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INEQUALITIES IN DIMENSION THEORY FOR POSETS

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ABSTRACT. The dimension of a poset (X, P), denoted $\dim(X, P)$, is the minimum number of linear extensions of P whose intersection is P. It follows from Dilworth's decomposition theorem that $\dim(X, P) \leq \text{width}(X, P)$. Hiraguchi showed that $\dim(X, P) \leq |X|/2$. In this paper, A denotes an antichain of (X, P) and E the set of maximal elements. We then prove that $\dim(X, P) \leq |X - A|$; $\dim(X, P) \leq 1 + \text{width}(X - E)$; and $\dim(X, P) \leq 1 + 2\,\text{width}(X - A)$. We also construct examples to show that these inequalities are sharp.

1. **Introduction.** Dushnik and Miller [4] defined the dimension of a poset, denoted $\dim(X, P)$ or $\dim X$, to be the minimum number of linear extensions of P whose intersection is P. Equivalently, Ore [7] defined $\dim(X)$ to be the smallest integer k such that (X, P) is isomorphic to a subposet of R^k. We refer the reader to [1], [2], and [8] for other definitions and preliminaries. In this paper we establish inequalities involving dimension, width, height, and cardinality. A number of such inequalities are known and we begin by stating a sampling of them.

Theorem. *For any posets X, Y, any chain $C \subseteq X$, and any point $x \in X$, the following inequalities hold.*

(1) $\dim(X - x) \leq \dim X \leq 1 + \dim(X - x)$ [5], [1],

(2) $\dim X \leq 2 + \dim(X - C)$ [5],

(3) $\dim X \leq \text{width } X$ [5],

(4) $\dim X \leq |X|/2$ (*Hiraguchi's theorem* [5], [1]),

(5) $\dim(X \times Y) \leq \dim X + \dim Y$.

A poset has dimension one iff it is a chain. If a poset consists of an antichain of at least two points, then its dimension is two. Throughout the remainder of this paper we will assume that X is a poset which is neither a chain nor an antichain. We will use the symbols A and E to denote an arbitrary antichain in X and the set of maximal elements respectively. If

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$|X - A| = 1$, but X is not a chain, then it is trivial to show that $\dim X = 2$. Therefore we will assume that for any antichain $A \subseteq X$, $|X - A| \geq 2$. Furthermore we do not distinguish between a poset and its dual.

2. **Some new inequalities.** In this section we establish some new inequalities for the dimension of a poset.

Lemma 1. *Suppose x and y are incomparable points in a poset X, but for every $z \in X - \{x, y\}$, $z > x$ iff $z > y$ and $z < x$ iff $z < y$. Then* $\dim(X - x) = \dim X$ *unless $X - x$ is a chain.*

Proof. If $X - x$ is not a chain then $\dim X - x \geq 2$; let $L_1, L_2, \cdots, L_t$ be linear extensions of $P \mid X - x = P'$ whose intersection is P'. In $L_1, L_2, \cdots, L_{t-1}$ insert y immediately over x, and in L_t insert y immediately under x. The resulting linear extensions of P intersect to give P, and thus $\dim X \leq \dim X - x$. We note that if $X - x$ is a chain, then $\dim X - x = \dim X - y = 1$, but $\dim X = 2$.

A trivial modification of this argument also proves the following statement.

Lemma 2. *Suppose $x > y$ in P but for every $z \in X - \{x, y\}$, $z > x$ iff $z > y$ and $z < x$ iff $z < y$. Then* $\dim X = \dim X - x = \dim X - y$.

Lemma 3. *If $|X - A| = 2$, then* $\dim X = 2$.

Proof. We may assume without loss of generality that X cannot be reduced by either of the preceeding lemmas to a poset with the same dimension as X by having fewer number of points. Then it is easy to see that X is isomorphic to a subposet of one of the following posets.

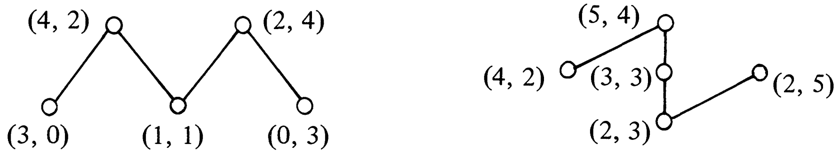

Figure 1

But the coordinatizations given in Figure 1 show that each of these has dimension 2.

Since the removal of a point cannot decrease the dimension more than one, we have proved the following result.

Theorem 2. *If* $|X - A| \geq 2$, *then* $\dim X \leq |X - A|$.

Combining this result with the easily obtained bound $\dim X \leq \text{width}\,(X)$, we have established Hiraguchi's theorem[1] that $\dim X \leq |X|/2$ when $|X| \geq 4$.

We also note that the standard examples of maximal dimensional posets, denoted S_n^0 [2], [8], show that the bounds $\dim X \leq \text{width}\,(X)$, $\dim X \leq |X - A|$ and $\dim X \leq |X|/2$ are best possible.

Theorem 3. $\dim X \leq \text{width}(X - E) + 1$.

Proof. Let $t = \text{width}(X - E)$; then by Dilworth's theorem [3], there is a partition $X - E = C_1 \cup C_2 \cup \cdots \cup C_t$, where each C_i is a chain. For each i, let L_i be a linear extension of P which is a lower extension [1] with respect to C_i. Form a linear extension L_{t+1} of X by placing all maximal elements on top of some linear extension M of $X - E$ and then ordering the maximal elements in L_{t+1} in the reverse order imposed on them by L_t. It is easy to see that $L_1 \cap L_2 \cap \cdots \cap L_{t+1} = P$, and the proof of our theorem is complete.

For $w = 1$ and $w = 2$, the following examples show that the bound is best possible.

Figure 2

For $n \geq 3$, we construct a poset Y_n as follows. Y_n has $3n + 2$ points $\{a_1, a_2, \cdots, a_n, a_{n+1}\} \cup \{y_1, y_2, \cdots, y_n\} \cup \{x_1, x_2, \cdots, x_n\} \cup \{p\}$. The points $\{a_i \mid i \leq n\}$, $\{y_i \mid i \leq n\}$ form a copy of S_n^0. Each y_i covers x_i; p covers $a_1, a_2, \cdots, a_n$ but $p \| a_{n+1}$; and a_{n+1} covers all x's. We illustrate this construction with the Hasse diagram for Y_3.

[1] K. P. Bogart first suggested that an elementary proof of Hiraguchi's theorem might be produced by considering the complement of the largest antichain. R. Kimble has independently discovered this result; his proof will appear in his thesis [6].

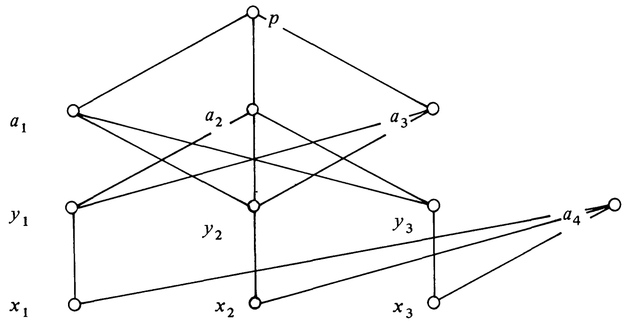

Figure 3

It is clear that if $E = \{p, a_{n+1}\}$, then $w(Y_n - E) = n$. We now show that $\dim Y_n = n + 1$.

Suppose $\dim Y_n \leq n$; let $L_1, L_2, \cdots, L_n$ be linear extensions of Y_n whose intersection is the partial ordering on Y_n. We may assume that the L's have been numbered so that x_i is over a_i in L_i. Now a_{n+1} is over all x's; since $y_i I a_{n+1}$ but $y_i < a_j$ for all $j \neq i$, $j \leq n$, y_i is under a_{n+1} in all lists except possibly L_i. Hence we must have y_i over a_{n+1} in L_i. Since $p > y_i$ for all i, this implies p is over a_{n+1} in every L_i. The contradiction shows that $\dim Y_n = n + 1$.

We note that it is straightforward to show that each Y_n is irreducible; i.e., the removal of any point from Y_n lowers the dimension to n. We refer the reader to [9] for details.

Theorem 4. $\dim X \leq 2 \operatorname{width}(X - A) + 1$.

Proof. Suppose $t = \operatorname{width}(X - A)$ and let $X - A = C_1 \cup C_2 \cup \cdots \cup C_t$ be a decomposition into chains. For each i, let L_{2i-1} and L_{2i} be upper and lower extensions, respectively, of C_i. Then let M be an ordering of A which is the reverse of ordering imposed on A by L_{2t}; then let L_{2t+1} be any linear extension of P whose restriction to A is M. Clearly $L_1 \cap L_2 \cap \cdots \cap L_{2t+1} = P$ and the proof of our theorem is complete.

To show that the inequality of Theorem 4 is best possible, we construct for each $n \geq 1$, $h \geq 1$ a poset $X(n, h)$ as follows. $X(n, h)$ contains a maximal antichain A, and $X(n, h) - A = X_U \cup X_L$ is the natural decomposition into upper and lower halves. X_U and X_L each consist of n incomparable chains with each chain containing h points. Every point in X_U is greater than every point in X_L. For each ordered pair (S, T) where S is an order ideal of $\hat{X}_U$ and T is an order ideal of X_L, there is a point in A which is less than all points in S and greater than all points in T. We illustrate this definition with the Hasse diagrams for $X(1, 2)$ and $X(2, 1)$.

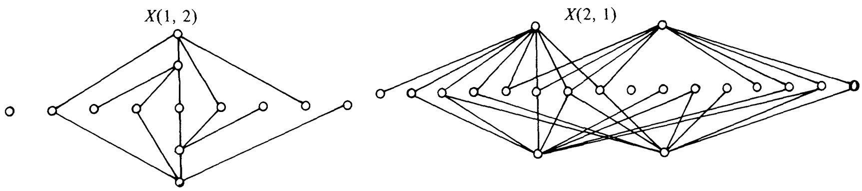

Figure 4

We note that the width of $X(n, h) - A$ is n. However, it can be shown [10] that for sufficiently large h, $\dim X(n, h) = 2n + 1$.

3. **Some open problems.** Although we have outlined in this paper an elementary proof of Hiraguchi's theorem: $\dim X \leq |X/2|$, it is not known whether or not every poset contains a pair of points whose removal lowers the dimension at most 1.

A second problem involves cartesian products. Although $\dim(X \times Y) \leq \dim X + \dim Y$, it is easy to construct posets X, Y for which $\dim(X \times Y) < \dim X + \dim Y$. (In fact $\dim(S_n^0 \times S_n^0) \leq 2n - 2$.) The question involves the accuracy of the lower bound $\dim(X \times Y) \geq \max\{\dim X, \dim Y\}$.

REFERENCES

1. K. Bogart, *Maximal dimensional partially ordered sets*. I, Discrete Math. 5 (1973), 21–32.

2. K. Bogart and W. Trotter, *Maximal dimensional partially ordered sets*. II, Discrete Math. 5 (1973), 33–44.

→ R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. (2) 51 (1950), 161–166. MR 11, 309.

→ B. Dushnik and E. Miller, *Partially ordered sets*, Amer. J. Math. 63 (1941) 600–610. MR 3, 73.

5. T. Hiraguchi, *On the dimension of orders*, Sci. Rep. Kanazawa Univ. 4 (1955), 1–20. MR 17, 1045; 19, 1431.

6. R. Kimble, Ph.D. Thesis, M.I.T., Cambridge, Mass.

7. O. Ore, *Theory of graphs*, Amer. Math., Soc. Colloq. Publ., vol. 38, Amer. Math. Soc., Providence, R.I., 1962. MR 27 #740.

8. W. T. Trotter, *Dimension of the crown* S_n^k, Discrete Math. 8 (1974), 85–103.

9. W. T. Trotter, *Some families of irreducible partially ordered sets*, U.S.C. Math. Tech. Rep. 06A10-2, 1974.

10. ———, *Irreducible posets with large height exist*, J. Combinatorial Theory Ser. A 17 (1974).

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