Article Title: A bound on the interval number of a complete multipartite graph

William Trotter (wt48)
School of Mathematics
Georgia Tech
Atlanta, GA 30332

Faculty
Math

COPYRIGHT NOTICE:
This material may be protected by copyright law (Title 17 U.S. Code).
A Bound on the Interval Number of a Complete Multipartite Graph

L.B. HOPKINS
W.T. TROTTER

Abstract

The interval number of a graph \( G \) denoted \( i(G) \), is the least positive integer \( t \) for which \( G \) is the intersection graph of a family of sets each of which is the union of \( t \) pairwise disjoint intervals of the real line. For example, a graph \( G \) is an interval graph if and only if \( i(G) = 1 \), while \( i(C_n) = 2 \) for all \( n \geq 4 \). Griggs showed that the maximum value of the interval number of a graph on \( n \) vertices is \( \lceil (n+1)/4 \rceil \) and Trotter and Harary showed that the interval number of the complete bipartite graph \( K(n_1,n_2) \) is given by the formula \( i(K(n_1,n_2)) = \lceil (n_1n_2+1)/(n_1+n_2) \rceil \). Several researchers have been investigating the problem of determining the interval number of complete multipartite graphs, and it was conjectured that the interval number of the complete multipartite graph \( K(n_1,n_2,n_3,\ldots,n_p) \), where \( n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_p \) and \( p \geq 3 \), equals the interval number of the complete bipartite graph \( K(n_1,n_2) \). In support of this conjecture, Matthews proved that for every \( p \geq 3 \), if \( n_1 = n_2 = n_3 = \ldots = n_p \), then \( i(K(n_1,n_2,\ldots,n_p)) = i(K(n_1,n_2)) \). However, D. West disproved the conjecture by showing that for each \( n \geq 3 \), there exists a constant \( c_n \) so that if \( n_1 = n^2 - n - 1 \), \( n_2 = n_3 = \ldots = n_p = n \), and \( p \geq c_n \), then \( i(K(n_1,n_2,n_3,\ldots,n_p)) = 1 + 391 \).
392 A Bound on the Interval Number of a Complete Multipartite Graph

\( i(K(n_1, n_2)) \). In view of West's counterexample, it was suggested that the interval number of a complete multipartite graph might exceed the interval number of the bipartite graph, formed by the largest two parts, by an arbitrarily large amount. In this paper, we prove to the contrary that \( i(K(n_1, n_2, n_3, \ldots, n_p)) \leq 1 + i(K(n_1, n_2)) \) for all \( p, n_1, n_2, \ldots, n_p \) with \( p \geq 3 \) and \( n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_p \).

1. Introduction.

In recent years, there has been considerable interest in generalizations of interval graphs. Much of the research is motivated by the wide range of interpretations which may be given to optimization and extremal problems involving interval graphs. In this paper, we consider the subject of t-interval graphs. For a positive integer \( t \), we represent a graph as the intersection graph of a family of sets each of which is the union of \( t \) pairwise disjoint intervals of the real line. Among the several extremal problems involving t-interval graphs, we will be concerned with minimizing \( t \) for a given graph or class of graphs. If we view a t-interval graph as a work schedule permitting cooperation between certain specified components of the work force while safeguarding against interference between other components, then the minimization of \( t \) yields a schedule in which each component has relatively few work periods. Consequently, the inherent inefficiency of starting up and closing down unnecessary work periods of short duration and limited productivity is avoided.

Among the classes of graphs for which this extremal problem is quite natural is the class of complete bipartite graphs where the work force is subdivided into two units with no interference permitted between any two components in the same unit, but cooperation required between any two components from different

Proceedings-Fourth International units. In this paper, we will consider complete multipartite graphs and reduce the problem in the multipartite case to that arising from the bipartite graph.

2. Notation and Terminology.

Trotter and Harary [4] defined the t-interval graph of a graph \( G \) as a function \( F(x) = \{ \text{intervals of the real line } \mathbb{R} \} \) of distinct vertices, we have \( x \in G \) if there exists a pair of intervals \( i(x) \cap i(y) \neq \emptyset \). For which \( G \) has a t-interval graph \( i(G) \) is the least integer \( t \) such that \( G \) is the intersection graph of a family of sets each of \( t \) closed intervals of \( \mathbb{R} \). Denote such an interval graph if and only if \( i(G) = t \).

We will find it convenient to use (degenerate) closed interval notation in specifying an interval graph. For example, Figure 1 provides a representation of the graph \( G \). Note that \( i(G) = 2 \).

Furthermore, we will deal with a single representation, all isolated intervals may be simplified as in Figure 2.

Throughout this paper, we will assume that, as shown in Figures 1 and 2, the intervals will be spread out with the reader should bear in mind that...
Considerable interest in such of the research is problems involving interval representations which may be subject of t-interval we represent a graph as a set of intervals of the real line. Involving t-interval graphs, for a given graph or interval graph as a work schedule certain specified components against interference optimization of t yields a relatively few work periods. The variety of starting up and of short duration and which this extremal problem complete bipartite graphs where units with no interference in the same unit, but components from different units. In this paper, we will discuss the natural generalization to complete multipartite graphs and will show that the extremal problem in the multipartite case does not differ substantially from the bipartite graph.

2. Notation and Terminology.

Trotter and Harary [4] defined a t-interval representation of a graph G as a function F which assigns to each vertex x ∈ G a sequence F(x)(1), F(x)(2), ..., F(x)(t) of closed intervals of the real line R so that for every pair x, y of distinct vertices, we have x adjacent to y in G if and only if there exists a pair of integers i, j with 1 ≤ i, j ≤ n so that F(x)(i) ∩ F(y)(j) ≠ ∅. The interval number of a graph G, denoted i(G), is then defined as the least positive integer t for which G has a t-interval representation. Alternately, i(G) is the least integer t for which G is the intersection graph of a family of sets each of which is the union of at most t closed intervals of R. In particular, a graph G is an interval graph if and only if i(G) = 1.

We will find it convenient to consider a point as a (degenerate) closed interval and will frequently use this convention in specifying an interval representation of a graph. For example, Figure 1 provides a 2-interval representation of the graph G. Note that G is not an interval graph so i(G) = 2.

Furthermore, we will delete from Figure 1 for an interval representation, all isolated intervals and points. So Figure 1 may be simplified as in Figure 2.

Throughout this paper, we will use diagrams similar to those shown in Figures 1 and 2 to illustrate interval representations. Intervals will be spread out vertically for clarity but the reader should bear in mind that all intervals are to be projected
A Bound on the Interval Number of a Complete Multipartite Graph

Figure 1.

onto a single horizontal line.

Figure 2.

We now present a brief summary of recent research involving interval numbers. We begin with the following elementary result due to Trotter and Harary [4].

\textbf{Theorem 1 [4].} If $T$ is a tree, then $i(T) \leq 2$.

J. Griggs [2] has established the following upper bound on the interval number as a function of the order of the graph.

\textbf{Theorem 2 [2].} If $G$ is a graph on $n$ vertices, then $i(G) \leq \lceil (n+1)/4 \rceil$.

Proceedings-Fourth Internation...

J. Griggs and D. West [1] bound on $i(G)$ as a function of $\Delta$ in $G$.

\textbf{Theorem 3 [1].} If $\Delta$ is the maximum degree of $G$, then $i(G) \leq \lceil (\Delta+1)/2 \rceil$.

Griggs and West [1] showed that if $G$ is triangle-free, then equality holds in their result as an immediate consequence.

\textbf{Corollary 4 [1].} For each $n$, the distance-2 graph $Q_n$ is given by $i(Q_n) = n$.

Trotter and Harary [4] determined the interval number of a complete bipartite graph $K(m,n)$.

\textbf{Theorem 4 [4].} The interval number of a graph $K(m,n)$ is given by:

$$i(K(m,n)) = \begin{cases} m & \text{if } m \leq \frac{n}{2} \\ n & \text{if } m > \frac{n}{2} \end{cases}$$

3. Interval Numbers of Complete Graphs

In the remainder of this section the computation of the interval number for a complete $p$-partite graph $K(n_1, n_2, \ldots, n_p)$ will require $n_1 \geq n_2 \geq \ldots \geq n_p$. To simplify notation we denote the interval number of $K(n_1, n_2, \ldots, n_p)$ by $i(n_1, n_2, \ldots, n_p)$.

To prove the following theorem we consider the construction due to M. Matthew for $K(n,n)$ be $A \cup B$ where $A = \{a_1, a_2, \ldots, a_n\}$ with each $a_i$ adjacent to every vertex in $B$.
J. Griggs and D. West [1] have also established an upper bound on $i(G)$ as a function of the maximum degree of a vertex in $G$.

**Theorem 3 [1].** If $\Delta$ is the maximum degree of a vertex in $G$, then $i(G) \leq \lceil (\Delta + 1)/2 \rceil$.

Griggs and West [1] showed that if $G$ is regular and triangle-free, then equality holds in Theorem 3. The following result follows as an immediate corollary.

**Corollary 4 [1].** For each $n \geq 1$, the interval number of the $n$-cube $Q_n$ is given by $i(Q_n) = \lceil (n+1)/2 \rceil$.

Trotter and Harary [4] developed a formula for the interval number of a complete bipartite graph.

**Theorem 4 [4].** The interval number of the complete bipartite graph $K(m,n)$ is given by:

$$i(K(m,n)) = \lceil (mn+1)/(m+n) \rceil.$$

3. **Interval Numbers of Complete Multipartite Graphs**

In the remainder of this paper, we will be concerned with the computation of the interval number of a complete multipartite graph $K(n_1, n_2, \ldots, n_p)$ where $p \geq 2$. By convention, we will require $n_1 \geq n_2 \geq \ldots \geq n_p$. For simplicity, we let $i(n_1, n_2, \ldots, n_p)$ denote the interval number of $K(n_1, n_2, \ldots, n_p)$; note that $i(n_1, n_2) = \lceil (n_1 n_2 + 1)/(n_1 + n_2) \rceil$ by Theorem 4. Since $K(n_1, n_2)$ is an induced subgraph of $K(n_1, n_2, \ldots, n_p)$, we always have $i(n_1, n_2) \leq i(n_1, n_2, \ldots, n_p)$. We now proceed to show that $i(n_1, n_2, \ldots, n_p)$ never exceeds $n_2$. We begin by presenting a construction due to M. Matthews [3]. We let the vertex set of $K(n,n)$ be $A \cup B$ where $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$ with each $a_i$ adjacent to every $b_j$.

First let $n$ be even, say $n = 2r$. Then the following
diagram provides an \( r + 1 \) - interval representation of \( K(n,n) \).

\[
\begin{array}{cccccccc}
  a_1 & r-1 & a_2 & r-1 & a_3 & r-1 & \ldots & a_n & r-1 & a_1 \\
  b_1 & r-1 & b_2 & r-1 & b_3 & r-1 & \ldots & b_n & r-1 & b_1 \\
\end{array}
\]

Figure 3.

In the gap between \( a_i \) and \( a_{i+1} \) (cyclically), occur the \( r - 1 \) points corresponding to \( a_{i+2}, a_{i+3}, \ldots, a_{i+r} \) (cyclically). Similarly, the gap between \( b_i \) and \( b_{i+1} \) contains points corresponding to \( b_{i+2}, b_{i+3}, \ldots, b_{i+r} \). Here is a diagram when \( r = 3 \). For simplicity, only the subscripts are given.

\[
\begin{array}{cccccccccccc}
  1 & 34 & 2 & 45 & 3 & 56 & 4 & 61 & 5 & 12 & 6 & 23 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
  1 & 34 & 2 & 45 & 3 & 56 & 4 & 61 & 5 & 12 & 6 & 23 & 1 \\
\end{array}
\]

Figure 4.

We shall continue to use the convention followed in Figures 3 and 4 for bipartite and multipartite graphs i.e., the diagram will be presented in "levels" with all intervals occurring in the same level corresponding to vertices in the same part.

The reader is encouraged to compare this example with the construction given by Trotter and Harary [4] for a \( 4 \)-representation of \( K_{6,6} \). The advantage of Matthews’ construction is that it can easily be extended to multipartite graphs. It suffices to add additional "levels" to the diagram following the same intersection pattern as determined by the first two. For example,
A Complete Multipartite Graph

Here is a 3-representation (with labels deleted) of \( K(6,6,6) \).

\[
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
1 & \cdots & b_{r-1} & b_1 \\
1 & \cdots & a_{r-1} & a_1 \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Cyclically), occur the \( r-1 \) \( \ldots, a_{i+r} \) (cyclically).

Here is a diagram where the subscripts are given.

\[
\begin{array}{rrrrrr}
5 & 12 & 6 & 23 & 1 & \cdots \\
5 & 12 & 6 & 23 & 1 & \cdots \\
\end{array}
\]

Convention followed in Figures 5

The diagrams i.e., the diagram of intervals occurring in the same part.

Here is an example with the reference [4] for a 4-representa-

The construction that they used is that of multipartite graphs. It suffices to use the following the same interval for the first two. For example,

\[
\begin{array}{rrrrrrrrrr}
1 & \cdots & 2 & 45 & 3 & 51 & 4 & 12 & 5 & 23 & 1 \\
1 & \cdots & 2 & 45 & 3 & 51 & 4 & 12 & 5 & 23 & 1 \\
\end{array}
\]

Figure 5.

More generally, it is easy to see that when \( n = 2r \), this construction produces for each \( p \geq 2 \), an \( r+1 \) -interval representation of \( K(n_1, n_2, \ldots, n_p) \) where \( n_1 = n_2 = \ldots = n_p = n \).

It follows that when \( n = 2r \), we have \( r+1 \) is \( \left[ \frac{4r^2 + 1}{4r} \right] = i(n,n) \leq i(n_1, n_2, \ldots, n_p) \leq r + 1 \), and thus \( i(n_1, n_2, \ldots, n_p) = r + 1 \).

When \( n \) is odd, say \( n = 2r + 1 \) then we construct an \( r+2 \) -interval representation of \( K(n,n) \) using the same scheme as above. For example, here is the diagram for \( K(5,5) \):

\[
\begin{array}{rrrrrrrrrr}
1 & \cdots & 34 & 2 & 45 & 3 & 51 & 4 & 12 & 5 & 23 & 1 \\
1 & \cdots & 34 & 2 & 45 & 3 & 51 & 4 & 12 & 5 & 23 & 1 \\
\end{array}
\]

Figure 6.

Reading the diagram from left to right, we then remove from each level the first occurrence of 1 as a point in a gap. The resulting diagram is an \( r+1 \) -interval representation of
A Bound on the Interval Number of a Complete Multipartite Graph

K(2r + 1, 2r + 1).

\[ \begin{array}{cccccccc}
1 & 34 & 2 & 45 & 3 & 5 & 12 & 5 \\
1 & \cdots & 34 & \cdots & 2 & \cdots & 45 & \cdots & 3 & \cdots & 5 & \cdots & 12 & \cdots & 5 & \cdots & 1
\end{array} \]

Figure 7.

As before, this construction is easily extended to show that whenever \( p \geq 2 \) and \( n = 2r + 1 = n_1 = n_2 = \ldots = n_p \), then \( r + 1 = i(n,n) = i(n_1,n_2,\ldots,n_p) \). We have then established the following result of Matthews [3].

**Theorem 5.** For every \( p \geq 2 \) and every \( n \geq 1 \), if \( n_1 = n_2 = \ldots = n_p = n \), then \( i(n_1,n_2,\ldots,n_p) = i(n_1,n_2) = \lceil (n^2 + 1)/2n \rceil = \lceil (n+1)/2 \rceil \).

From Theorem 5 we obtain the following upper bound.

**Corollary 6.** If \( p \geq 2 \) and \( n_1 \geq n_2 \geq \ldots \geq n_p \), then \( i(n_1,n_2,\ldots,n_p) \leq n_2 \).

**Proof.** It suffices to establish the result when \( p \geq 3 \) and \( n_2 = n_3 = \ldots = n_p \). Set \( n = n_2 \) and then choose a \( \lceil (n+1)/2 \rceil \) -interval representation of the complete \( p - 1 \) partite graph \( K(n,n,n,\ldots,n) \) as provided in the preceding theorem.

We then observe that for each \( i \) with \( 1 \leq i \leq n \), there are \( p - 1 \) intervals, one from each level of the diagram, each of which has label \( i \) so that the intersection of these \( p - 1 \) intervals is a nondegenerate interval. We may then insert in each of these intervals, \( n_1 \) points – one for each element in the part of size \( n_1 \). The resulting diagram is a \( n_2 \) -interval representation of \( K(n_1,n_2,\ldots) \).

We illustrate this result below:

\[ \begin{array}{cccc}
1 & 34 & 2 & 45 \ldots \\
1 & \cdots & 34 & \cdots & 2 & \cdots & 45 & \cdots & \ldots & 1
\end{array} \]

**Figure 8.**

For the remainder of the paper, we will assume the convention that \( m \geq n_1 \) and \( p \geq 1 \). If \( n \geq 1 \), into the subsets \( |A| = m \) and \( |B_i| = n_i \) for \( i = 1 \) to \( p \), we label the vertices in \( A \) with the symbols \( a_1, a_2, \ldots, a_m \) and the vertices in \( B_i \) with \( b_{i1}, b_{i2}, \ldots, b_{in_i} \). When presenting a representation, we will present the intervals (or points) corresponding to the vertices in the highest level which we consider downwards, the intervals (or points) corresponding to the vertices in \( B_1 \) will be displayed in \( B_1 \) first.

When \( m \geq n \geq 1 \) and \( p \geq 1 \), the complete \( p + 1 \) -partite graph \( i(m,n,p) \) denote the interval \( i(m,n,p) \) define \( i(m,n,p) = \sup \{ i(m,n,p) \} \) and then follows trivially.

**Theorem 7.** For every \( m, n \), we have \( i(m,n,p) \leq n \).

We next describe a construction.
representation of \( K(n_1, n_2, \ldots, n_p) \).

We illustrate this result for a diagram for \( K(4, 3, 3, 3) \).

\[
\begin{array}{ccc}
1234 & 1234 & 1234 \\
1 & 2 & 3 \\
\end{array}
\]

Figure 8.

For the remainder of the paper, we will adopt the following conventions. We partition the vertex set of the complete \( p + 1 \)-partite graph \( K(m, n_1, n_2, \ldots, n_p) \), where \( m \geq n_1 \geq n_2 \geq \ldots \geq n_p \) and \( p \geq 1 \), into the subsets \( A, B_1, B_2, \ldots, B_p \) where \( |A| = m \) and \( |B_i| = n_i \) for \( i = 1, 2, \ldots, p \). We label the vertices in \( A \) with the symbols \( a_1, a_2, \ldots, a_m \). For each \( i \), we label the vertices in \( B_i \) with the symbols \( b_{i1}, b_{i2}, \ldots, b_{in_i} \). When providing a diagram for an interval representation, we will present the intervals in levels. The intervals (or points) corresponding to vertices in \( A \) will be in the highest level which we call level zero. Then proceeding downwards, the intervals (or points) corresponding to vertices in \( B_i \) will be displayed in level \( i \).

When \( m \geq n \geq 1 \) and \( p \geq 1 \), we let \( K(m, n \cdot p) \) denote the complete \( p + 1 \)-partite graph \( K(m, n, n, \ldots, n) \) and let \( i(m, n \cdot p) \) denote the interval number of \( K(m, n \cdot p) \). We then define \( i(m, n \cdot \infty) = \sup \{ i(m, n \cdot p) : p \geq 1 \} \). The following result then follows trivially.

**Theorem 7.** For every \( m, n \) with \( m \geq n \geq 1 \), \( \frac{mn + 1}{m + n} \leq i(m, n) \leq i(m, n \cdot \infty) \leq n \).

We next describe a construction generalizing the technique.
A Bound on the Interval Number of a Complete Multipartite Graph

used in Theorem 5. Let \( \sigma \) be a sequence (with repetition allowed) of length \( \ell \) and let \( D \) be a subset of \( \{1, 2, 3, \ldots, \ell\} \) with \( \{1, \ell\} \in D \). Then we refer to the pair \((\sigma, D)\) as a sequence with distinguished positions, or DP-sequence for short.

To define such a pair, we will find it convenient to list in order the terms of \( \sigma \) and underline the distinguished positions, e.g., \((1, 2, 3, 2, 4, 2, 3, 2, 4, 1)\). For a DP-sequence \((\sigma, D)\) and an integer \( p \geq 1 \), we then associate an interval representation having \( p \) levels as illustrated below for the DP-sequence given above and \( p = 3 \).

\[
\begin{array}{cccccc}
1 & 23 & \cdots & 2 & 42 & 3 & 2 & 4 & 1 \\
1 & 23 & 2 & 42 & 3 & 2 & 4 & 1 \\
1 & 23 & 2 & 42 & 3 & 2 & 4 & 1
\end{array}
\]

Figure 9.

Suppose we have a DP-sequence \((\sigma, D)\) with the symbols in \( \sigma \) selected from \( \{1, 2, 3, \ldots, n\} \). Then it is elementary to determine when \((\sigma, D)\) produces an interval representation of \( K(n,n,\ldots,n) \). (Note that the question does not depend on the number of parts.) For emphasis, we state the characterization of such DP-sequences as a theorem, but we leave it to the reader to supply the straightforward proof.

**Theorem 8.** Let \((\sigma, D)\) be a DP-sequence of length \( \ell \) with the symbols in \( \sigma \) selected from \( \{1, 2, 3, \ldots, n\} \). Also let \( D = \{k_1, k_2, \ldots, k_d\} \) where \( 1 = k_1 < k_2 < k_3 < \ldots < k_d = \ell \). Then \((\sigma, D)\) produces an interval representation of \( K(n,n,\ldots,n) \) if and only if for every ordered pair \((j_1, j_2)\) from \( \{1, 2, \ldots, n\} \), there exists an integer \( \beta \) with one of the following statements:

- \( j_1 = \sigma(k_{\beta+1}) \) and \( j_2 = \sigma(k_{\beta}) \) and \( j_1 \neq j_2 \)

An essential feature of the proof depends on an Euler circuit in \( T(n) \). Let \( T(n) \) denote the graph on the vertex set \( \{1, 2, 3, \ldots, n\} \), i.e., \( ((i,j) : 1 \leq i, j \leq n, i \neq j) \in E \) \( \leq \binom{n-1}{2} \). We let \( T(n,s) \) denote the graph on the vertex set \( \{1, 2, \ldots, n\} \), whose edge set is \( ((i,j) : 1 \leq i, j \leq n + i - s \text{ cyclically}) \). Note that in which each vertex has degree \( 1 \).

It follows easily that \( T(n,s) \) has a directed Eulerian cycle. However, in \( T(n,s) \) we require an Euler circuit of \( T(n,s) \) with the additional property, namely that we select a sequence of vertices which begins with the first and ends with the last consecutive vertices in this sequence, and which begins with the following element.

**Lemma 9.** Let \( n \geq 2, s \geq 1 \), and let \( (1, 2, \ldots, n) \) be the following sequence of vertices which begin at \( 1 \) and end with \( n \). If \( s + 1 \) consecutive vertices in this sequence begin with \( 1 \), there are \( s + 1 \) consecutive vertices in this sequence which begin with \( 1 \), conclude with \( n \).

**Proof.** We note that the proof is straightforward.

There are then two cases. When \((\sigma, D)\) does not contain a directed cycle and the given sequence is not an Euler circuit of \( T(n,s) \), then we can choose a \( \delta \) such that \( \delta \) is an \( s + 1 \) consecutive vertices in this sequence.

On the other hand, when the given sequence is still...
there exists an integer \( \beta \) with \( 1 \leq \beta < d \) so that at least one of the following statements hold:

\begin{enumerate}
  \item \( j_1 = \sigma(k_\beta+1) \) and \( j_2 \notin \{ \sigma(k) : k_\beta \leq k \leq k_\beta+1 \} \)
  \item \( j_2 = \sigma(k_\beta) \) and \( j_1 \notin \{ \sigma(k) : k_\beta \leq k \leq k_\beta+1 \} \).
\end{enumerate}

An essential feature of the construction we are building depends on an Euler circuit in a directed graph. For an integer \( n \geq 2 \), let \( T(n) \) denote the complete directed graph with vertex set \( \{1,2,3,\ldots,n\} \), i.e., the edge set of \( T(n) \) is \( \{ (i,j) : 1 \leq i,j \leq n, i \neq j \} \). For an integer \( s \) with \( 1 \leq s \leq \left\lfloor \frac{(n-1)}{2} \right\rfloor \), we let \( T(n,s) \) denote the spanning subgraph of \( T(n) \) whose edge set is \( \{ (i,j) : 1 \leq i,j \leq n, i+s+1 \leq j \leq n+1-s \text{ (cyclically)} \} \). Note that \( T(n,s) \) is a regular graph in which each vertex has indegree and outdegree \( n-2s \).

It follows easily that \( T(n,s) \) has an Euler circuit (in the directed sense). However, in a construction to follow, we shall require an Euler circuit of \( T(n,s) \) which satisfies an additional property, namely that when the circuit is specified by a sequence of vertices which begins and ends at 1, any \( s+1 \) consecutive vertices in this sequence are distinct. To this end we proceed to explicitly construct such an Euler circuit. We begin with the following elementary result.

**Lemma 9.** Let \( n \geq 2, s \geq 1, \text{ and } s = \left\lfloor \frac{(n-1)}{2} \right\rfloor \). Then the following sequence of vertices is an Euler circuit of \( T(n,s) \) in which each \( s+1 \) consecutive vertices are distinct:

\[ 1, s+2, 2, s+3, 3, s+4, \ldots, n-1, s, n, s+1, 1. \]

**Proof.** We note that the hypothesis requires that \( n \geq 3 \). There are then two cases. When \( n \) is odd, \( T(n,s) \) is a directed cycle and the given sequence is easily seen to be an Euler circuit of \( T(n,s) \). Since \( s+1 \leq n \), we know that any \( s+1 \) consecutive vertices in the sequence are distinct.

On the other hand, when \( n \) is even, \( T(n,s) \) has \( 2n \) edges but the given sequence is still an Euler circuit. A set of \( s+1 \)
consecutive vertices in this sequence has the following form:

\{i, i+1, i+2, \ldots, i+s_1-1\} \cup \{i+s+1, i+s+2, \ldots, i+s+s_2\}

where \(s_1, s_2 > 0\) and \(s_1 + s_2 = s + 1\). Since these \(s + 1\) integers are distinct, the desired result follows.

**Lemma 10.** Let \(n \geq 2\) and \(1 \leq s < \left\lfloor \frac{(n-1)/2} \right\rfloor\). Then the sequence: \(1, s + 2, 2, s + 3, 3, \ldots, n - 1, s, n, s + 1, 1\) traverses a set of \(2n\) edges in \(\Gamma(n,s)\). If these \(2n\) edges are removed from \(\Gamma(n,s)\), then the remaining graph is \(\Gamma(n,s+1)\).

**Proof.** It suffices to observe that the sequence traverses exactly the edges in the following sets: \(\{(i, i+s+1): 1 \leq i \leq n\} \cup \{(i, 1-s): 1 \leq i \leq n\}\). But this set consists of precisely those edges which belong to \(\Gamma(n,s)\) but not \(\Gamma(n,s+1)\).

**Lemma 11.** Let \(n \geq 2, s \geq 1, \) and \(1 \leq s \leq \left\lfloor \frac{(n-1)/2} \right\rfloor\). Then \(\Gamma(n,s)\) has an Euler circuit in which each \(s + 1\) consecutive vertices are distinct.

**Proof.** The result follows from Lemma 9 when \(s = \left\lfloor \frac{(n-1)/2} \right\rfloor\). So we may assume that \(s < \left\lfloor \frac{(n-1)/2} \right\rfloor\). We then construct an Euler circuit \(\sigma\) by recursively applying Lemma 10. It remains only to show that every set of \(s + 1\) vertices in \(\sigma\) is distinct. Let \(S = \{\sigma(j): j_0 \leq j \leq j_0 + s\}\) be a set of \(s + 1\) consecutive vertices in \(\sigma\). Then let \(S_1 = \{\sigma(j): j_0 \leq j \leq j_0 + s, j\) odd\} and \(S_2 = \{\sigma(j): j_0 \leq j \leq j_0 + s, j\) even\}. Note that \(S_1\) is always a set of consecutive integers (cyclically). However, for some values of \(j_0\), \(S_2\) is a set of consecutive integers (cyclically), and for other values of \(j_0\), \(S_2\) is "almost" a set of consecutive integers with only a single missing integer preventing it from being a set of consecutive integers.

Suppose first that \(j_0\) is odd. If we let \(\sigma(j_0) = 1\) and \(s_1 = |S_1|\), then \(s_1 = \left\lceil \frac{(s+1)/2} \right\rceil\) and \(S_1 = \{i, i+1, i+2, \ldots, i+s_1-1\}\). Now let \(\sigma(j_0+1) = i + s_3 + 1\) and \(S_2 = \{i, i+1, \ldots, i+s_2-1\}\). Then \(S_2 = \{i+s_3+1, i+s_3+2, \ldots, i+s_2\}\). Then \(S_2 = \{i, i+1, \ldots, i+s_2-1\}\). Then \(S_2 = \{i+s_3+1, i+s_3+2, \ldots, i+s_2\}\).

Now consider the case where \(S_2 = \{i+s_3+1, i+s_3+2, \ldots, i+s_2\}\). Then \(S_2 = \{i, i+1, \ldots, i+s_2-1\}\). Then \(S_2 = \{i+s_3+1, i+s_3+2, \ldots, i+s_2\}\).

At the risk of belaboring the sequence determines an Euler circuit.

At the risk of belaboring the sequence determines an Euler circuit.

At the risk of belaboring the sequence determines an Euler circuit.

At the risk of belaboring the sequence determines an Euler circuit.
has the following form:

\[ s+1, i+s+2, \ldots, i+s+s_{2} \]

1. Since these \( s+1 \) elements are distinct, the following result follows.  

Then the graph \( T(n,s+1) \) is a complete multipartite graph, and thus, the \( s+1 \) vertices in \( S \) are distinct.

Now consider the case when \( j_{0} \) is even. As before let \( s_{1} = |S_{1}| \); also let \( i = o(j_{0}+1) \). Then \( s_{1} = \lfloor (s+1)/2 \rfloor \), and \( S_{1} = \{i, i+1, i+2, \ldots, i+s_{1}-1\} \). Now let \( o(j_{0}) = i+s_{3} \) and \( s_{2} = |S_{2}| \). Then \( s_{2} = \lfloor (s+1)/2 \rfloor \), \( s \leq s_{3} \leq (n-1)/2 \), and \( S_{2} \) is a subset of the following set of \( s_{2}+1 \) consecutive integers:

\[ \{i+s_{3}+1, i+s_{3}+2, \ldots, i+s_{3}+s_{2}+1\} \]

Since \( s_{1}-1 < s_{3}+1 \) and \( s_{3}+s_{2}+1 < n \), it follows that \( S_{1} \cap S_{2} = \emptyset \) and thus, the \( s+1 \) vertices in \( S \) are distinct. With this observation, the proof is complete.

At the risk of belaboring an obvious point, the following sequence determines an Euler circuit of \( T(n,3) \) in which every set of 3 consecutive vertices is distinct: \( 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5 \).

We then construct an \( \sigma \)-sequence for \( T(n,3) \). It remains for us to show that \( \sigma \) is distinct: \( i, i+1, i+2, \ldots, i+s_{1}-1 \).

Next, we define \( S_{1} = \{i, i+1, i+2, \ldots, i+s_{1}-1\} \).

If we let \( o(j_{0}) = i \) and \( S_{1} = \{i, i+1, i+2, \ldots, i+s_{1}-1\} \), then \( s = |S_{1}| \).

Then \( s_{2} = |S_{2}| \). Then \( s_{2} = \lfloor (s+1)/2 \rfloor \), \( s \leq s_{3} \leq (n-1)/2 \), and \( S_{2} \) is a subset of the following set of \( s_{2}+1 \) consecutive integers:

\[ \{i+s_{3}+1, i+s_{3}+2, \ldots, i+s_{3}+s_{2}+1\} \]

Since \( s_{1}-1 < s_{3}+1 \) and \( s_{3}+s_{2}+1 < n \), it follows that \( S_{1} \cap S_{2} = \emptyset \) and thus, the \( s+1 \) vertices in \( S \) are distinct. With this observation, the proof is complete.

**Theorem 12.** Let \( m \geq n \geq 1 \). Then \( 1(m,n) \leq 1 + \frac{1}{m,n} \).

**Proof.** It suffices to show that \( 1(m,n) \leq 1 + \frac{1}{m,n} \) for every \( p \geq 2 \). Choose an arbitrary \( p \geq 2 \). The desired result follows from Theorem 5 when \( m = n \) so we may assume that
m > n.

Now let \( t = \lfloor m/n \rfloor \), i.e., \( t = \left\lfloor (mn+1)/(m+n) \right\rfloor \). Then \( \left( n+1 \right)/2 \leq t \leq n \). If \( t \geq n-1 \), the result follows from Theorem 7 since \( i(m,n,p) \leq i(m,n,w) \leq n \). So we may also assume that \( t < n-1 \). Then let \( s = n - t \). We observe that \( 1 \leq s \leq \left\lfloor (n-1)/2 \right\rfloor \), and in fact \( s \geq 2 \).

We now construct a DP-sequence \((c_1,D_1)\) of length \( ns + 1 \) using the symbols \( \{1,2,3,\ldots,n\} \). The DP-sequence \((c_1,D_1)\) has \( n + 1 \) distinguished positions \( D_1 = \{(i-1)(s+1) + 1 \mid 1 \leq i \leq n + 1 \} \). The symbol 1 occurs in the distinguished position \((i-1)s + 1\) for \( i = 1,2,3,\ldots,n \) and the symbol 1 occurs in the distinguished position \( ns + 1 \); note that the symbol 1 is both the first and last symbol in \( c_1 \) and that both of these positions are distinguished.

For each \( i = 1,2,3,\ldots,n \), and each \( j = 1,2,3,\ldots,s-1 \), \( c_1 \) has the symbol \( i + j + 1 \) in position \((i-1)s + j + 1\); the position is not distinguished. We illustrate the definition of \((c_1,D_1)\) when \( n = 12 \) and \( s = 4 \). In this case, \((c_1,D_1)\) is: \( 1,3,4,5,2,4,5,6,3,5,6,7,4,6,7,8,5,7,8,9,5,8,9,10,7,9,10,11,8,10,11,12,9,11,12,1,10,12,1,2,3,12,2,3,4,1 \).

The construction of \((c_2,D_2)\) is simple. We let \( c_2 \) be a sequence from \( \{1,2,3,\ldots,n\} \) which begins and ends with 1, determines an Euler circuit of \( T(n,s) \), and satisfies the requirement that every \( s + 1 \) consecutive symbols in \( c_2 \) are distinct. Note that the length of \( c_2 \) is \( n(n-2s) + 1 \). We then let \( D_2 = \{1,2,3,4,\ldots,n(n-2s)+1\} \), i.e., every position in \((c_2,D_2)\) is distinguished.

Now let \((c,D)\) be the splice \((c_1,D) + (c_2,D_2)\). Note that the length of \( c \) is \( ns + n(n-2s) + 1 = n(n-s) + 1 = nt + 1 \). Also note that \( D = \{(i-1)s + 1 \mid 1 \leq i \leq n+1\} \cup \{i \mid n(n-2s) + 1 \leq i \leq nt+1\} \).

The next step in the argument uses the criteria given in (c,D) with the criteria given in \((c,D)\) produces an interval representation of \( (j_1,j_2) \) from \( \{1,2,3,\ldots,n\} \).

To see that this statement holds, \( (j_1,j_2) \) is an edge in \( T(n,s) \). If \( \{j_2 - j \mid 0 \leq j \leq s - 1\} \), then we may set \( n = j_2 \) and observe that \( j_1 \in \{o(k) \mid k \leq j_2 \leq k_{j_2+1}\} \). If \( \{j_2 - j \mid 0 \leq j \leq s - 1\} \), then we may set \( n = j_2 \) and \( \beta = n \) when \( j_1 = 1 \) and \( o(k_{j_2+1}) \) and \( j_1 = o(k_{j_2+1}) \). If \( \{j_2 - j \mid 0 \leq j \leq s - 1\} \), then we may set \( n = j_2 \) and \( \beta = n \) when \( j_1 = 1 \) and \( o(k_{j_2+1}) \) and \( j_1 = o(k_{j_2+1}) \). If \( \{j_2 - j \mid 0 \leq j \leq s - 1\} \), then we may set \( n = j_2 \) and \( \beta = n \) when \( j_1 = 1 \) and \( o(k_{j_2+1}) \) and \( j_1 = o(k_{j_2+1}) \).

This completes the proof of the theorem and shows that \( (c_1,D) \) determines an interval representation of \( (c,D) \) in order to obtain a 

K(m,n,p). For each vertex \( \{j \in (c,D) \cap K(m,n,p) \mid j \}

overlaps intervals for \( s + 1 \) unless these \( s + 1 \) vertices will be on the other \( t - 1 \) intervals when \( I(a) \) will overlap exactly one interval and the vertex to which each \( i \).

For \( j = 1,2,\ldots,m \) with \( c(js+1) \) overlaps the intervals corresponding to each of the levels.
The next step in the argument is to compare the DP-sequence \((\sigma, D)\) with the criteria given in Theorem 8 and observe that \((\sigma, D)\) produces an interval representation of \(K(n, n \cdot (p - 1))\). To see that this statement holds, consider an ordered pair \((j_1, j_2)\) from \((1, 2, 3, \ldots, n)\). If \(j_1 \in \{j + j : 0 \leq j \leq s\}\), then we may set \(\beta = j_2\) and observe that \(j_2 = \sigma(k_\beta)\) and \(j_1 \in \{\sigma(k) : k_\beta \leq k \leq k_{\beta + 1}\}\). Similarly, if \(j_1 \notin \{j_2 - j : 0 \leq j \leq s - 1\}\) then we may set \(\beta = j_1 - 1\) when \(j_1 > 1\) and \(\beta = n\) when \(j_1 = 1\) and observe that \(j_2 \in \{\sigma(k) : k_\beta \leq k \leq k_{\beta + 1}\}\) and \(j_1 = \sigma(k_{\beta + 1})\). If neither of these conditions hold, then \((j_2, j_1)\) is an edge in \(T(n, s)\) and there exists an integer \(j\) with \(1 \leq j \leq n(n - 2s)\) so that \(\sigma_2(j) = j_2\) and \(\sigma_2(j + 1) = j_1\). We may then set \(\beta = n + j\) and observe that \(\sigma(k_{\beta}) = j_2\) and \(\sigma(k_{\beta + 1}) = j_1 \in \{\sigma(k) : k_\beta \leq k \leq k_{\beta + 1}\}\).

This completes the proof of our claim that \((\sigma, D)\) determines an interval representation of \(K(n, n \cdot (p - 1))\).

Furthermore, we observe that each symbol in \([1, 2, 3, \ldots, n]\), except 1, is used exactly \(t\) times in \(\sigma\), and the symbol 1 is used \(t + 1\) times in \(\sigma\). We now show how to add intervals corresponding to vertices in \(A\) to an interval representation of \((\sigma, D)\) in order to obtain an interval representation of \(K(m, n \cdot p)\). For each vertex \(a \in A\), we will assign \(t\) intervals. One of these intervals, which we denote \(I(a)\), will overlap intervals for \(s + 1\) distinct vertices from each \(B_\frac{1}{2}\); these \(s + 1\) vertices will be the same for each \(i\). Each of the other \(t - 1\) intervals which correspond to \(a\) (other than \(I(a)\)) will overlap exactly one interval from each \(B_\frac{1}{2}\); this interval and the vertex to which it corresponds are the same for each \(i\).

For \(j = 1, 2, \ldots, m\) we choose an interval \(I(a_j)\) which overlaps the intervals corresponding to \(\sigma((j - 1)s + 1)\) and \(\sigma(js + 1)\) for each of the levels in the representation. Note
that $I(a_j)$ overlaps a set of $s + 1$ intervals corresponding

to $s + 1$ distinct vertices in $E_i$ for each $i$.

For each $j = 1, 2, \ldots, m$, we then choose $t - 1$
"points" which overlap intervals corresponding to the
$n - (s + 1) = t - 1$ vertices in $E_i$ not already overlapped by
$I(a_j)$. This assignment is easily accomplished since the first $n$
distinguished positions in $(a, D)$ contain \{1, 2, 3, \ldots, n\}.
With this observation, the proof is complete.

We illustrate the preceding theorem for $m = 7$, $n = 5$, and $p = 2$. For clarity, the points corresponding to vertices in $A$ are omitted.

\begin{center}
\begin{tabular}{cccccccc}
  1 & 2 & 3 & 4 & 5 & 6 & 7 \\
  \hline
  1 & 3 & 2 & 4 & 3 & 5 & 4 & 1 & 5 & 2 & 1 & 4 & 2 & 5 & 3 & 1 \\
  \hline
  1 & 3 & 2 & 4 & 3 & 5 & 4 & 1 & 5 & 2 & 1 & 4 & 2 & 5 & 3 & 1
\end{tabular}
\end{center}

Figure 10.

For emphasis, we also state as a formal theorem the
following alternate form of Theorem 12.

\textbf{Theorem 13.} If $p, n_1, n_2, n_3, \ldots, n_p$ are integers with
$p \geq 2$ and $n_1 \geq n_2 \geq \ldots \geq n_p$, then

$$i(n_1, n_2, n_3, \ldots, n_p) \leq 1 + i(n_1, n_2).$$


It should be noted that the inequality in Theorems 12 and 13 is best possible as the following result due to D. West [5]

\textbf{REFERENCES}


4. William T. Trotter, Jr. and D. West, Personal Communication.

University of South Carolina
A Complete Multipartite Graph

Let \( i \) be an interval corresponding to each \( i \).

Then choose \( t - 1 \) intervals corresponding to the edges that are not already overlapped by intervals corresponding to the edges that complete the graph.

For \( m = 7, n = 5 \), the intervals corresponding to vertices are:

\[
\begin{array}{ccccccc}
6 & 7 & 1 & 4 & 2 & 5 & 3 \\
\hline
2 & 1 & 4 & 2 & 5 & 3 & 1
\end{array}
\]

Formal theorem: Let \( m, n \) be integers with \( m \geq n \). Then

\[ i(m, n) = \begin{cases} 
1 + i(m, n) & \text{if } n \geq 3 \text{ and } m = n^2 - n - 1 \\
1 + i(m, n) & \text{if } m = 7 \text{ and } n = 5 \\
i(m, n) & \text{otherwise.}
\end{cases} \]

The construction used in Theorem 12 suggests strongly that the determination of \( i(m, n \cdot \infty) \) for \( m \geq n \) rests on properties of DP-sequences. We announce that the authors and D. West have solved completely the problem of determining \( i(m, n \cdot \infty) \) for all values of \( m \) and \( n \). The proof of the following result will appear elsewhere.

Theorem 15. Let \( m, n \) be integers with \( m \geq n \). Then

\[ i(m, n \cdot \infty) = \begin{cases} 
1 + i(m, n) & \text{if } n \geq 3 \text{ and } m = n^2 - n - 1 \\
1 + i(m, n) & \text{if } m = 7 \text{ and } n = 5 \\
i(m, n) & \text{otherwise.}
\end{cases} \]

References

5. D. West, Personal Communication.

University of South Carolina