which joining sets of
(and diverting) paths,
r 1977), 18-28.

n which players of
gger game, *Scientific*
.

Wesley, Reading, (1969).

of the acquaintance graph,

wright, *Structural Models:*
Directed Graphs, Wiley,

. Wormald, Isomorphic
s, *Trans. Amer. Math. Soc.*,

Solution of Irving's
to appear.

roblem and the Zarankiewicz
78), 13-26.

al logic, *Proc. London*

g strategy for the second
Tenth S.E. Conference on
h Theory, Utilitas,

Recreational Math., 2

A Bound on the Interval Number of a Complete Multipartite Graph

L.B. HOPKINS
W.T. TROTTER*

ABSTRACT

The *interval number* of a graph G denoted $i(G)$, is the least positive integer t for which G is the intersection graph of a family of sets each of which is the union of t pairwise disjoint intervals of the real line. For example, a graph G is an interval graph if and only if $i(G) = 1$, while $i(C_n) = 2$ for all $n \geq 4$. Griggs showed that the maximum value of the interval number of a graph on n vertices is $\lceil (n+1)/4 \rceil$ and Trotter and Harary showed that the interval number of the complete bipartite graph $K(n_1,n_2)$ is given by the formula $i(K(n_1,n_2)) = \lceil (n_1 n_2 + 1)/(n_1 + n_2) \rceil$. Several researchers have been investigating the problem of determining the interval number of complete multipartite graphs, and it was conjectured that the interval number of the complete multipartite graph $K(n_1,n_2,n_3,\ldots,n_p)$, where $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_p$ and $p \geq 3$, equals the interval number of the complete bipartite graph $K(n_1,n_2)$. In support of this conjecture, Matthews proved that for every $p \geq 3$, if $n_1 = n_2 = n_3 = \ldots = n_p$, then $i(K(n_1,n_2,\ldots,n_p)) = i(K(n_1,n_2))$. However, D. West disproved the conjecture by showing that for each $n \geq 3$, there exists a constant c_n so that if $n_1 = n^2 - n - 1$, $n_2 = n_3 = \ldots = n_p = n$, and $p \geq c_n$, then $i(K(n_1,n_2,n_3,\ldots,n_p)) = 1 +$

$i(K(n_1,n_2))$. In view of West's counterexample, it was suggested that the interval number of a complete multipartite graph might exceed the interval number of the bipartite graph, formed by the largest two parts, by an arbitrarily large amount. In this paper, we prove to the contrary that $i(K(n_1,n_2,n_3,\ldots,n_p)) \leq 1 + i(K(n_1,n_2))$ for all $p, n_1, n_2, \ldots, n_p$ with $p \geq 3$ and $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_p$.

1. Introduction.

In recent years, there has been considerable interest in generalizations of interval graphs. Much of the research is motivated by the wide range of interpretations which may be given to optimization and extremal problems involving interval graphs. In this paper, we consider the subject of t-interval graphs. For a positive integer t, we represent a graph as the intersection graph of a family of sets each of which is the union of t pairwise disjoint intervals of the real line. Among the several extremal problems involving t-interval graphs, we will be concerned with minimizing t for a given graph or class of graphs. If we view a t-interval graph as a work schedule permitting cooperation between certain specified components of the work force while safeguarding against interference between other components, then the minimization of t yields a schedule in which each component has relatively few work periods. Consequently, the inherent inefficiency of starting up and closing down unnecessary work periods of short duration and limited productivity is avoided.

Among the classes of graphs for which this extremal problem is quite natural is the class of complete bipartite graphs where the work force is subdivided into two units with no interference permitted between any two components in the same unit, but cooperation required between any two components from different

units. In this paper, we will
to complete multipartite graph
problem in the multipartite ca
from the bipartite graph.

2. Notation and Terminology.
Trotter and Harary [4] de
of a graph G as a function
$x \in G$ a sequence $F(x)(1)$, F
intervals of the real line $\mathbb{R}$
distinct vertices, we have x
if there exists a pair of inte
that $F(x)(i) \cap F(y)(j) \neq \emptyset$.
denoted $i(G)$, is then defin
for which G has a t-interval
$i(G)$ is the least integer t
graph of a family of sets each
t closed intervals of $\mathbb{R}$. I
interval graph if and only if
We will find it convenien
(degenerate) closed interval a
vention in specifying an inter
For example, Figure 1 provid
of the graph G. Note that
$i(G) = 2$.
Furthermore, we will dele
representation, all isolated i
may be simplified as in Figure
Throughout this paper, we
shown in Figures 1 and 2 to il
Intervals will be spread out
reader should bear in mind tha

erexample, it was suggested
e multipartite graph might
artite graph, formed by the
large amount. In this
$i(K(n_1,n_2,n_3,\ldots,n_p)) \leq$
$\ldots, n_p$ with $p \geq 3$ and

considerable interest in
Much of the research is
retations which may be
oblems involving interval
he subject of t-interval
we represent a graph as
sets each of which is the
als of the real line.
nvolving t-interval graphs,
t for a given graph or
rval graph as a work sched-
rtain specified components
against interference
nimization of t yields a
relatively few work periods.
cy of starting up and
of short duration and

which this extremal problem
lete bipartite graphs where
units with no interference
in the same unit, but
omponents from different

units. In this paper, we will discuss the natural generalization to complete multipartite graphs and will show that the extremal problem in the multipartite case does not differ substantially from the bipartite graph.

2. Notation and Terminology.

Trotter and Harary [4] defined a t-*interval representation* of a graph G as a function F which assigns to each vertex $x \in G$ a sequence $F(x)(1), F(x)(2), \ldots, F(x)(t)$ of closed intervals of the real line $\mathbb{R}$ so that for every pair x, y of distinct vertices, we have x adjacent to y in G if and only if there exists a pair of integers i, j with $1 \leq i, j \leq n$ so that $F(x)(i) \cap F(y)(j) \neq \emptyset$. The *interval number* of a graph G, denoted $i(G)$, is then defined as the least positive integer t for which G has a t-interval representation. Alternately, $i(G)$ is the least integer t for which G is the intersection graph of a family of sets each of which is the union of at most t closed intervals of $\mathbb{R}$. In particular, a graph G is an interval graph if and only if $i(G) = 1$.

We will find it convenient to consider a point as a (degenerate) closed interval and will frequently use this convention in specifying an interval representation of a graph. For example, Figure 1 provides a 2-interval representation of the graph G. Note that G is not an interval graph so $i(G) = 2$.

Furthermore, we will delete from Figure 1 for an interval representation, all isolated intervals and points. So Figure 1 may be simplified as in Figure 2.

Throughout this paper, we will use diagrams similar to those shown in Figures 1 and 2 to illustrate interval representations. Intervals will be spread out vertically for clarity but the reader should bear in mind that all intervals are to be projected

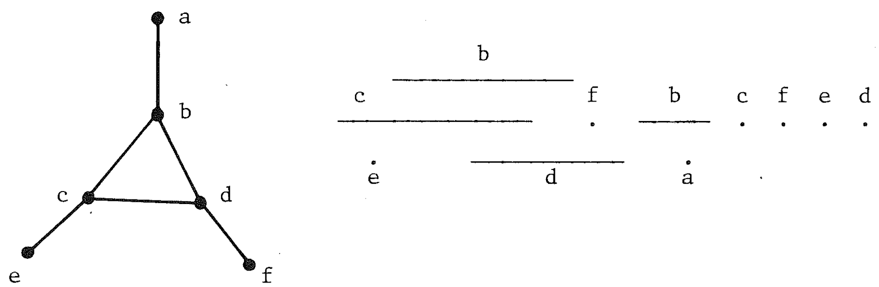

Figure 1.

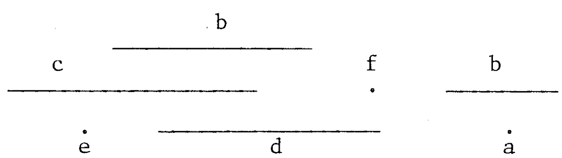

Figure 2.

onto a single horizontal line.

We now present a brief summary of recent research involving interval numbers. We begin with the following elementary result due to Trotter and Harary [4].

Theorem 1 [4]. If T is a tree, then $i(T) \leq 2$.

J. Griggs [2] has established the following upper bound on the interval number as a function of the order of the graph.

Theorem 2 [2]. If G is a graph on n vertices, then $i(G) \leq \lceil (n+1)/4 \rceil$.

J. Griggs and D. West [1] have also established an upper bound on $i(G)$ as a function of the maximum degree of a vertex in G.

Theorem 3 [*1*]. If Δ is the maximum degree of a vertex in G, then $i(G) \leq \lceil (\Delta+1)/2 \rceil$.

Griggs and West [1] showed that if G is regular and triangle-free, then equality holds in Theorem 3. The following result follows as an immediate corollary.

Corollary 4 [*1*]. For each $n \geq 1$, the interval number of the n-cube Q_n is given by $i(Q_n) = \lceil (n+1)/2 \rceil$.

Trotter and Harary [4] developed a formula for the interval number of a complete bipartite graph.

Theorem 4 [*4*]. The interval number of the complete bipartite graph $K(m,n)$ is given by:

$$i(K(m,n)) = \lceil (mn+1)/(m+n) \rceil .$$

3. <u>Interval Numbers of Complete Multipartite Graphs</u>

In the remainder of this paper, we will be concerned with the computation of the interval number of a complete multipartite graph $K(n_1,n_2,\ldots,n_p)$ where $p \geq 2$. By convention, we will require $n_1 \geq n_2 \geq \ldots \geq n_p$. For simplicity, we let $i(n_1,n_2,\ldots,n_p)$ denote the interval number of $K(n_1,n_2,\ldots,n_p)$; note that $i(n_1,n_2) = \lceil (n_1n_2+1)/(n_1+n_2) \rceil$ by Theorem 4. Since $K(n_1,n_2)$ is an induced subgraph of $K(n_1,n_2,\ldots,n_p)$, we always have $i(n_1,n_2) \leq i(n_1,n_2,\ldots,n_p)$. We now proceed to show that $i(n_1,n_2,\ldots,n_p)$ never exceeds n_2. We begin by presenting a construction due to M. Matthews [3]. We let the vertex set of $K(n,n)$ be $A \cup B$ where $A = \{a_1,a_2,\ldots,a_n\}$, $B = \{b_1,b_2,\ldots,b_n\}$ with each a_i adjacent to every b_j.

First let n be even, say $n = 2r$. Then the following diagram provides an $r + 1$ - interval representation of $K(n,n)$.

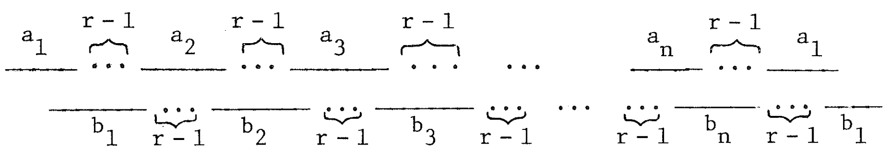

Figure 3.

In the gap between a_i and a_{i+1} (cyclically), occur the $r - 1$ points corresponding to $a_{i+2}, a_{i+3}, \ldots, a_{i+r}$ (cyclically). Similarly, the gap between b_i and b_{i+1} contains points corresponding to $b_{i+2}, b_{i+3}, \ldots, b_{i+r}$. Here is a diagram when $r = 3$. For simplicity, only the subscripts are given.

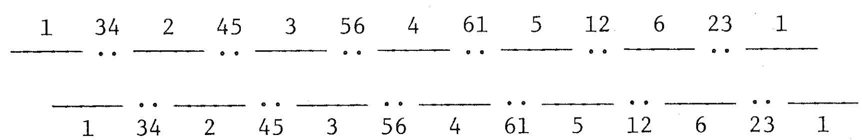

Figure 4 .

We shall continue to use the convention followed in Figures 3 and 4 for bipartite and multipartite graphs i.e., the diagram will be presented in "levels" with all intervals occurring in the same level corresponding to vertices in the same part.

The reader is encouraged to compare this example with the construction given by Trotter and Harary [4] for a 4-representation of $K_{6,6}$. The advantage of Matthews' construction is that it can easily be extended to multipartite graphs. It suffices to add additional "levels" to the diagram following the same intersection pattern as determined by the first two. For example,