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TOLERANCE GRAPHS

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Tolerance graphs arise from the intersection of intervals with varying tolerances in a way that generalizes both interval graphs and permutation graphs. In this paper we prove that every tolerance graph is perfect by demonstrating that its complement is perfectly orderable. We show that a tolerance graph cannot contain a chordless cycle of length greater than or equal to 5 nor the complement of one. We also discuss the subclasses of bounded tolerance graphs, proper tolerance graphs, and unit tolerance graphs and present several possible applications and open questions.

1. Introduction

An undirected graph $G=(V,E)$ is called a *tolerance graph* if there exists a collection $\mathcal{I}=\{I_x|x\in V\}$ of closed intervals on a line and a set $t=\{t_x|x\in V\}$ of positive numbers satisfying

$$xy\in E \Leftrightarrow |I_x\cap I_y|\geq \min\ \{t_x,t_y\},$$

where $|I|$ denotes the length of interval I. The pair $\langle\mathcal{I},t\rangle$ is called a *tolerance representation* of G. A tolerance representation $\langle\mathcal{I},t\rangle$is called *bounded* if $t_x\leq|I_x|$ for all $x\in V$. A tolerance graph is a *bounded tolerance* graph if it admits a bounded tolerance representation. See Fig. 1.

Tolerance graphs were introduced in [4] where the following results were shown. If we restrict all the tolerances t_x to be equal to any fixed positive constant c, then we obtain exactly the class of interval graphs. If we restrict the tolerances such that $t_x=|I_x|$ for all vertices x, then we obtain exactly the class of permutation graphs (or, equivalently, the interval containment graphs). Thus, interval graphs and permutation graphs are all bounded tolerance graphs. Furthermore, the following theorem was proved.

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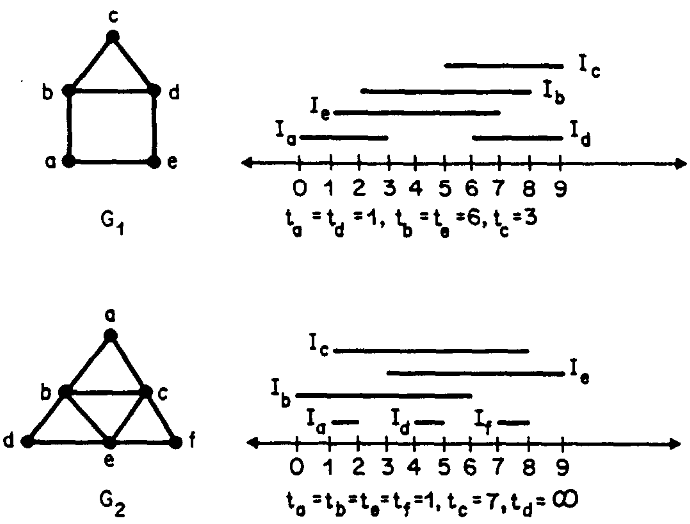


Fig. 1. A bounded tolerance (G_1) and unbounded tolerance (G_2) graph with their representations.

Theorem 1. *Every bounded tolerance graph is the complement of a comparability graph (called a cocomparability graph).*

Tolerance graphs clearly satisfy the hereditary property, that is, every induced subgraph is also a tolerance graph. (All definitions not explicitly given here can be found in [3].)

In Section 2 we investigate the differences between bounded tolerance, tolerance and nontolerance graphs. We obtain necessary conditions for a graph to be a tolerance graph. In particular, we prove that tolerance graphs may contain neither a chordless cycle (C_n) of length greater than or equal to 5 nor the complement ($\bar{C}_n$) of one; these are commonly called odd holes and odd antiholes, respectively. A general technique is presented for constructing nontolerance graphs. We also characterize those trees which are bounded tolerance graphs and those trees which are tolerance graphs.

A graph G is called *perfect* [1, 3] if for every induced subgraph H of G, the chromatic number of H equals the size of the largest clique in H. The perfect graph theorem [8, 9] states that G is perfect if and only if its complement $\bar{G}$ is perfect. Chvátal [2] has shown that perfectly orderable graphs are perfect; perfectly orderable graphs are defined in Section 3 where we prove that tolerance graphs are perfect by showing that their complements are perfectly orderable.

In Section 4 we introduce proper tolerance graphs. We prove that the complete bipartite graph $K_{3,3}$ is not a proper tolerance graph. Finally, we suggest some possible applications and open questions in Section 5.

2. Some necessary conditions for tolerance graphs

In this section we provide some forbidden subgraph configurations for tolerance graphs, and we investigate the differences between tolerance graphs and bounded tolerance graphs. We note that if any tolerance representation exists, one exists satisfying any or all of the following five properties: (a) the tolerances are all strictly positive; (b) the tolerances are all distinct (except those set to infinity); (c) the end points of the intervals are distinct; and (d) the intersection of all of the intervals is a nonempty interval (To see this, note that for any positive number M, we may increase each interval in length by M symmetrically about its center and add M to its tolerance.); and (e) any tolerance which is larger than the length of its corresponding interval is set to be infinity; such a tolerance is called unbounded. A tolerance representation satisfying these five properties is called a *regular representation*. For an interval I_x, we denote the right and left end points by $R(x)$ and $L(x)$.

A vertex z of G is called *assertive* if for every tolerance representation $\langle \mathcal{I}, t \rangle$ of G replacing t_z by $\min\{t_z, |I_z|\}$ leaves the tolerance graph unchanged. An assertive vertex is one which *never* requires unbounded tolerance. The following observation is immediate.

Remark 1. If every vertex of a tolerance graph G is assertive, then G is a bounded tolerance graph.

It follows from the definition that a vertex x is *nonassertive* if there exists a tolerance representation $\langle \mathcal{I}, t \rangle$ for G with $t_x = \infty$ such that reducing t_x to $|I_x|$ would create a new edge xy in G. In such a case $xy \notin E$, and $I_x \subseteq I_y$. We say that *x is dominated by y* in $\langle \mathcal{I}, t \rangle$. Thus, *assertive vertices* are never *dominated* whereas *nonassertive vertices are sometimes dominated.*

In the representation of graph G_2 of Fig. 1, vertex d is dominated by vertex c. By symmetry, vertices a, d and f are nonassertive. By Lemma 2, vertices b, c and e are assertive.

Let Adj(x) denote the set of vertices adjacent to x by an edge of G.

Lemma 1. *Let $\langle \mathcal{I}, t \rangle$ be a tolerance representation of $G = (V, E)$. If x is dominated by y in $\langle \mathcal{I}, t \rangle$, then* $\mathrm{Adj}(x) \subseteq \mathrm{Adj}(y)$.

Proof. If x is dominated by y, then $xy \notin E$ and $I_x \subseteq I_y$, and we may assume that $t_x = \infty$. Suppose that there exists a vertex $z \in \mathrm{Adj}(x) - \mathrm{Adj}(y)$. Then we obtain the inequalities

$$t_z = \min\{t_x, t_z\} \le |I_x \cap I_z| \le |I_y \cap I_z| < t_z,$$

a contradiction. Therefore, $\mathrm{Adj}(x) \subseteq \mathrm{Adj}(y)$. □

The following restatement of Lemma 1 is immediate and will be useful.

Lemma 2. *Let x by a vertex of a tolerance graph. If* $\mathrm{Adj}(x)-\mathrm{Adj}(y)\neq\emptyset$ *for all* $y\neq x$, *then x is assertive.*

We now obtain a sufficient condition for tolerance graphs to be bounded.

Lemma 3. *Let* $G=(V,E)$ *be an undirected graph satisfying*

$$\mathrm{Adj}(x)-\mathrm{Adj}(y)\neq\emptyset \quad \textit{for all } x,y\in V (x\neq y). \tag{1}$$

Then G is tolerant if and only if G is bounded tolerant.

Proof. Suppose G is tolerant. By Lemma 2, every vertex is assertive. Hence, G is bounded tolerant. The converse is trivial. □

As we mentioned earlier, tolerance graphs may be regarded as a generalization of interval graphs. An interval graph may not contain any chordless cycle of length 4 or more. The analogous result for tolerance graphs is the following.

Theorem 2. *A tolerance graph may not contain a chordless cycle of length greater than or equal to* 5.

Proof. By the hereditary property of tolerance graphs, it is sufficient to show that C_n is not tolerant, for any $n\geq 5$, including n even. It is well known that its complement $\bar{C}_n$ is not a comparability graph for any $n\geq 5$, so, by Theorem 1, C_n is not bounded tolerant. Therefore, since C_n satisfies (1), Lemma 3 implies that C_n is not tolerant.

The same proof would show that the complements of odd length chordless cycles are not tolerant. In fact a stronger result holds; namely, a tolerance graph may not contain $\bar{C}_n$ for $n\geq 5$, including n even. To show this it is necessary to introduce a few more concepts.

We define F to be the *tolerance orientation* associated with a regular representation $\langle \mathcal{I}, t\rangle$ for a tolerance graph $G=(V,E)$, by

$$xy\in F \quad \text{iff} \quad xy\in E \text{ and } t_x<t_y.$$

Clearly, a tolerance orientation is acyclic. In general, a tolerance orientation need not be transitive, e.g., for a path on four vertices. However, it is transitive for C_4, as we show next.

Lemma 4. *The tolerance orientation F of* C_4 *is acyclic and transitive. Therefore, any tolerance orientation is unique, i.e., it is isomorphic to the directed graph in Fig.* 2.

Proof. Consider a regular tolerance representation $\langle \mathcal{I}, t\rangle$ of C_4. By the symmetry of C_4, we may assume that $t_a<t_b<t_c$ and $t_a<t_d$ in Fig. 2. To prove the lemma, it suffices to show that $t_d<t_b$. In order to obtain a contradiction, we assume that $t_d>t_b$. We note that $I_a \not\subseteq I_d$, since otherwise

$$|I_a \cap I_d| = |I_a| \leq t_a,$$

a contradiction. Also, $I_d \not\subseteq I_a$, since otherwise

$$|I_c \cap I_d| \leq |I_d| = |I_d \cap I_a| < t_a < \min(t_c, t_d),$$

a contradiction. By the symmetry of $\mathcal{I}$, we may assume that $L(d)<R(a)<R(d)$ and $L(a)<L(d)<R(a)$.

It must be the case that $I_c \cap (I_a - I_d) \neq \emptyset$, since otherwise $(I_c \cap I_a) \subseteq (I_a \cap I_d)$ would imply

$$t_a \leq |I_c \cap I_a| \leq |I_a \cap I_d| < t_a,$$

a contradiction. Similarly, $I_b \cap (I_a - I_d) \neq \emptyset$. Also, $I_c \cap (I_d - I_a) \neq \emptyset$, since otherwise $(I_c \cap I_d) \subseteq (I_a \cap I_d)$ would imply

$$t_a < \min(t_c, t_d) \leq |I_c \cap I_d| \leq |I_a \cap I_d| < t_a,$$

a contradiction. Similarly, $I_b \cap (I_d - I_a)$. Therefore, $L(c)<L(d)$, $L(b)<L(d)$, $R(c)>R(a)$ and $R(b)>R(a)$.

Now, if $R(b)>R(c)$ then $(I_c \cap I_d) \subseteq (I_b \cap I_c)$ which leads to

$$\min(t_b, t_c) \leq \min(t_d, t_c) \leq |I_c \cap I_d| \leq |I_b \cap I_c|,$$

a contradiction. On the other hand, if $R(b)<R(c)$ then $(I_b \cap I_d) \subseteq (I_b \cap I_c)$ which leads to

$$t_b = \min(t_b, t_d) \leq |I_b \cap I_d| \leq |I_b \cap I_c| < t_b,$$

a contradiction. This completes the proof of the lemma. □

We are now ready to show that tolerance graphs may not contain $\bar{C}_n$, $n \geq 5$.

Theorem 3. *A tolerance graph may not contain the complement of a chordless cycle of length greater than or equal to 5.*

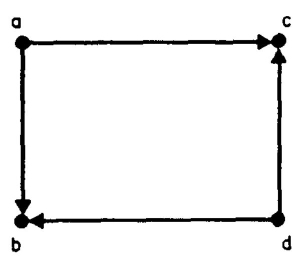


Fig. 2. Unique tolerance orientation of C_4.

Proof. Again by the hereditary property it is sufficient to show that $G=\bar{C}_n$ $(n\geq 5)$ is not tolerant. Since $\bar{C}_5=C_5$ we may assume by Theorem 2 that $n\geq 6$. Furthermore, since $\bar{C}_n$ satisfies (1), we need only show that it is not bounded tolerant.

Suppose that $\bar{C}_n=(V,E)$ is bounded tolerant for some $n\geq 6$ with regular representation $\langle \mathcal{I}, t\rangle$. Let the vertices be numbered cyclically $v_1, v_2, \ldots, v_n$ where $v_i v_j \in E$ if and only if i and j differ by more than 1 (modulo n). Let F be the tolerance orientation of $\bar{C}_n$ corresponding to $\langle \mathcal{I}, t\rangle$.

Consider the subgraph C_4 induced by vertices v_2, v_3, v_n and v_{n-1}. By symmetry and Lemma 4, we may assume that the tolerance orientation is as shown in Fig. 3(a).

Now consider the subgraph C_4 induced by vertices v_1, v_3, v_4 and v_n. Since the edge $v_n v_3$ is already oriented, it follows from Lemma 4 that the subgraph is oriented as in Fig. 3(b).

The contradiction now follows from the fact that this argument can be continued to obtain vertex v_i oriented by tolerance towards vertex v_{i+2} for all $i=1,2,\ldots,n$. This is clearly impossible since it violates the acyclic nature of F. □

Let $N(x)=\mathrm{Adj}(x)\cup\{x\}$ denote the *neighborhood* of a vertex x in G. A set of vertices $\{u_1,u_2,u_3\}$ is called an *asteroidal* triple if u_i and u_k are in the same connected component of $G-N(u_j)$ for all permutations (i,j,k) of $\{1,2,3\}$. In other words, any two of them are connected by a path in G which avoids the neigh-

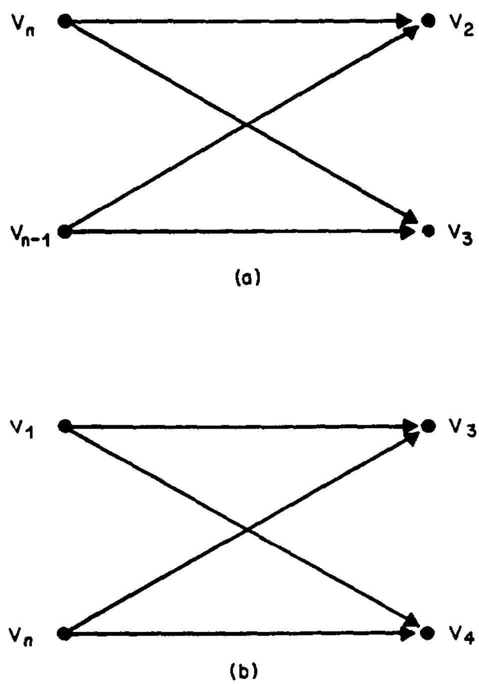


Fig. 3. Tolerance orientations of subgraphs for proof of Theorem 3.

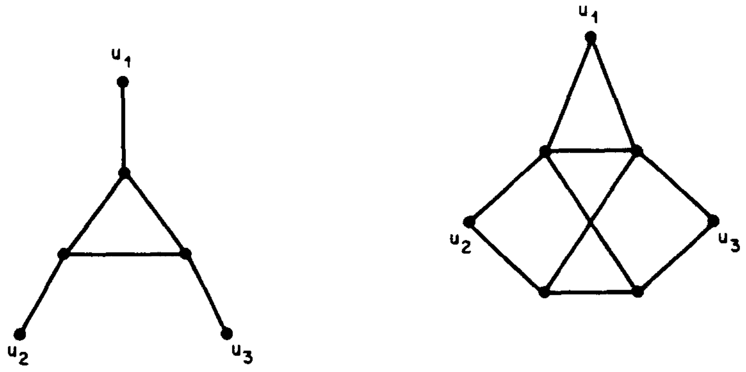


Fig. 4. Two tolerance graphs which are not bounded since they contain an asteroidal triple $\{u_1, u_2, u_3\}$.

borhood of the remaining vertex. Clearly, the vertices of an asteroidal triple are pairwise nonadjacent. See Fig. 4. It is well known [7] that an interval graph contains no asteroidal triple. Although this is not true for tolerance graphs, it is true for bounded tolerance graphs which follows from Theorem 1 and the next result.

Theorem 4. *If G is the complement of a comparability graph, then G contains no asteroidal triple.*

Proof. It has been shown in [5] that a graph $G=(V,E)$ is the complement of a comparability graph if and only if G is the intersection graph of a function diagram. A *function diagram* for G consists of the curves $\{\bar{v} \mid v \in V\}$ obtained from a family of real-valued continuous functions $f_v: [0,1] \to \mathbb{R}$ for all $v \in V$, where $vw \in E$ if and only if $\bar{v}$ and $\bar{w}$ intersect. (See Fig. 5.)

Let D be a function diagram for G, and suppose that G has an asteroidal triple $\{u_1, u_2, u_3\}$. Since the vertices u_1, u_2, u_3 are pairwise nonadjacent, the curves $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$ are disjoint and one of them, say $\bar{u}_2$, lies between the other two. Now if we remove $\bar{u}_2$ and all curves which intersect it, we will obtain a function diagram for $G - N(u_2)$ in which u_1 and u_3 are separated into distinct connected components. This contradicts the assumption that $\{u_1, u_2, u_3\}$ is asteroidal. □

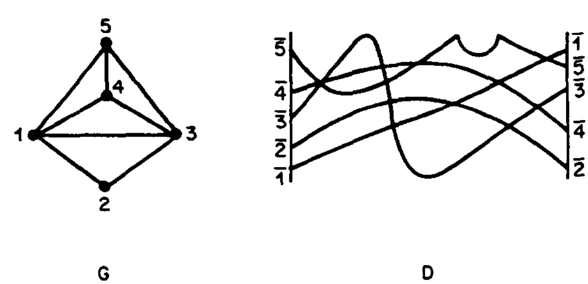


Fig. 5. A function graph G and a function diagram D for G.

Theorem 4 shows that from any connected graph G_0 we may construct a graph G which is not bounded tolerant simply by 'growing' three new paths of two edges anywhere on G_0. The simplest example of this grows an isolated vertex into the tree T_2 in Fig. 6, which is a tolerance graph but is not a bounded tolerance graph. We now show that it is just as easy to construct nontolerance graphs.

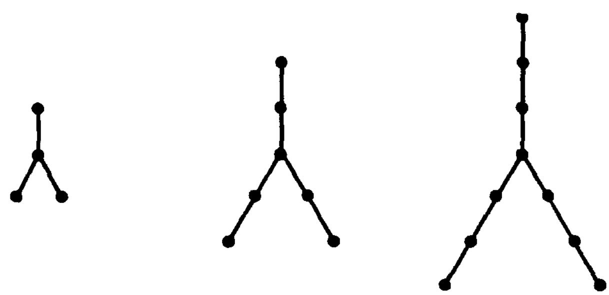
Theorem 5. *Let G be a tolerance graph which is not bounded, and let X be the set of all nonassertive vertices. Then the graph H formed by adding a new pendant vertex onto every member of X, is not a tolerance graph.*

Proof. Suppose H has a tolerance representation $\langle \mathcal{I}, t \rangle$. By Lemma 2, each $x \in X$ is assertive in H since each such x has an exclusive new neighbor. Therefore, we may assume that $t_x \leq |I_x|$ for all $x \in X$. Now if we restrict $\mathcal{I}$ to only the members of G, we obtain a tolerance representation for G in which all nonassertive vertices have bounded tolerance. But this contradicts the assumption that G is not a bounded tolerance graph. □

Example. In Fig. 6, the nonassertive vertices of T_2 are its three leaves. Therefore by Theorem 5, T_3 is not a tolerance graph.

In the remainder of this section we prove that the trees which are bounded tolerance graphs and those which are tolerance graphs are characterized by the forbidden subtrees T_2 and T_3, respectively.

Theorem 6. *If T is a tree, then the following conditions are equivalent:*
(i) *T is a bounded tolerance graph.*
(ii) *T contains no subtree isomorphic to the tree T_2 in Fig. 6.*
(iii) *T is an interval graph.*



T_1 T_2 T_3

Fig. 6. A bounded tolerance (T_1), unbounded tolerance (T_2) and nontolerance (T_3) graph.

Proof. Trees which satisfy (ii) have also been called *caterpillars* in the literature[1]. The equivalence of (ii) and (iii) was established in [7]. See also [11]. The implication (iii)$\Rightarrow$(i) [4] was mentioned in Section 1, and the implication (i)$\Rightarrow$(ii) follows from the fact that T_2 is not bounded tolerance graph. □

Theorem 7. *If T is a tree, then the following conditions are equivalent:*
(i) *T is a tolerance graph.*
(ii) *T contains no subtree isomorphic to the tree T_3 in Fig.* 6.

Proof. Trees which satisfy (ii) may be called caterpillars with toes[2]. The implication (i)$\Rightarrow$(ii) follows from the fact that T_3 is not a tolerance graph. To show (ii)$\Rightarrow$(i) suppose that T is a caterpillar with toes. Let the body of T be the path $[x_1, x_2, \ldots, x_k]$, and let the feet attached to x_i be $y_1^i, \ldots, y_{d_i}^i$. First, we assign the intervals $I_{x_i} = [a_i, b_i]$ as follows:

$$a_1 = 0, \qquad b_1 = d_1 + 4,$$

$$a_i = b_{i-1} - 2, \quad b_i = a_i + d_i + 4 \quad \text{for } i = 2, \ldots, k.$$

We set the tolerances $t_{x_i} = 2$ for all i, and note that $|I_{x_i} \cap I_{x_j}| \geq 2$ if and only if i and j differ by one. Second, we let $I_{y_j^i} = [a_i + 1 + j, a_i + 2 + j]$ and assign $t_{y_j^i} = 1$. Finally, for each toe z attached to the foot y_j^i we associate $I_z = I_{y_j^i}$ and set $t_z = \infty$. See Fig. 7. It is easily verified that this is a tolerance representation for T. □

3. Tolerance graphs are perfect

An *obstruction* [2] of an oriented graph is an induced subgraph isomorphic to the graph in Fig. 8. An undirected graph is called *perfectly orderable* if it admits an acyclic orientation which contains no obstruction.

Theorem 8. (Chvátal [2]). *Perfectly orderable graphs are perfect.*

Examples of perfectly orderable graphs include all comparability graphs, triangulated graphs, and the complements of triangulated graphs. We will prove that the complements of tolerance graphs are perfectly orderable, and hence, by Theorem 8 and the perfect graph theorem, tolerance graphs are perfect.

Let $\langle \mathcal{I}, t \rangle$ be a regular tolerance representation of $G = (V, E)$. We define F to be the *right end point* (REP) *orientation* of $\bar{G}$, that is,

$$xy \in F \quad \Leftrightarrow \quad xy \notin E \text{ and } R(x) < R(y).$$

[1] A caterpillar is a graph consisting of one chordless path (the body) and an arbitrary number of pendant vertices (the feet) attached to the path.

[2] *A caterpillar with toes* consists of a caterpillar and an arbitrary number of pendant vertices (the toes) attached to the feet.

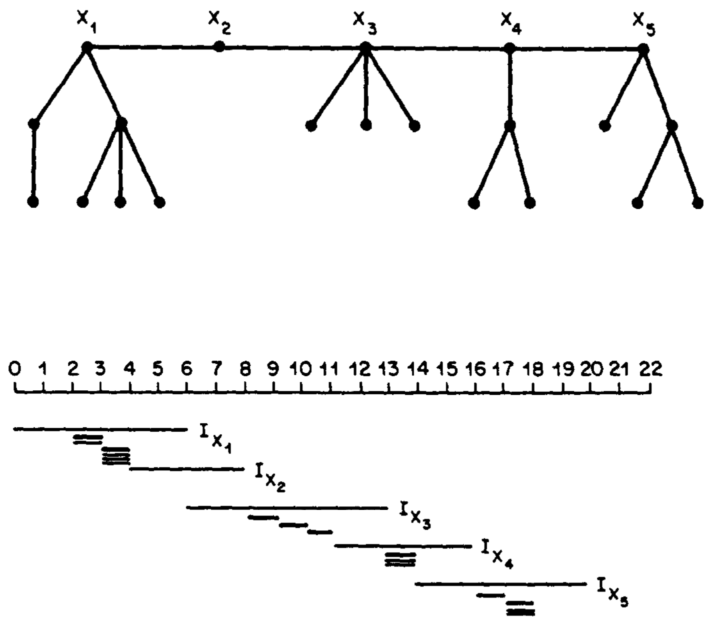


Fig. 7. A caterpillar with toes and a tolerance representation for it.

Furthermore, we label $xy \in F$ as *Type* 1 if $I_x \nsubseteq I_y$ and as *Type* 2 if $I_x \subseteq I_y$. Clearly, the REP orientation is acyclic.

Remark 2. We observe that xy is of Type 2 if and only if x is dominated by y in $\langle \mathscr{I}, t \rangle$. Hence, by Lemma 1, if xy is of Type 2, then $\mathrm{Adj}(x) \subseteq \mathrm{Adj}(y)$.

In general, the REP orientation F of $\bar{G}$ is not transitive. However, it is 'almost' transitive.

Lemma 5. *If xy, $yz \in F$ and it is not the case that xy is Type* 1 *and yz is Type* 2, *then $xz \in F$.*

Proof. If xy is of Type 2, then, by Remark 2, $\mathrm{Adj}(x) \subseteq \mathrm{Adj}(y)$ so $xz \notin E$. Since F is acyclic, it follows that $xz \in F$.

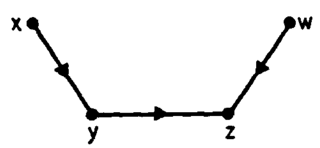


Fig. 8. An obstruction.

If xy and yz are both of Type 1, then we have the following:

$$|I_x \cap I_z| < \min\{|I_x \cap I_y|, |I_y \cap I_z|\} < \min\{t_x, t_z\}.$$

The first inequality follows since $R(x) < R(y) < R(z)$ and $I_x \subsetneq I_y \subsetneq I_z$. The second follows since xy and $yz \notin E$. Thus, $xz \notin E$, and again $xz \in F$, (in fact xz is of Type 1). □

From Lemma 5 we obtain an alternate proof of Theorem 1, i.e., *the complement of a bounded tolerance graph is a comparability graph.*

Proof of Theorem 1. Let F be the REP orientation of $\bar{G}$ obtained from a bounded tolerance representation of G. Since F has no Type 2 edges, Lemma 5 implies that F is a transitive orientation of $\bar{G}$. Therefore, $\bar{G}$ is a comparability graph.

We now present the main result of this section.

Theorem 9. *A tolerance graph is perfect since the* REP *orientation of its complement is perfectly ordered.*

Proof. Suppose that the REP orientation F of $\bar{G}$ contains an obstruction as illustrated in Fig. 8. Since $xz \notin F$ we know, from Lemma 5, that xy is of Type 1 and yz is of Type 2. By Remark 2, $\mathrm{Adj}(y) \subseteq \mathrm{Adj}(z)$. However, $w \in \mathrm{Adj}(y)$ and $w \notin \mathrm{Adj}(z)$, a contradiction. Therefore, F contains no obstruction and G is perfectly orderable. □

4. Proper tolerance graphs

A graph G is called a *proper tolerance graph* if G admits a tolerance representation in which no interval properly contains another interval. It is immediate that the class of proper interval graph is contained in the class of proper tolerance graph; $K_{1,3}$ is an example in the difference.

Theorem 10. *Every proper tolerance graph G is a bounded tolerance graph.*

Proof. Let $\langle \mathcal{I}, t \rangle$ be a proper tolerance representation for G. Without loss of generality we may assume that $\langle \mathcal{I}, t \rangle$ is regular. Suppose that G is not a bounded tolerance graph, and let x be a nonassertive vertex dominated by y in $\langle \mathcal{I}, t \rangle$. Then $I_x \subseteq I_y$. By regularity, $I_x \neq I_y$; therefore, I_x is properly contained in I_y, a contradiction. □

We now show that the graph $K_{3,3}$ is a forbidden subgraph for proper tolerance graphs.

Theorem 11. *If G is a proper tolerance graph, then G contains no induced copy of* $K_{3,3}$.

Proof. It suffices to show that $K_{3,3}$ is not a proper tolerance graph. Suppose that $K_{3,3}$ has a proper tolerance representation $\langle \mathscr{I}, t\rangle$. Without loss of generality we may assume $\langle \mathscr{I}, t\rangle$ is regular and, by symmetry, that the left end points are related by $L(1)<L(2)<L(3)$ and $L(4)<L(5)<L(6)$ and $t_1<t_6$ and $L(1)<L(6)$. Since (1,2) is not an edge in $K_{3,3}$, it is easy to see that $L(6)<L(2)$. Therefore,

$$L(4)<L(5)<L(6)<L(2)<L(3)$$

and, since the representation is proper,

$$R(4)<R(5)<R(6)<R(2)<R(3).$$

Now,

$$\min\{t_4, t_3\}<|I_4\cap I_3|<|I_2\cap I_3|\le t_3$$

$$\min\{t_4, t_3\}<|I_4\cap I_3|<|I_4\cap I_5|\le t_4,$$

a contradiction. □

5. Applications and open questions

Interval graphs capture the notion of objects conflicting because they overlap in time or space. These graphs have found application in areas ranging from scheduling and data storage to archeology and genetics. We refer to [3; Chapter 8] for a discussion of various applications.

Tolerance graphs extend this notion of conflict by incorporating a 'tolerance for overlap'. Two objects conflict only if their overlap meets or exceeds one of their tolerances. This introduces a flexibility into the situations which can be modeled.

As an example, consider a group of employees each scheduled to work for a fixed interval of time at work stations. Employees are each assigned to a single work station for their entire work interval. A conflict arises if two employees are assigned to work on the same work station at the same time. Tolerances arise in a natural way. In addition to working at their stations, employees perform other administrative functions during their day. An employee's tolerance arises from work time that can be spent away from the work station. Employees assigned to the same station who overlap in time alternate performing their administrative duties from day to day. So two employees conflict only if their work intervals overlap by an amount exceeding either of their tolerances.

Another situation arises in the scheduling of meeting rooms. Usually, two meetings are thought to conflict if any part of their meeting times coincide and so they must be assigned to different rooms. However, in order to increase their chances of obtaining a room, some groups might tolerate a certain amount of overlap in time with other meetings. This overlap time could alternatively be used by one group or another from meeting to meeting or the room could actually be shared during this time.

A number of interesting open questions remain unanswered. The most important of which are the characterizations of tolerance and bounded tolerance graphs. Fig. 9 shows the relationships of these graphs to the classes of triangulated, comparability and cocomparability graphs. An example is known for every region shown except for the one containing a question mark. This leads us to the conjecture: "A tolerance graph is bounded if and only if its complement is a comparability graph." An initial step under consideration is to study complements of trees to see if these graphs are tolerance graphs or not.

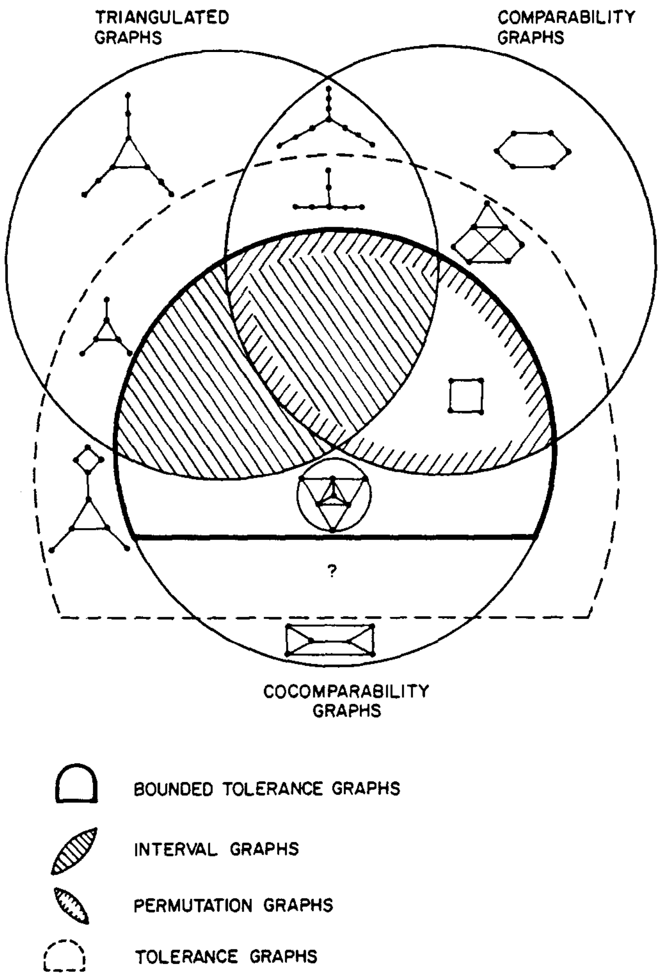


Fig. 9. Relations of tolerance graphs to other classes of graphs.

A graph G is called a *unit tolerance graph* if G admits a tolerance representation in which every interval is of unit length. Clearly, every unit tolerance graph is proper. We pose the question: "Is a graph proper tolerant if and only if is a unit tolerance graph?" The analogous statement for proper and unit interval graphs is true [10].

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