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problem domains - especially those in which the cardinality of the resource set is bound above by a constant - which are not included in the domains Π or Π'_M. Results involving these more general unifications are the subject of our next report[7].

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The Maximum Number of Edges in a Strongly Connected Oriented Graph Without Long Directed Paths

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Abstract

For positive intergers n and k with $n \geq 3$ and $n \geq k$, let $h(n,k)$ be the maximum number of edges in a strongly connected oriented graph on n vertices which does not contain a directed path on $k + 1$ points. M. C. Heydemann gave an explicit formula for $h(n,k)$ when $\lceil (n+5)/2 \rceil \leq k$ and gave a conjecture for $h(n,k)$ when $3 \leq k < \lceil (n+5/2 \rceil$. In this paper, we settle Heydemann's conjecture in the affirmative. In each case, the extremal graphs are oriented complete multipartite graphs although not all part sizes are equal.

1. Introduction

In this paper, we investigate a natural extremal problem for oriented graphs: For integers n and k with $n \geq k \geq 3$, determine the maximum number $h(n,k)$ of edges in a strongly connected oriented graph on n vertices which does not contain a directed path on $k + 1$ points. This problem was first studied by M. C. Heydemann [2] who derived an explicit formula for $h(n,k)$ when $\lceil (n+5)/2 \rceil \leq k$ and constructed a family of extremal graphs. Heydemann also made a conjecture for $h(n,k)$ when $3 \leq k < \lceil (n+5)/2 \rceil$. The principal result of this paper yields a complete determination of $h(n,k)$ for all $n \geq k \geq 3$ and in the process settles Heydemann's conjecture in the affirmative.

2. Notation and Terminology

All graphs considered in this paper are finite and have no loops or multiple edges. An oriented graph is a graph with an orientation

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assigned to each edge. When xy is an edge, we will write (x,y) to denote that the edge has been oriented from x to y. An oriented graph is strongly connected when there exists a directed path from x to y and a directed path from y to x for every pair of distinct vertices.

3. Two Related Extremal Problems

For integers n and s with $n \geq s \geq 1$, let $T(n,s)$ denote the complete multipartite graph having s parts as equal in size as is possible. More formally, if $n = sq + r$ with $0 \leq r < s$, then the vertex set of $T(n,s)$ consists of s disjoint sets of independent vertices $B_1, B_2, \ldots, B_s$. When $r = 0$, each B_i contains q points. When $r > 0$, $B_1, B_2, \ldots, B_r$ each contain $q + 1$ points while B_{r+1}, $B_{r+2}, \ldots, B_s$ each contain q points. Whenever $1 \leq i < j \leq s$, $x \in B_i$ and $y \in B_j$ we have an edge xy. Let $t(n,s)$ denote the number of edges in $T(n,s)$. The following well known theorem is due to P. Turan [3].

Theorem 1: The maximum number of edges in a graph on n vertices in which there is no complete subgraph on $s + 1$ vertices is $t(n,s)$. Furthermore, $T(n,s)$ is the unique extremal graph. □

Theorem 1 together with the elementary result that every orientation of a complete graph admits a directed hamiltonian path explains why the condition that the graph be strongly connected is included in the definition of $h(n,k)$; else the answer would simply be $h(n,k) = t(n,k)$.

The following theorem is due to M. C. Heydemann [1].

Theorem 2: Let n and s be integers with $n \geq s + 1 \geq 3$. The maximum number of edges in a strongly connected oriented graph G on n vertices which has no directed cycle on more than $s + 1$ vertices is $n - 1 + t(n-1,s)$. □

In order to present Heydemann's proof that theorem 2 is best possible and to motivate the proof of our principal theorem, we present the following construction for a strongly connected oriented graph $H(n, s_1, s_2)$ when $n \geq s_1 + 1 + s_2$, $s_1 \geq s_2$, $s_1 \geq 2$ and $s_2 \geq 0$. The vertex set of $H(n, s_1, s_2)$ contains a distinguished vertex x_0 which is adjacent to all other vertices. The remaining $n - 1$ vertices form a

complete s_1-partite graph with parts labelled $B_1, B_2, \ldots, B_{s_1}$. When $s_2 < s_1$, the parts $B_{s_2+1}, B_{s_2+2}, \ldots, B_{s_1}$ are singletons. When $s_2 > 0$, the vertices in $B_1 \cup B_2 \cup \ldots \cup B_{s_2}$ induce an oriented copy of the Turán graph $T(n-1-s_1+s_2, s_2)$. When $1 \leq i < s_1$ and $x \in B_i$ the edge x_0x is oriented (x_0,x); when $x \in B_{s_1}$, the edge x_0x is oriented (x,x_0). When $1 \leq i < j \leq s_1$, $x \in B_i$ and $y \in B_j$, the edge xy is oriented (x,y).

The following result follows immediately from the preceding definition.

Lemma 1: The maximum number of points in a directed cycle in $H(n, s_1, s_2)$ is $s_1 + 1$; and the maximum number of points in a directed path in $H(n, s_1, s_2)$ is $s_1 + s_2 + 1$. □

We let $h(n, s_1, s_2)$ denote the number of edges in $H(n, s_1, s_2)$. Since $h(n,s,s) = n - 1 + t(n-1,s)$, the bound provided by Theorem 2 is best possible as was noted by Heydemann [2].

4. The Principal Theorem

For integers n and k with $n \geq k \geq 3$, let $h^*(n,k) = \max\{h(n, s_1, s_2): s_1 + s_2 + 1 \leq k\}$. Since the maximum number of points in a directed path in $H(n, s_1, s_2)$ is $s_1 + s_2 + 1$, it is clear that $h(n,k) \geq h^*(n,k)$. The principal result of this paper will be to show that $h(n,k) = h^*(n,k)$.

Theorem 3: Let $n \geq k \geq 3$. Then $h(n,k) = h^*(n,k)$.

Proof: Since we always have $h(n,k) \geq h^*(n,k)$, it suffices to show that $h(n,k) \leq h^*(n,k)$. To accomplish this, we consider a strongly connected oriented graph G on n vertices not having a directed path on more than k points. We let E denote the edge set of G and show that $|E| \leq h^*(n,k)$. Without loss of generality. we may assume that $k < n$ since $h^*(n,n) = h(n,n-1,0) = \binom{n}{2}$.

Now let $C_1 = (x_0, x_1, x_2, \ldots, x_{s_1})$ be a directed cycle of maximum size in G. Note that $s_1 \geq 2$ since G is strongly connected. Then let $Y = G - C_1$. It is easy to see that every vertex in Y has at most s_1 neighbors on the cycle C_1 (see Lemma 2, [1]). Now let $Y_1 = \{y \in Y: y$ has a neighbor in $Y\}$ and $Y_2 = Y - Y_1$.

Suppose first that $Y_1 = \phi$. Then $Y_2 \neq \phi$, and every vertex $y \in Y_2$ is incident with an edge to C_1 and an edge from C_1; otherwise G fails to be strongly connected. If $y \in Y_2$ and (y, x_i) is an oriented edge, then $P = (y, x_i, x_{i+1}, x_{i+2}, \ldots, x_{s_1+1}, x_0, x_1, \ldots, x_{i-1})$ is a directed path on $s_1 + 2$ vertices. Thus $s_1 + 2 \leq k$. Furthermore $|E| \leq \binom{s_1+1}{2} + (n-s_1-1)\, s_1 = h(n, s_1, 1) \leq h^*(n,k)$. We may therefore assume that $Y_1 \neq \phi$.

We now proceed to construct a strongly connected oriented graph G' by a procedure which can loosely be described as deleting the vertices in Y_2 and contracting the cycle C_1 to a single vertex. More formally, we choose a symbol y_0 not used to label any vertex in G. The vertex set of G' is $\{y_0\} \cup Y_1$. The restriction of G' to Y_1 is the same as the restriction of G to Y_1. In order to complete the definition of G', we must describe the adjacencies and orientations between y_0 and vertices in Y_1. This will be accomplished by an "orienting algorithm" which is applied to each component (in the undirected sense) of Y_1. The algorithm will satisfy the following properties.

P1: If $y \in Y_1$, yy_0 is an edge in G' and yy_0 is oriented (y, y_0), then there is some $x_i \in C_1$ so that yx_i is an edge in G and is oriented (y, x_i).

P2: If $y \in Y_1$, yy_0 is an edge in G' and yy_0 is oriented (y_0, y), then there is some $x_i \in C_1$ so that yx_i is an edge in G and is oriented (x_i, y).

Now let F be a component of Y_1. Then let $S_1 = \{y \in F$: y has at least one neighbor on C_1 but all edges between y and C_1 are oriented from y to $C_1\}$. Also let $S_2 = \{y \in F$: y has at least one neighbor on C_1, but all edges between y and C_1 are oriented from C_1 to $y\}$. In G', we have edges yy_0 for every $y \in S_1 \cup S_2$. In view of P_1 and P_2, yy_0 is oriented (y, y_0) when $y \in S_1$ and as (y_0, y) when $y \in S_2$.

Let F' denote those vertices in F which have at least one neighbor on C_1 but do not belong to $S_1 \cup S_2$. If $F' = \phi$, we observe that the oriented graph induced by $\{y_0\} \cup F$ is strongly connected so we may turn our attention to another component of Y_1. Now suppose $F' \neq \phi$. Of course each vertex must actually have at least two neighbors on C_1.

It is easy to see that we may choose an edge y_2y_1 in F oriented (y_2, y_1) and a vertex $x_i \in C_1$ so that y_1x_i is an edge oriented (y_1, x_i). Then set $T_1 = S_1 \cup \{y_1\}$. Note that it may happen that $y_1 \in S_1$ so $S_1 = T_1$. If $y_1 \notin S_1$, put the edge y_1y_0 in G' and orient it (y_1, y_0). Then set $T_2 = S_2 \cup \{y \in F'$: $y \neq y_1$, there exists a directed path in Y_1 from y to a vertex in $T_1\}$. For each $y \in T_2$, we add the edge yy_0 to G' and orient it as (y_0, y). Now suppose for some even $\alpha \geq 2$ we have defined pairwise disjoint subsets $T_1, T_2, T_3, T_4, \ldots, T_\alpha$. If these sets cover F', stop and go to another component of Y_1. Else define $W_\alpha = T_1 \cup T_2 \cup \ldots \cup T_\alpha$, $T_{\alpha+1} = \{y \in F' - W_\alpha$: There exists a directed path in Y_1 from a vertex in T_α to $y\}$ and $T_{\alpha+2} = \{y \in F' - (W_\alpha \cup T_{\alpha+1})\}$: There exists a directed path in Y_1 from y to a vertex in $T_{\alpha+1}\}$. For each $y \in T_{\alpha+1} \cup T_{\alpha+2}$, we add the edge yy_0 to G' and orient it (y, y_0) when $y \in T_{\alpha+1}$ and (y_0, y) when $y \in T_{\alpha+2}$.

It is an easy exercise to show that this algorithm results in a strongly connected oriented graph G' satisfying properties P_1 and P_2.

Now let $C_2 = (z_0, z_1, z_2, \ldots, z_{s_2})$ be a cycle of maximum size in G'. We show that $s_1 + s_2 + 1 \leq k$. To accomplish this, we first consider the case where $y_0 \notin C_2$. Then C_1 and C_2 are disjoint cycles in the original graph G. Take the shortest directed path between C_1 and C_2 in G. We can relabel the vertices on C_1 and C_2 so that this shortest path is $(w_0, w_1, w_2, \ldots, w_r)$ where $w_0 = x_0$ and $w_r = z_0$. Of course, $r \geq 1$. It follows that $(x_1, x_2, \ldots, x_{s_1}, x_0, w_1, \ldots, w_{r-1}, z_0, z_1, \ldots, z_{s_2})$ is a directed path in G having at least $s_1 + s_2 + 2$ vertices. Thus $s_1 + s_2 + 2 \leq k$ which is stronger than the desired inequality.

Now suppose $y_0 \in C_2$. We may assume that $y_0 = w_0$. Since $(w_0, w_1) = (y_0, w_1)$ is an edge in G', we know there exists a vertex $x_i \in C_1$ so that x_iw_1 is an edge in G and is oriented (x_i, w_1). Then $(x_{i+1}, x_{i+2}, \ldots, x_{s_1}, x_0, x_1, \ldots, x_{i-1}, x_i, w_1, w_2, \ldots, w_{s_2})$ is directed path on $s_1 + s_2 + 1$ points in G. Thus $s_1 + s_2 + 1 \leq k$ as desired.

We now apply Theorem 2 to the graph G'. Let $n_1 = |Y_1|$ and $n_2 = |Y_2|$. Note that G' has $n_1 + 1$ vertices and that $n_1 + n_2 + s_1 + 1 = n$. Now G' has no directed cycle on more than $s_2 + 1$ points. So if we let E' denote the edge set of G', we have the inequality:

$$|E'| \le n_1 + t(n_1, s_2).$$

Now each vertex in Y_1 has at most s_1 neighbors in C and at most $s_1 - 1$ of these are not reflected in the inequality for F'. Allowing for C_1 to induce a complete graph and for each vertex in Y_2 to be adjacent of s_1 vertices on C_1, we have:

$$\begin{aligned}|E| &\le |E'| + n_1(s_1-1) + n_2(s_1) + \binom{s_1+1}{2}\\ &\le n_1 + t(n_1,s_2) + n_1(s_1-1) + n_2 s_1 + \binom{s_1+1}{2}\\ &= t(n_1,s_2) + (n_1+n_2)\, s_1 + \binom{s_1+1}{2}\\ &\le t(n_1+n_2,s_2) + (n_1+n_2)\, s_1 + \binom{s_1+1}{2}\\ &= h(n,s_1,s_2)\end{aligned}$$ □

At this point, the reader may be mildly dissatisfied with the form of Theorem 3 since it does not explicitly determine the appropriate values of s_1 and s_2 which maximize $h(n,s_1,s_2)$.

Lemma 2: Let $n \ge k \ge 3$. Then $h^*(n,k) = \max\{h(n,s_1,s_2)\colon s_1 + s_2 + 1 \le k\} = \max\{h(n,s_1,s_2)\colon s_1 + s_2 + 1 = k\}$ □

Lemma 3: Suppose $n \ge k \ge 3$, $s_1 + s_2 + 1 = k$ and $s_2 > 0$. In $H(n,s_1,s_2)$, let $b_i = |B_i|$ for $i = 1, 2, \ldots, s_2$.

(1) If $b_{s_2} = 2$, then $h(n,s_1,s_2) < h(n,s_1+1,s_2-1)$.

(2) If $b_{s_2} = b_{s_2-1} = 3$, then

$$h(n,s_1,s_2) = h(n,s_1+1,s_2-1).$$

(3) In all other cases,

$$h(n,s_1,s_2) > h(n,s_1+1,s_2-1).$$ □

Lemma 2 explains the subtle difficulty behind Theorem 3. Specifically, the form of the conjectured extremal graph changes depending on the relative size of n and k.

We illustrate Lemma 2 with the following specific examples.

Example 1: $h(14,8) = h(14,4,3) = 73$

Example 2: $H(20,13) = h(20,6,6) = h(20,7,5) = h(20,8,4) = 169$

5. Open Problems

We close with two open problems.

(1) Characterize the extremal graphs for Theorem 3.

(2) Determine the maximum number of edges in a strongly connected digraph on n vertices which does not have a directed path on more than k vertices.

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