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The Ramsey Number of a Graph with Bounded Maximum Degree

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The Ramsey number of a graph G is the least number t for which it is true that whenever the edges of the complete graph on t vertices are colored in an arbitrary fashion using two colors, say red and blue, then it is always the case that either the red subgraph contains G or the blue subgraph contains G. A conjecture of P. Erdös and S. Burr is settled in the affirmative by proving that for each $d \geqslant 1$, there exists a constant c so that if G is any graph on n vertices with maximum degree d, then the Ramsey number of G is at most cn.

1. INTRODUCTION

If F, G, and H are graphs, we write $F \to (G, H)$ when the following condition is satisfied: If the edges of F are colored in any fashion with two

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colors, say red and blue, then either the red subgraph contains a copy of G or the blue subgraph contains a copy of H. Now let K_m denote the complete graph on m vertices. Then it follows easily from Ramsey's theorem that for every pair (G, H) there is a least positive integer m for which $K_m \to (G, H)$. This integer m is called the Ramsey number $r(G, H)$. When $G = H$, we write only $r(G)$. An excellent survey of results concerning Ramsey numbers can be found in the book [3]. Here, we will be concerned with the following conjecture of Burr and Erdös [2]:

Conjecture. For each $d \geqslant 1$, there exists a constant c, depending only on d, so that if G is a graph on n vertices in which each vertex has at most d neighbors, then $r(G) \leqslant cn$.

Recently, Beck [1] has made some progress on this conjecture by showing that $r(G) < (2n)^c$, where $c = (2d)^{2d-1}$. In this paper, we settle the above conjecture in the affirmative. Our proof will depend heavily on the "regularity" lemma of Szemerédi [4]. The presentation of this lemma requires some preliminary definitions.

Let H be a graph and let A and B be disjoint subsets of the vertex set of H. Then the *density* of (A, B), denoted $\delta(A, B)$, is the ratio n_1/n_2, where $n_1 = |\{(a, b)\colon a \in A,\ b \in B, \text{ and } a \text{ is adjacent to } b \text{ in } H\}|$ and $n_2 = |A \times B|$. The density of (A, B) measures the probability that a pair (a, b) selected at random from $A \times B$ determines an edge in H. Of course, we always have $0 \leqslant \delta(A, B) \leqslant 1$.

Now let ε be a positive number. Then the pair (A, B) is said to be ε-*regular* if whenever we have two subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geqslant \varepsilon\,|A|$ and $|B'| \geqslant \varepsilon\,|B|$, then the following inequalities hold:

$$\delta(A, B) - \varepsilon \leqslant \delta(A', B') \leqslant \delta(A, B) + \varepsilon.$$

Next, let $V(H) = A_1 \cup A_2 \cup \cdots \cup A_k$ be a partition of the vertex set of H into disjoint subsets. The partition is said to be *equipartite* if $\big|\,|A_i| - |A_j|\,\big| \leqslant 1$ for all $i, j = 1, 2, \ldots, k$. With these definitions, we can now state the following lemma whose proof is given in [4]:

LEMMA. *For every $\varepsilon > 0$ and every integer $m \geqslant 0$, there exist integers N_1 and N_2 (depending on ε and m) so that if H is a graph having at least N_2 vertices, then there exists an equipartite partition $V(H) = A_1 \cup A_2 \cup \cdots \cup A_k$, where*

(i) *$m \leqslant k \leqslant N_1$, and*

(ii) *all but at most $\varepsilon\binom{k}{2}$ of the pairs (A_i, A_j) are ε-regular.*

2. The Principal Result

Our goal in this section is to prove the following result:

THEOREM. *For each positive integer d, there exists a constant c, depending only on d, so that if G is a graph on n vertices with maximum degree at most d, then $r(G) \leqslant cn$.*

Proof. Let d be any positive integer. Choose the least positive integer t so that if we define $\varepsilon = 1/t$, then $\frac{1}{2}\log(1/3\varepsilon) \geqslant d+1$. Observe that with this choice, we also know that $1/3^d > 2d^2\varepsilon$. Next, set $m = 1/\varepsilon$. Then let N_1 and N_2 be the values determined by these values of ε and m in the regularity lemma. Then set $c = \max\{N_2, N_1/d^2\varepsilon\}$. Note that c is a constant depending only on d.

Next, let G be a graph having n vertices $x_1, x_2, \ldots, x_n$ and maximum degree at most d. We show that $r(G) \leqslant cn$. Consider an arbitrary coloring of the edges of the complete graph K_{cn} using two colors, say red and blue. Then let H denote the graph on cn vertices determined by the red edges. The complement of H, denoted by $\bar{H}$, is the graph determined by the blue edges. Note that if A and B are disjoint sets of vertices, then $\delta_H(A, B) = 1 - \delta_{\bar{H}}(A, B)$. Furthermore, (A, B) is ε-regular in H if and only if it is ε-regular in $\bar{H}$.

Since H has cn vertices and $cn \geqslant N_2$, we know that there exists an equipartite partition, $V(H) = A_1 \cup A_2 \cup \cdots \cup A_k$ as guaranteed by the regularity lemma. Then let H^* denote the graph whose vertex set is $\{1, 2, \ldots, k\}$ with edges (i, j), where (A_i, A_j) is ε-regular in H for $1 \leqslant i \leqslant j \leqslant k$. The graph H^* has at least $(1-\varepsilon)\binom{k}{2}$ edges and thus by Turán's theorem has a complete subgraph H^{**} of size (being generous) at least $1/2\varepsilon$. Without loss of generality, we may assume that the subsets in the partition have been labelled so that (A_i, A_j) is ε-regular whenever $1 \leqslant i \leqslant j \leqslant 1/2\varepsilon$. Now we two color the edges of H^{**} using the colors green and white. We color (i, j) green if $\delta_H(A_i, A_j) \geqslant \frac{1}{2}$ and color (i, j) white if $\delta_H(A_i, A_j) < \frac{1}{2}$. We pause to recall that $\frac{1}{2}\log(1/2\varepsilon) \geqslant d+1$. Then it follows from Ramsey's theorem that we have (again being generous) a monochromatic complete subgraph H^{***} having $d+1$ vertices.

Assume first that H^{***} has all of its edges colored green. Then we may relabel the sets in the partition so that

(i) (A_i, A_j) is ε-regular, and

(ii) $\delta_H(A_i, A_j) \geqslant \frac{1}{2}$

for all i, j with $1 \leqslant i < j \leqslant d+1$. We now proceed to show that the red subgraph H contains a copy of G. (If the edges of H^{***} are white, then we replace H by $\bar{H}$ in the second condition and proceed to show that the blue subgraph $\bar{H}$ contains a copy of G.)

To construct a copy of G in H, we will proceed inductively to choose vertices $y_1, y_2, \ldots, y_n$ from H so that the map $x_i \to y_i$ is an isomorphism. Furthermore, we will choose these points so that for each $i = 1, 2, \ldots, n$, the following conditions are satisfied:

(a) If $1 \leqslant \alpha \leqslant i$, then $y_\alpha \in A_\beta$ for some β with $1 \leqslant \beta \leqslant d+1$.

(b) If $1 \leqslant \alpha_1 < \alpha_2 \leqslant i$ and x_{α_1} is adjacent to x_{α_2} in G, then y_{α_1} and y_{α_2} come from distinct sets in the partition and y_{α_1} is adjacent to y_{α_2} in H.

(c) If $i < \alpha' \leqslant n$, $V(\alpha', i) = \{y_\alpha : 1 \leqslant \alpha \leqslant i,\ x_\alpha \text{ adjacent to } x_{\alpha'}\}$, and $v = |V(\alpha', i)|$, then for each β with $1 \leqslant \beta \leqslant d+1$ so that A_β contains no y_α in $V(\alpha', i)$, A_β contains a subset A'_β having at least $|A_\beta|/3^v$ points so that every point in A'_β is adjacent to every y_α in $V(\alpha', i)$.

At first, condition (c) may seem hopelessly complicated to the reader. Upon reflection, however, it will be clear that this condition is precisely what is needed to ensure that the selection of the vertices $y_1, y_2, \ldots, y_n$ can proceed indutively as claimed. Here are the details.

Suppose that for some nonnegative integer i with $i < n$, the points y_α for $1 \leqslant \alpha \leqslant i$ have been chosen so that conditions (a)–(c) are satisfied. We show how to make a suitable choice for y_{i+1}. (Note that this definition allows $i = 0$ because the rule for choosing y_1 is the same as for all other values of i.)

First choose some β_0 with $1 \leqslant \beta_0 \leqslant d+1$ so that A_{β_0} does not contain a point from $V(i+1, i)$, i.e., we choose a set among the first $d+1$ in the partition which does contain y_α with $1 \leqslant \alpha \leqslant i$ for which x_α is adjacent to x_{i+1}. This is possible because x_{i+1} has at most d neighbors. Then let A'_{β_0} be the subset of A_{β_0} consisting of those points adjacent to every y_α in $V(i+1, i)$. By condition (c), we know that $|A'_{\beta_0}| \geqslant |A_{\beta_0}|/3^v$, where $v = |V(i+1, i)|$. Also note that $1/3^v \geqslant 1/3^d \geqslant \varepsilon$.

With the choice of any points from A'_{β_0} as y_{i+1}, we would satisfy conditions (a) and (b). However, some care must be taken to insure that condition (c) is satisfied. It is clear that we need only be concerned with those values $\alpha' > i+1$ in which x_{i+1} is adjacent to $x_{\alpha'}$. There are at most d such values. Choose one, say α', arbitrarily. Then choose a β with $\beta \neq \beta_0$ so that A_β does not contain any y_α from $V(\alpha', i)$ and let $v' = |V(\alpha', i+1)| = 1 + V(\alpha', i)$. We already know that A_β contains a subset A'_β containing at least $|A_\beta|/3^{v'-1}$ points so that every point in A'_β is adjacent to every point in $V(\alpha', i)$. Note that $|A'_\beta| \geqslant \varepsilon\,|A_\beta|$. Furthermore, it is clear that at most $\varepsilon\,|A_{\beta_0}|$ of the points in A'_{β_0} are adjacent to less than one-third of the points in A'_β. Fixing α' and proceeding through all values of β, we would then eliminate at most $d\varepsilon\,|A_{\beta_0}|$ of the points in A'_{β_0} as candidates for y_{i+1}. If we then range over all possible values for α', we would eliminate at most $d^2\varepsilon\,|A_{\beta_0}|$ of the points in A'_{β_0}. In addition, we cannot select any of the points in A'_{β_0} which have been selected previously. This eliminates at most n additional points. Since the number k of sets in the partition satisfies $k \leqslant N_1$ and $c \geqslant N_1/d^2\varepsilon$, we know that $|A_{\beta_0}| \geqslant cn/N_1$ and thus $n \leqslant d^2\varepsilon\,|A_{\beta_0}|$.

In order to ensure that the point y_{i+1} can successfully be chosen from A'_{β_0}, we require only that $2d^2\varepsilon\,|A_{\beta_0}| < |A'_{\beta_0}|$. However, this inequality is satisfied since $|A'_{\beta_0}|/|A_{\beta_0}| > 1/3^d > 2d^2\varepsilon$. With this observation, the proof of our theorem is complete.

3. Concluding Remarks

Although we do not include the details here, the theorem in Section 2 can be modified to allow for more than two colors. Specifically, for each pair (d, t), there exists a constant c depending only on d and t so that if G is a graph with n vertices and maximum degree at most d, then any coloring of the edges of the complete graph on cn vertices using t colors has a monochromatic copy of G. A complication arises from the fact that we no longer have the complementary relationship between the red and blue graph which preserves regularity, which in turn requires a generalization of the regularity lemma.

However, our methods are not sufficient to settle the following strong form of the conjecture of Burr and Erdös:

Conjecture. For each d, there exists a constant c depending only on d so that if G is a graph on n vertices for which for every subgraph G' of G, the *average* degree of a vertex in G' is at most d, then the Ramsey number $r(G)$ is at most cn.

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