Call #: QA166 .C95 1982
Location: 4

ILL Number: 94223202

Patron: Trotter, William

Journal Title: Graphs and other combinatorial topics; proceedings of the third Czechoslovak Symposium on Graph Theory, held in Prague, August 24th to 27th, 1982 /

Volume: 59 Issue: 
Month/Year: 1983
Pages: 316 - 319

Article Info: Recent progress in problems in discrete geometry

This material may be protected by US copyright law (Title 17 U.S.C.).
Recent Progress in Extremal Problems in Discrete Geometry

Endre Szemerédi
and
William T. Trotter, Jr.*
Department of Mathematics and Statistics
University of South Carolina
Columbia, SC 29208, U.S.A.

1. Introduction.

Extremal problems in discrete geometry constitute a fascinating and surprisingly difficult area of research in combinatorial mathematics. The reader is encouraged to consult P. Erdős' survey papers [5], [6], [7], [8] and W. Moser's summary [11] of recent progress on a wide range of problems in this area. In this paper, we concentrate on extremal problems in the Euclidean plane and announce several new results which have been obtained by the authors and other researchers within the last year. Proofs will appear elsewhere.

The starting point is the authors' covering lemma [12] which, despite its somewhat technical flavor, may be of independent interest. Suppose that a pair of coordinate axes has been chosen. When we use the term square, it will also be assumed that the sides are parallel to the coordinate axes. The following lemma asserts the existence of a family of squares covering a positive fraction of a set of \( n \) points with an additional restriction on the number of points contained in each square.

**Lemma (Szemerédi and Trotter [12]):** Let \( r_1, r_2 \) be integers with \( r_2 \geq 256 r_1 \) and let \( \mathcal{P} \) be any set of \( n \) points in the plane. Then there exists a family \( \mathcal{Q} \) of squares so that:
1. No point in the plane is in the interior of more than one square.
2. Each square contains at least \( r_1 \) but no more than \( r_2 \) points of \( \mathcal{P} \).
3. At least \( n/16 \) of the points in \( \mathcal{P} \) are covered by the squares in \( \mathcal{Q} \).

A second corollary is the following result which is a partial solution to Dirac's conjecture [8].

**Theorem 1 (Szemerédi and Trotter [13]):** There exists a constant \( c_1 \) so that if \( \mathcal{P} \) is a set of \( n \) points and \( \mathcal{L} \) is a family of \( t \) lines in the Euclidean plane, then the number of incidences between points in \( \mathcal{P} \) and lines in \( \mathcal{L} \) is at most \( c_1 n^{2/3} t^{2/3} \) whenever \( \sqrt{n} \leq t \leq \frac{n}{2} \).

P. Erdős [5] had conjectured the conclusion of Theorem 1 in the special case \( t = n \). From Theorem 1, the following theorem, settling in the affirmative a conjecture of Erdős and Purdy [4], follows as an immediate corollary.

**Theorem 2 (Szemerédi and Trotter [13]):** There exists a constant \( c_2 \) so that if \( \mathcal{P} \) is a set of \( n \) points and \( \mathcal{L} \) is a family of \( t \) lines each containing at least \( k \) points from \( \mathcal{P} \) where \( k \leq \sqrt{n} \), then \( t < c_2 n^{2/k} \).

Let \( \mathcal{P} \) be a set of \( n \) points and let \( \mathcal{L} = \{l_1, l_2, \ldots, l_t\} \) be a family of \( t \) lines each containing at least two points from \( \mathcal{P} \). For each \( j = 1, 2, \ldots, t \), let \( y_j \) count the number of points from \( \mathcal{P} \) which are on line \( l_j \). Assume that the lines have been labelled so that \( y_1 \leq y_2 \leq \ldots \leq y_t \). Then \( \sum_{j=1}^{t} (y_j) \leq \binom{n}{2} \). However, this condition is not sufficient to insure that an arbitrary nondecreasing sequence arises in this fashion. Let \( \xi(n) \) count the number of distinct nondecreasing sequences \( y_1 \leq y_2 \leq \ldots \leq y_t \) determined by all possible configurations of \( n \) points in the plane. Theorem 1 can be used to prove the following result which also settles in the affirmative a conjecture of P. Erdős:

*Research supported in part by NSF Grants ISM-8011451 and MCS-8202172.
Theorem 4 (Szemerédi and Trotter [13]): There exists a constant $c_4$ so that $e(n) < 2^{4/3}$ for all $n \geq 1$.

3. Inequalities involving distance.

Two of the oldest problems in discrete geometry are:

a. Find the maximum number $f(n)$ of pairs of points in the Euclidean plane which can be at unit distance apart.

b. Find the minimum number $g(n)$ of distinct distances determined by $n$ points in the Euclidean plane.

P. Erdős conjectures that $f(n) < n^{1/2} \log \log n$ and $g(n) > \frac{cn}{\log n}$ and has offered substantial sums for the solutions to these problems. Until recently progress has been slow. Szemerédi [8] proved $f(n) = o(n^{3/2})$. Beck and Spencer [2] proved $f(n) < n^{1.499}$. Recently, Spencer, Szemerédi and Trotter used the technique employed in proving the theorems in Section 1 to improve this to $f(n) < (n)$.

For the other problem Erdős [4] proved $g(n) > n^{1/2}$ and Moser [10] improved this to $g(n) > n^{2/3}$. Recently, Fan Chung [3] proved that $g(n) > n^{5/7}$ and it appears that further improvements can be obtained by utilizing the incidence inequalities. In any case, it is clear that these problems and similar problems in discrete geometry will continue to attract the attention of researchers in future years.

References


