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EVERY t-IRREDUCIBLE PARTIAL ORDER IS A SUBORDER OF A $t+1$-IRREDUCIBLE PARTIAL ORDER

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The dimension of a partial order $(X, \leqslant)$ is the least integer t for which there exist linear extensions $X_1, X_2, \ldots, X_t$ of X so that $x_1 \leqslant x_2$ in X if and only if $x_1 \leqslant x_2$ in X_i for each $i = 1, 2, \ldots, t$. For an integer $t \geqslant 2$, a partial order is said to be t-irreducible if it has dimension t and every proper nonempty subpartial order has dimension less than t. We answer a natural question concerning dimension by proving that for each $t \geqslant 2$, every t-irreducible partial order is a subpartial order of a $t+1$-irreducible partial order.

1. Introduction

In this paper, we answer one of the most natural questions that can be asked concerning the dimension of partially ordered sets. Utilizing a construction whose origins lie in chromatic graph theory, we prove that for each $t \geqslant 2$, every t-irreducible partial order can be embedded in a $t+1$-irreducible partial order. The construction also relies on two fundamental concepts in dimension theory: the structure of nonforced pairs and realizers of irreducible partial orders. Nevertheless, for the reader who is familiar with little more than the most basic concepts concerning partial orders, the paper is entirely self contained, and it is only necessary to present a few definitions and preliminary lemmas before proceeding to the principal result. The reader who desires additional background material on the dimensional theory of posets is referred to the survey article [4] which also contains an extensive bibliography of papers on this subject.

A *partially ordered set* (poset) is a set X equipped with a reflexive antisymmetric and transitive binary relation $\leqslant$. If $x_1, x_2 \in X$, $x_1 \not\leqslant x_2$ and $x_2 \not\leqslant x_1$, then x_1 and x_2 are *incomparable* and we write $x_1 \| x_2$. For each point $x_1 \in X$, we let $D_X(x_1) = \{x_2 \in X : x_2 < x_1\}$, $U_X(x_1) = \{x_2 \in X : x_1 < x_2\}$, and $I_X(x_1) = \{x_2 \in X : x_1 \| x_2\}$. We let $I_X = \{(x_1, x_2) : x_1 \| x_2\}$. We say X is a *linear order* if $I_X = \emptyset$.

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If X_1 and X_2 are partial orders on the same set and $x_1 < x_2$ in X_2 whenever $x_1 < x_2$ in X_1, we say X_2 is an extension of X_1; if X_2 is a linear order and an extension of X_1, X_2 is called a *linear extension* of X_1. Dushnik and Miller [1] defined the *dimension* of a poset X, denoted $\dim(X)$, as the least positive integer t for which there exist t linear extensions $X_1, X_2, \ldots, X_t$ of X such that $x_1 \leq x_2$ in X if and only if $x_1 \leq x_2$ in X_i for each $i = 1, 2, \ldots, t$.

If X_1 and X_2 are posets and the point set of X_1 is a subset of the point set of X_2, the poset X_1 is called a *subposet* of X_2 when $x_1 \leq x_2$ in X_1 if and only if $x_1 \leq x_2$ in X_2 for all $x_1, x_2 \in X_1$. For each point $x \in X$, we let $X - \{x\}$ denote the subposet of X whose point set contains all points in X except x. Of course, $\dim(X - \{x\}) \leq \dim X$ for each $x \in X$. For an integer $t \geq 2$, a poset X is *t-irreducible* if $\dim(X) = t$ and $\dim(X - \{x\}) < t$ for each $x \in X$. A poset has dimension one if and only if it is a linear order (a chain) so the only 2-irreducible poset is a two point antichain. There are infinitely many 3-irreducible posets, and a complete listing of these posets has been made by Trotter and Moore [7] and by Kelly [3]. These posets can be conveniently grouped into 9 infinite families with 18 odd examples left over.

An incomparable pair $(x_1, x_2) \in I_X$ is called a *nonforced pair* if $x_3 < x_1$ implies $x_3 < x_2$ and $x_2 < x_4$ implies $x_1 < x_4$ for all $x_3, x_4 \in X$. We let N_X denote the set of all nonforced pairs. For the poset X shown in Fig. 1, $N_X = \{(2,3), (3,2), (6,1), (5,6), (2,4), (3,4)\}$.

It is customary to consider N_X as a directed graph whose vertex set is the point set of X. When $(x_1, x_2) \in N_X$, we draw an edge from x_2 to x_1. For the poset X in Fig. 1, we have the digraph shown in Fig. 2.

The properties of the digraph N_X are central to the theory of rank for partial orders and we refer the reader to [5] and [6] for additional material on this subject. In this paper we will need only a few basic facts concerning N_X. We state these elementary results without proof. The reader may enjoy providing the arguments, although full details are given in [5].

Lemma 1. *As a binary relation $X \cup N_X$ is transitive, that is, if $\{x_i : 1 \leq i \leq m\}$ is a subset of X and for each $i = 1, 2, \ldots, m-1$, either $x_i < x_{i+1}$ in X or $(x_i, x_{i+1}) \in N_X$, then either $x_1 < x_m$ in X or $(x_1, x_m) \in N_X$.*

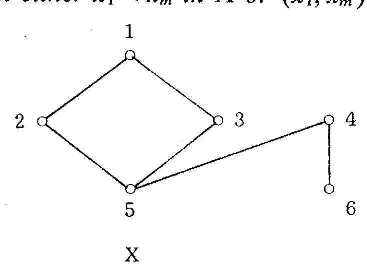


Fig. 1.

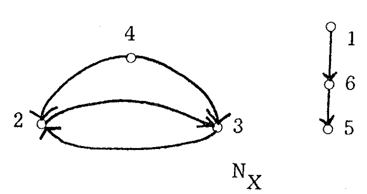


Fig. 2.

Lemma 2. *If $A = \{a_1, a_2, \ldots, a_n\}$ is a sul $\{(a_i, a_{i+1}) : 1 \leq i < n\} \cup \{(a_n, a_1)\}$, then the if $x \in X - A$, then $x > a_i$ if and only ij Dually, $x < a_i$ if and only if $x < a_j$ for*

If $t \geq 3$, a t-irreducible partial order is sum [2] so in particular, it never contains the preceding lemma. A 2-irreducible p an antichain and has a directed cycle of pairs.

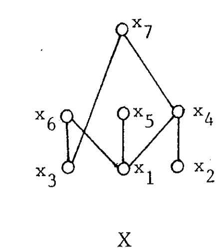
However, when $t \geq 3$ the digraph of contains no directed cycles. In this cas write $X \cup N_X$ to denote the set X equi $x_1 \leq x_2$ in $X \cup N_X$ if and only if $x_1 \leq x$

Lemma 3. *If $t \geq 3$ and X is a t-irreduc partial order.*

We illustrate the preceding lemma f A set $R = \{X_1, X_2, \ldots, X_t\}$ of linear when $x_1 \leq x_2$ in X if and only if $x_1 \leq$

Lemma 4. *A set $R = \{X_1, X_2, \ldots, X_t\}$ realizer of X if and only if for each nonf $i \leq t$ for which $x_2 < x_1$ in X_i.*

Note in the preceding lemma that the $x_2 < x_1$ in X_i, so it is natural to say tha The dimension of a partial order X extensions of X required to reverse t



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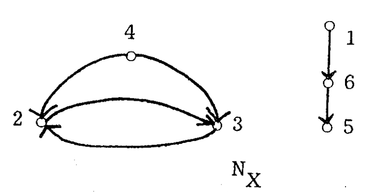

Fig. 2.

Lemma 2. *If $A = \{a_1, a_2, \ldots, a_n\}$ is a subset of X and N_X contains a directed cycle $\{(a_i, a_{i+1}) : 1 \leqslant i < n\} \cup \{(a_n, a_1)\}$, then the set A is an antichain in X. Furthermore, if $x \in X - A$, then $x > a_i$ if and only if $x > a_j$ for each i, j with $1 \leqslant i < j \leqslant n$. Dually, $x < a_i$ if and only if $x < a_j$ for each i, j with $1 \leqslant i < j \leqslant n$.*

If $t \geqslant 3$, a t-irreducible partial order is indecomposable with respect to ordinal sum [2] so in particular, it never contains an antichain satisfying the conclusion of the preceding lemma. A 2-irreducible poset (a two point antichain) is itself such an antichain and has a directed cycle of length two for its digraph of nonforced pairs.

However, when $t \geqslant 3$ the digraph of nonforced pairs of a t-irreducible poset contains no directed cycles. In this case, we abuse terminology somewhat and write $X \cup N_X$ to denote the set X equipped with the binary relation defined by $x_1 \leqslant x_2$ in $X \cup N_X$ if and only if $x_1 \leqslant x_2$ in X or $(x_1, x_2) \in N_X$.

Lemma 3. *If $t \geqslant 3$ and X is a t-irreducible partial order, then $X \cup N_X$ is also a partial order.*

We illustrate the preceding lemma for a 3-irreducible poset (Fig. 3).

A set $R = \{X_1, X_2, \ldots, X_t\}$ of linear extensions of X is called a *realizer* of X when $x_1 \leqslant x_2$ in X if and only if $x_1 \leqslant x_2$ in X_i for $i = 1, 2, \ldots, t$.

Lemma 4. *A set $R = \{X_1, X_2, \ldots, X_t\}$ of linear extensions of a poset X is a realizer of X if and only if for each nonforced pair $(x_1, x_2) \in N_X$, there exists some $i \leqslant t$ for which $x_2 < x_1$ in X_i.*

Note in the preceding lemma that the emphasis is on a linear extension X_i with $x_2 < x_1$ in X_i, so it is natural to say that X_i *reverses* the nonforced pair (x_1, x_2). The dimension of a partial order X is then the minimum number of linear extensions of X required to reverse the nonforced pairs of X. It is therefore

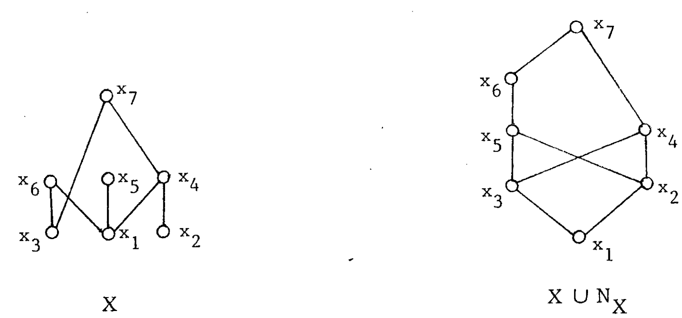

Fig. 3.

natural to associate with a partial order X a hypergraph H_X whose vertices are the nonforced pairs in N_X. A subset $N \subseteq N_X$ is an edge in the hypergraph H_X when there is no linear extension of X which reverses all the nonforced pairs in N, but if N' is a nonempty proper subset of N, then there is a linear extension of X reversing the nonforced pairs in N'. It follows immediately that the dimension of X is the chromatic number of the hypergraph H_X, that is, the least number of colors required to color the vertices of H_X so that no edge of H_X has all of its vertices assigned the same color. For the posets in Figs. 1 and 3, the associated hypergraphs are illustrated in Figs. 4a and 4b, respectively. Note that the graph in Fig. 4a is 2-colorable and that the graph in Fig. 4b is 3-colorable as it contains an odd cycle on seven points.

Example 5. For the poset X shown in Fig. 3, the following three linear extensions realize X:

$$X_1 = \{x_2 < x_1 < x_4 < x_5 < x_3 < x_6 < x_7\},$$

$$X_2 = \{x_3 < x_1 < x_6 < x_5 < x_2 < x_4 < x_7\},$$

$$X_3 = \{x_1 < x_2 < x_3 < x_4 < x_7 < x_5 < x_6\}.$$

Note that X_1 reverses the nonforced pairs in $\{(x_3, x_5), (x_3, x_4), (x_1, x_2)\}$, X_2 reverses $\{(x_5, x_6), (x_1, x_3), (x_2, x_6), (x_2, x_5)\}$, and X_3 reverses $\{(x_6, x_7), (x_5, x_7)\}$. Also note that deleting x_7 from X_1 and X_2 leaves two linear extensions which realize $X - \{x_7\}$.

Hiraguchi [2] proved that removing a point from a poset decreases the dimension by at most one. Here we will require a specialized version of this result.

Lemma 6. *Let X be a t-irreducible poset where $t \geq 3$ and let x be a maximal element of $X \cup N_X$. Then there exists a linear extension X_0 of $X \cup N_X$ in which x is the largest element and $x_1 < x_2$ in X_0 for every $x_1 \in D_X(x)$ and $x_2 \in I_X(x)$.*

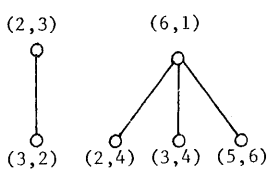

Fig. 4a.

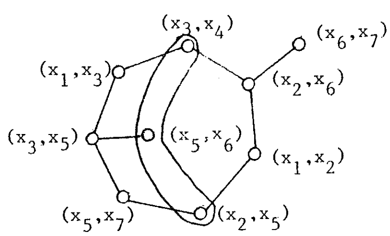

Fig. 4b.

Proof. It suffices to observe that if $x_1 \in$
$X \cup N_X$. □

Let X be an irreducible poset of dim
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extension of X (with respect to the ma
$x_3 < \cdots < x_n\}$ is a consistent linear
$\{x_1, x_2, \ldots, x_s\}$ and $I_X(x_n) = \{x_{s+1}, x_{s+2}, .$
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consistent linear extension X_0. Note tha
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Lemma 7. *Let X be a t-irreducible po*
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For the 3-irreducible poset X show
$\{x_1 < x_2 < \cdots < x_7\}$ is consistent with re
The linear extensions $\{X_1, X_2, X_3\}$ def
Note that X_3 is the reverse of X_0.

2. The embedding theorem

In this section, we use the concep
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Theorem. *If $t \geq 2$ and X is a t-irre*
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Proof. The result is trivial when $t = 2$ s
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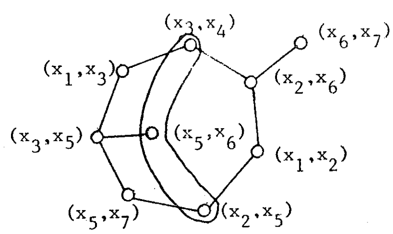

Fig. 4b.

Proof. It suffices to observe that if $x_1 \in D_X(x)$ and $x_2 \in I_X(x)$, then $x_2 \not< x_1$ in $X \cup N_X$. □

Let X be an irreducible poset of dimension at least 3. A maximal element of $X \cup N_X$ is called a *strongly maximal* element of X, and a linear extension X_0 of $X \cup N_X$ satisfying the conclusion of Lemma 6 is called a *consistent* linear extension of X (with respect to the maximal element of X_0). If $X_0 = \{x_1 < x_2 < x_3 < \cdots < x_n\}$ is a consistent linear extension of X, so that $D_X(x_n) = \{x_1, x_2, \ldots, x_s\}$ and $I_X(x_n) = \{x_{s+1}, x_{s+2}, \ldots, x_{n-1}\}$, then the linear order $X_0^* = \{x_1 < x_2 < x_3 < \cdots < x_s < x_n < x_{s+1} < x_{s+2} < \cdots < x_{n-1}\}$ is called the *reverse* of the consistent linear extension X_0. Note that X_0^* is a linear extension of X but not of $X \cup N_X$. The linear order X_0^* will play an important role in the proof of our principal theorem. At this point, we note that X_0^* can be used to form a realizer of X.

Lemma 7. *Let X be a t-irreducible poset, where $t \geq 3$, and let x be a strongly maximal element of X. Also let X_0 be a consistent linear extension with respect to x. Furthermore, let $\{X_1', X_2', \ldots, X_{t-1}'\}$ be a realizer of $X - \{x\}$, and for each $i = 1, 2, \ldots, t-1$, let X_i be the linear order formed by adding x to X_i' as the largest element. Then $\{X_0^*, X_1, X_2, \ldots, X_{t-1}\}$ is a realizer of X.*

For the 3-irreducible poset X shown in Fig. 3, the linear extension $X_0 = \{x_1 < x_2 < \cdots < x_7\}$ is consistent with respect to the strongly maximal element x_7. The linear extensions $\{X_1, X_2, X_3\}$ defined in Example 5 illustrate Lemma 7. Note that X_3 is the reverse of X_0.

2. The embedding theorem

In this section, we use the concept of a consistent linear extension of a t-irreducible partial order X to construct a $t+1$-irreducible partial order containing X as a subposet. The reader who is familiar with chromatic graph theory will recognize the flavor of the construction, since its roots lie in that subject.

Theorem. *If $t \geq 2$ and X is a t-irreducible poset, then there exists a $t+1$-irreducible poset containing X as a subposet.*

Proof. The result is trivial when $t = 2$ so we assume that $t \geq 3$. We then let X be an arbitrary t-irreducible poset and choose a consistent linear extension $X_0 = \{x_1 < x_2 < x_3 < \cdots < x_n\}$. As in Section 1, we let $D_X(x_n) = \{x_1, x_2, \ldots, x_s\}$