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FOR $t \geqslant 3$, EVERY t-DIMENSIONAL PARTIAL ORDER CAN BE EMBEDDED IN A $t+1$-IRREDUCIBLE PARTIAL ORDER

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ABSTRACT

The dimension of a partial order X is the least integer t for which there exist linear extensions $X_1, X_2, \dots, X_t$ so that $x_1 \leqslant x_2$ in X if and only if $x_1 \leqslant x_2$ in X_i for each $i = 1, 2, \dots, t$. For an integer $t \geqslant 2$, a partial order is t-irreducible if it has dimension t and every proper nonempty subpartial order has dimension less than t. The only 2-irreducible partial order is a 2-element antichain. There are infinitely many 3-irreducible partial orders, and they may be conviently grouped into 9 infinite families with 18 odd examples left over. There are many 2-dimensional partial orders which cannot be embedded in a 3-irreducible partial order, for example, any 2-dimensional partial order whose length and width both exceed 4. However, when $t \geqslant 3$, we prove that every t-dimensional partial order can be embedded in a $t+1$-irreducible partial order.

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1. INTRODUCTION

In this paper, we prove that if $t \geqslant 3$, every t-dimensional partially ordered set can be embedded in a $t+1$-irreducible partially ordered set. This result is false when $t = 2$ since there are many 2-dimensional posets which cannot be embedded in 3-irreducible posets. The construction used to establish this theorem has its origins in chromatic graph theory, and an elementary version had been previously used by the authors [12] to show that for each $t \geqslant 2$, every t-irreducible poset is a subposet of a $t+1$-irreducible poset. For the sake of completeness, we present in Section 2 the fundamental definitions and summarize some preliminary material on the dimension of partially ordered sets. For additional background material, we refer the reader to the survey articles [6] and [10]. In particular, [6] contains an extensive bibliography of papers on dimension theory.

2. NOTATION, TERMINOLOGY AND PRELIMINARY RESULTS

A *partially ordered set (poset)* is a set X equipped with a reflexive, antisymmetric and transitive binary relation $\leqslant$. If $x_1, x_2 \in X$, $x_1 \nleqslant x_2$ and $x_2 \nleqslant x_1$, then x_1 and x_2 are *incomparable* and we write $x_1 \parallel x_2$. For each point $x_1 \in X$, we let $D_X(x_1) = \{x_2 \in X\colon x_2 < x_1\}$, $U_X(x_1) = \{x_2 \in X\colon x_1 < x_2\}$, and $I_X(x_1) = \{x_2 \in X\colon x_1 \parallel x_2\}$. We let $I_X = \{(x_1, x_2)\colon x_1 \parallel x_2\}$. X is a *linear order* if $I_X = \phi$. If X_1 and X_2 are partial orders on the same set and $x_1 < x_2$ in X_2 whenever $x_1 < x_2$ in X_1, we say X_2 is an extension of X_1; if X_2 is a linear order and an extension of X_1, then X_2 is called a *linear extension* of X_1. Dushnik and Miller [1] defined the *dimension* of a poset X, denoted $\dim(X)$, as the least positive integer t for which there exist t linear extensions $X_1, X_2, \ldots, X_t$ of X so that $x_1 \leqslant x_2$ in X if and only if $x_1 \leqslant x_2$ in X_i for each $i = 1, 2, \ldots, t$.

If X_1 and X_2 are posets and the point set of X_1 is a subset of the point set of X_2, the poset X_1 is called a *subposet* of X_2 when $x_1 \leqslant x_2$ in X_1 if and only if $x_1 \leqslant x_2$ in X_2 for all $x_1, x_2 \in X_1$. For each point $x \in X$, we let $X - \{x\}$ denote the subposet of X whose point set contains all points in X except x. Of course, $\dim(X - \{x\}) \leqslant \dim X$ for

each $x \in X$. For an integer $t \geqslant 2$, a poset X is *t-irreducible* if $\dim(X) = t$ and $\dim(X - \{x\}) < t$ for each $x \in X$. A poset has dimension one if and only if it is a linear order (a chain), so the only 2-irreducible poset is a two point antichain. There are infinitely many 3-irreducible posets, and a complete listing of these posets has been made by Trotter and Moore [10] and by Kelly [4]. These posets can be conviently grouped into 9 infinite families with 18 odd examples left over.

In Section 3, we will make extensive use of one of the infinite families of 3-irreducible posets. In Section 4, we will use D. Kelly's dimension product construction to obtain for each $t \geqslant 4$ a particular family of t-dimensional posets. But before we proceed to those results, we need to develop additional material for working with the dimension of a poset.

A set $R = \{X_1, X_2, \ldots, X_t\}$ of t (not necessarily distinct) linear extensions of a poset X is called a *realizer* of X if $x_1 \leqslant x_2$ in X if and only if $x_1 \leqslant x_2$ in X_i for each $i = 1, 2, \ldots, t$. The dimension of X is then the minimum size of a realizer of X. Let I_X denote the set of all incomparable pairs of X. Then it is easy to see that a collection $R = \{X_1, X_2, \ldots, X_t\}$ of linear extensions of X is a realizer of X if and only if for each $(x_1, x_2) \in I_X$, there exists some $X_i \in R$ so that $x_2 < x_1$ in X_i. If $I \subseteq I_X$ and $R = \{X_1, X_2, \ldots, X_t\}$ is a collection of linear extensions of X, then we say R *reverses* I if for every $(x_1, x_2) \in I$, there exists some $X_i \in R$ with $x_2 < x_1$ in X_i. The dimension of a poset X with $I_X \neq \phi$ is then the minimum number of linear extensions of X required to reverse all incomparable pairs.

For a binary relation R, we let $\hat{R} = \{(r_1, r_2)\colon (r_2, r_1) \in R\}$. $\hat{R}$ is called the *dual* or *reverse* of R. When X is not a chain, it follows that $\dim(X)$ is the least t for which there exists a partition $\hat{I}_X = I_1 \cup I_2 \cup \ldots \cup I_t$ so that for each $i = 1, 2, \ldots, t$, there exists a linear extension X_i of X with $x_1 < x_2$ in X_i for every $(x_1, x_2) \in I_i$. It is therefore natural to consider the following question:

If $I \subseteq I_X$, under what conditions does there exist a linear extension X_0 of X with $x_1 < x_2$ in X_0 for every $(x_1, x_2) \in I$?

The answer to this question is easy to provide. A set $\{(c_i, d_i): 1 \leqslant i \leqslant m\} \subseteq I_X$ is called a *TM-cycle of length* m when $d_i \leqslant c_{i+1}$ for $i = 1, 2, \dots, m-1$ and $d_m \leqslant c_1$. It is easy to show that these sets provide an answer to this question (see [11], for example).

Lemma 1. *Let* X *be a poset and let* $I \subseteq I_X$. *Then there exists a linear extension* X_0 *of* X *with* $x_1 < x_2$ *in* X_0 *for every* $(x_1, x_2) \in I$ *if and only if* I *does not contain a TM-cycle.* ∎

In many cases it is convenient to have a somewhat more technical version of this result at our disposal. A TM-cycle $\{(c_i, d_i): 1 \leqslant i \leqslant m\}$ is said to be *strong* if $d_i \leqslant c_j$ if and only if $j = i + 1$ for $1 \leqslant i \leqslant m-1$, and $d_m \leqslant c_j$ if and only if $j = 1$. It is straightforward to verify that if $I \subseteq I_X$ and I contains a TM-cycle, then I also contains a strong TM-cycle. Furthermore, if I is a strong TM-cycle, then no proper subset of I contains a TM-cycle.

Lemma 2. *Let* X *be a poset and let* $I \subseteq I_X$. *Then there exists a linear extension* X_0 *of* X *with* $x_1 < x_2$ *in* X_0 *for every* $(x_1, x_2) \in I$ *if and only if* I *does not contain a strong TM-cycle.* ∎

In view of the preceding result, it is natural to associate with a poset X a hypergraph G_X so that the dimension of X is the same as the chromatic number of G_X. Here we use the definition of the chromatic number of G_X as the least number of colors required to assign colors to the vertices of G_X so that no edge of G_X has all of its vertices assigned the same color. The scheme for defining G_X is immediate. The vertex set of G_X is the set I_X of incomparable pairs and a subset $I \subseteq I_X$ is an edge if its reverse $\hat{I}$ is a strong TM-cycle.

From a practical view point, the hypergraph G_X contains too many vertices to be of much value in determining the dimension of X. However, there is a natural way to determine a subhypergraph H_X of G_X so that H_X and G_X have the same chromatic number, and in many cases the combinational structure of H_X is more readily analyzed.

An incomparable pair $(x_1, x_2) \in I_X$ is called a *nonforced* pair if $x_3 < x_1$ implies $x_3 < x_2$ for all $x_3 \in X$ and $x_2 < x_4$ implies $x_1 < x_4$

for all $x_4 \in X$. We let N_X denote the set of all nonforced pairs. It is customary to treat N_X as both a binary relation and a directed graph. In the latter interpretation, we draw an edge from x_2 to x_1 whenever $(x_1, x_2) \in N_X$. In Figures 1a and 1b we show a 2-dimensional poset X and its digraph N_X of nonforced pairs.

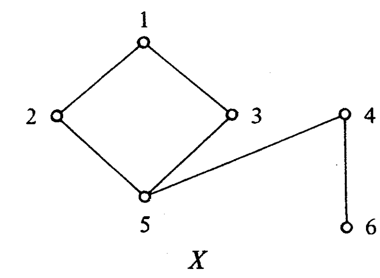

Figure 1a

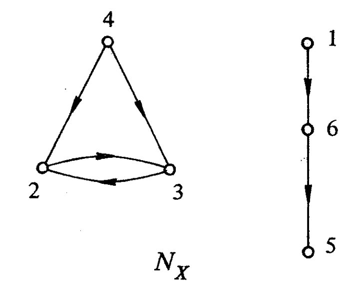

Figure 1b

The graph theoretic properties of the digraph N_X are central to the theory of rank for partially ordered sets, and we refer the reader to [6], [7], [8] and [9] for additional material on this topic. In this paper we will require some elementary properties of N_X. We state these results without proof and refer the reader to [7] for details.

The initial advantage gained from considering the set of nonforced pairs is that they are useful in identifying realizers.

Lemma 3. *Let* $R = \{X_1, X_2, \dots, X_t\}$ *be a set of linear extensions of a poset* X. *Then* R *is a realizer of* X *if and only if for each* $(x_1, x_2) \in N_X$, *there exists some* $X_i \in R$ *so that* $x_2 < x_1$ *in* X_i. ∎

It follows immediately from the preceding lemma that if X is not a chain, then the dimension of X is the minimum number of linear extensions of X required to reverse the nonforced pairs of X. This observation allows us to determine a subhypergraph H_X of G_X which has the same chromatic number as G_X. The vertex set of H_X is the set N_X of nonforced pairs of X. A subset $N \subseteq N_X$ is an edge in H_X if and only if $\hat{N}$ is a TM-cycle. It is often the case that the hypergraph H_X has relatively simple structure; in particular, it is frequently a simple graph whose coloring properties can be easily determined.

If $I \subseteq I_X$, we abuse notation somewhat and denote by $X \cup I$ the binary relation of the point set of the poset X defined by $(x_1, x_2) \in X \cup I$ if and only if $x_1 \leqslant x_2$ in X or $(x_1, x_2) \in I$. Note that an incomparable pair $(x_1, x_2) \in I$ is a nonforced pair if and only if $X \cup \{(x_1, x_2)\}$ is transitive.

Lemma 4. *If X is a poset and N_X is its set of nonforced pairs, then the binary relation $X \cup N_X$ is transitive.* ∎

A subposet Y of a poset X is said to be *partitive* (in X) if the following two conditions are satisfied:

(i) If $x \in X - Y$ and $x > y$ for some $y \in Y$, then $x > y$ for all $y \in Y$.

(ii) If $x \in X - Y$ and $x < y$ for some $y \in Y$, then $x < y$ for all $y \in Y$.

A partitive subset Y in a poset X is nontrivial when $2 \leqslant |Y| < |X|$. The following result is a special case of the formula for the dimension of an ordinal sum (see [3] or [7]).

Lemma 5. *If Y is a nontrivial partitive subsposet of a poset X and $y_0 \in Y$, then* $\dim(X) = \max\{\dim(X - (Y - \{y_0\})), \dim(Y)\}$. ∎

In particular, it follows that if $t \leqslant 2$ and X is t-irreducible, then X contains no nontrivial partitive subposets.

Lemma 6. *Let X be a poset and N_X the set of nonforced pairs of X. If the binary relation $X \cup N_X$ contains a directed cycle $\{(x_i, x_{i+1}): 1 \leqslant i < m\} \cup \{(x_m, x_1)\}$ where $m \geqslant 2$, then the subposet $\{x_1, x_2, \ldots, x_m\}$ is a partitive antichain in X.* ∎

The only 2-irreducible poset is a two element antichain. For this poset $I_X = N_X = X \cup N_X$ and each of these binary relations is a directed cycle of length two. But for $t \geqslant 3$, no such pathology can occur.

Lemma 7. *Let $t \geqslant 3$, let X be a t-irreducible poset, and let N_X be the set of nonforced pairs of X. Then the binary relation $X \cup N_X$ is acyclic – that is, it contains no directed cycles.* ∎

For any poset X for which $X \cup N_X$ is acyclic, it is therefore permissible to consider $X \cup N_X$ as a partially ordered set. With this interpretation, $X \cup N_X$ is an extension of the poset X. For such posets, a linear extension $X_0 = \{x_1 < x_2 < x_3 < \ldots < x_n\}$ of $X \cup N_X$ is said to be *consistent* if $i < j$ whenever $x_i \in D_X(x_n)$ and $x_j \in I_X(x_n)$. A maximal element x of the poset $X \cup N_X$ is called a *strongly maximal* element of X.

Lemma 8. *Let X be a poset and let N_X be the set of nonforced pairs of X. If $X \cup N_X$ is acyclic and x is a strongly maximal element of X, then there exists a consistent linear extension $X_0 = \{x_1 < x_2 < x_3 < \ldots < x_n\}$ of $X \cup N_X$ with $x = x_n$.* ∎

If X is a t-dimensional poset and X_0 is a consistent linear extension of $X \cup N_X$, then X_0 cannot belong to any realizer of size t of X since X_0 reverses no nonforced pairs. On the other hand, we will frequently make minor modifications in a consistent linear extension to obtain one which does belong to a realizer of size t. Here is one such instance; others will be discussed in Section 3.

If $X_0 = \{x_1 < x_2 < x_3 < \ldots < x_n\}$ is a consistent linear extension of $X \cup N_X$, where $D_X(x_n) = \{x_1, x_2, \ldots, x_s\}$, and $I_X(x_n) = \{x_{s+1}, x_{s+2}, \ldots, x_{n-1}\}$, then the linear order

$$X_0^* = \{x_1 < x_2 < x_3 < \ldots < x_s < x_n < \\ < x_{s+1} < x_{s+2} < \ldots < x_{n-1}\}$$

is called the *reverse* of X_0. Note that X_0^* is a linear extension of X but that X_0^* is not in general a linear extension of $X \cup N_X$. The following lemma shows that X_0^* belongs to a realizer of size t for X when X is t-irreducible. The result is a special case of the theorem due to Hiraguchi [3] which states that the removal of a point from a poset decreases the dimension by at most one.

Lemma 9. *Let $t \geqslant 3$ and let X be a t-irreducible poset. Also let X_0 be a consistent linear extension of $X \cup N_X$. Then let x be the strongly maximal element of X which is the greatest element in X_0.*

If $\{X'_1, X'_2, \ldots, X'_{t-1}\}$ *is a realizer of* $X - \{x\}$ *and for each* $i = 1, 2, \ldots, t-1$, *we form a linear extension* X_i *of* X *by adding* x *to* X'_i *as the largest element, then* $\{X^*_0, X_1, X_2, \ldots, X_{t-1}\}$ *is a realizer of* X. ∎

3. THE EMBEDDING POSET WHEN $t = 3$

In this section, we present an infinite family of 3-irreducible posets $\{X(n, 3)\colon n \geqslant 1\}$ whose special properties will be particularly useful in the proof of our principal theorem in the case $t = 3$. Is is of secondary importance that these posets are irreducible. What actually matters is that they each have a consistent linear extension (which of course cannot belong to any realizer of size 3) such that if any one of a large number of minor modifications is made, the resulting linear extension belongs to a realizer of size 3.

For each $n \geqslant 1$, the poset $X(n, 3)$ is a 3-irreducible poset for which the linear extension $X_0 = \{x_1, x_2, x_3, \ldots, x_{2n+5}\}$ is consistent. In Figure 2, we show a diagram for $X(n, 3)$; for clarity, only the subscripts are shown.

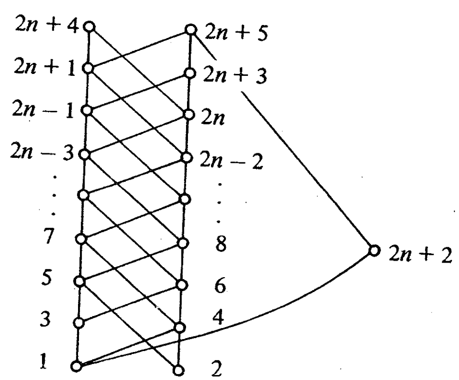

Figure 2

Let $X = X(n, 3)$; then it is straightforward to verify that the set N_X of nonforced pairs is the union of the two sets N'_X and N''_X where

$$N'_X = \{(x_i, x_{i+1})\colon 1 \leqslant i \leqslant 2n,\ 2n+2 \leqslant i \leqslant 2n+4\},$$

$$N''_X = \{(x_2, x_{2n+2}), (x_3, x_{2n+2}), (x_{2n+2}, x_{2n+4})\}.$$

The following lemma gives important information on the structure of the hypergraph H_X.

Lemma 10. *If* $n \geqslant 2$ *and* $X = X(n, 3)$, *then the hypergraph* H_X *is a simple graph.*

Proof. Let N be an edge in the hypergraph H_X and suppose that N contains at least three vertices of H_X. After relabelling, we may assume that $\hat{N} = \{(c_i, d_i)\colon 1 \leqslant i \leqslant m\}$ is a strong TM-cycle in X with $m = |N| \geqslant 3$. Since $C = \{c_1, c_2, \ldots, c_m\}$ and $D = \{d_1, d_2, \ldots, d_m\}$ are antichains in X, we may conclude that $m = 3$ and that $x_{2n+2} \in C \cap D$. Without loss of generality, we may assume that $x_{2n+2} = c_1$ and $x_{2n+2} = d_3$. Then since $(d_1, c_1) = (d_1, x_{2n+2}) \in N$, we know that either $d_1 = x_2$ or $d_1 = x_3$. Similarly, either $c_3 = x_{2n+3}$ or $c_3 = x_{2n+4}$. In any case, we would conclude that $d_1 < c_3$ which contradicts the assumption that $\hat{N}$ is a strong TM-cycle. ∎

Lemma 11. *If* $n \geqslant 2$ *and* $X = X(n, 3)$, *then the graph* H_X *is a 3-colorable graph. Furthermore, the removal from* H_X *of any one of the* $2n$ *vertices in the set* $\{(x_i, x_{i+1})\colon 1 \leqslant i \leqslant 2n\}$ *leaves a 2-colorable graph.* ∎

We illustrate the preceding lemma when $n = 2$. For clarity only the subscripts are shown.

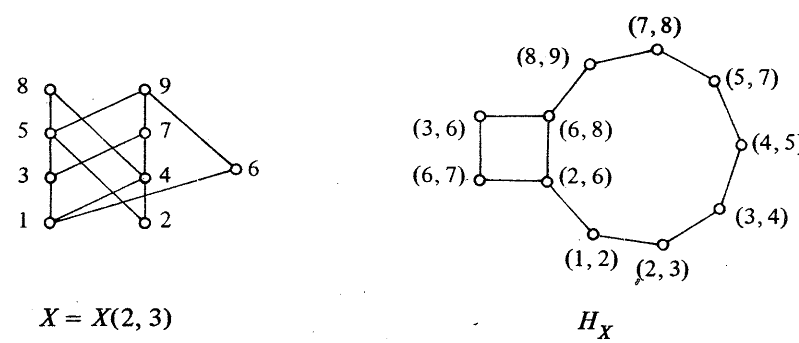

Figure 3

The reader may note that Lemma 11 has not been presented in the strongest possible form, but as will become clear, we are only interested in certain vertices in the graph H_X. Recall that the consistent linear extension X_0 reverses no nonforced pairs of X, so there is no realizer of size 3 to which X_0 belongs. Now let i be any integer with $1 \leqslant i \leqslant 2n$ and let X_0^i be the linear extension of X obtained by interchanging x_i and x_{i+1} in X_0 that is

$$X_0^i = \{x_1 < x_2 < \ldots < x_{i-1} < x_{i+1} < x_i < \\ < x_{i+2} < x_{i+3} < \ldots < x_{2n+5}\}.$$

We now show that each X_0^i belongs to a realizer of size 3 of X.

Lemma 12. *Let* $n \geqslant 2$ *and let* $X = X(n, 3)$. *Then for each* $i = 1, 2, \ldots, 2n$, *there exist linear extensions* X_1^i, X_2^i *so that* $\{X_0^i, X_1^i, X_2^i\}$ *is a realizer of* X.

Proof. From Lemma 11, we note that for each $i = 1, 2, \ldots, 2n$, the graph $H_X - \{(x_i, x_{i+1})\}$ can be 2-colored using the colors $\{1, 2\}$. Then for each $j = 1, 2$, let X_j^i be a linear extension of X which reverses all nonforced pairs which have been assigned color j. Since X_0^i reverses the nonforced pair (x_i, x_{i+1}), it follows that $\{X_0^i, X_1^i, X_2^i\}$ reverses N_X and these linear extensions are a realizer of X. ∎

4. THE EMBEDDING THEOREM WHEN $t = 3$

Suppose that X and Y are disjoint subposets of a poset Z. We say that Y is an *upper filter* of X if the following two conditions are satisfied:

(i) For every $(x_1, x_2) \in N_X$, there exists $y \in Y$ with $x_1 < y$ and $x_2 \parallel y$.

(ii) For every $x \in X$ and every $y \in Y$, $y \nless x$.

A linear extension Z_0 of Z is called an *injection of* X *over* Y when $x \in X$, $y \in Y$, and $x \parallel y$ imply that $x > y$ in Z_0. The concepts of upper filters and injections are related by the following elementary result.

Lemma 13. *Let* X *and* Y *be disjoint subposets in a poset* Z, *and let* Y *be an upper filter of* X. *If* $\dim(X) = t$ *and* Z_0 *is a linear extension of* Z *which is an injection of* X *over* Y, *then* Z_0 *cannot belong to a realizer of size* t *for* Z.

Proof. Any realizer of Z must reverse all incomparable pairs in Z and must therefore reverse the incomparable pairs in N_X. Now let $(x_1, x_2) \in N_X$; choose an element $y \in Y$ so that $x_1 < y$ and $x_2 \parallel y$ in Z. Since Z_0 is an injection of X over Y, we must have $x_1 < y < x_2$ in Z_0. Thus Z_0 reverses no pairs in N_X. If R is a realizer of size t for Z and $Z_0 \in R$, then the restrictions of the other $t - 1$ linear extensions to X must reverse N_X. Since $\dim X = t$, this is impossible. ∎

If X and Y are disjoint subposets of a poset Z, we write $X < Y$ when $x < y$ for every $x \in X$ and $y \in Y$. Similarly, we write $X \parallel Y$ when $x \parallel y$ for every $x \in X$ and $y \in Y$.

We are now ready to present the proof of our principal theorem for the case $t = 3$.

Theorem 14. *Let* P *be a poset with* $\dim P \leqslant 3$. *Then there exists a 4-irreducible poset* R *containing* P *as a subposet.*

Proof. We first construct a 4-dimensional poset S containing P as a subposet. In general, S will not be irreducible, but we will prove that S contains a 4-irreducible subposet R which also contains P as a subposet. The poset S is the union of five disjoint subposets P, X, Y, U, and V, with $X \parallel U$, $X < V$, $U < (P \cup Y)$, $P < Y$, and $V \parallel (P \cup Y)$. Furthermore, $P \cup Y$ will be an upper filter of X and V will be an upper filter of U. The posets X and U will be 3-irreducible posets.

Since $\dim P \leqslant 3$, there exists linear extensions P_1, P_2, and P_3 of P so that $p_1 \leqslant p_2$ in P if and only if $p_1 \leqslant p_2$ in P_i for $i = 1, 2, 3$. These linear extensions need not be distinct. We suppose that $|P| = m$ and let $P_1 = \{p_1 < p_2 < p_3 < \ldots < p_m\}$. Next choose an integer n so that $n \geqslant 2$ and $2n \geqslant m$. Then the subposet X is $X(n, 3)$.

The subposet Y is a chain containing $2n + 4 - m$ points $\{y_{m+1} < < y_{m+2} < \ldots < y_{2n+4}\}$. For each $j = 1, 2, \ldots, m$, $x_i < p_j$ in S if and