i \in A(x) \cap B(y) = (g - B(x)) \cap B(y) \), so \( i \in B(y) \). Dually, \((y,x) \in I \) too, and this forces some \((a_i,b_j) \), so \( j \in B(y) \). To prove the reverse implication, if \((x,y) \in P \), then clearly \( B(y) \subseteq B(x) \), and if \((y,x) \in P \), then \( B(x) \cap B(y) \). These facts also prove iii), for of the exhaustive possibilities \((x,y) \in I \), \((x,y) \in P \), \((y,x) \in P \), the only one compatible with \( B(x) \cap B(y) \) is \((y,x) \in P \).

References