TILING BOUNDED OPEN SETS WITH SQUARES THAT TOUCH THE BOUNDARY[a]

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1. Introduction

In this paper, we discuss the problem of tiling a bounded open subset S of the Euclidean plane with a (possibly infinite) family of squares each of which touches the boundary of S. The sides of the squares are required to be parallel to the coordinate axes. With these restrictions, we show that if S is an open subset of $\mathbb{R}^2$ and S is either a homeomorph of the unit disk or a homeomorph of an annulus, then S can be tiled with a family of squares each of which touches the boundary of S. We construct an open subset S which is a homeomorph of a disk with two holes which cannot be tiled with a family of squares each of which touches the boundary of S. We then show that every bounded open subset $S \subseteq \mathbb{R}^2$ can be tiled with a family of rectangles each of which touches the boundary of S. We also consider the obvious generalization of these problems to $\mathbb{R}^n$ and show that for every $n \geq 3$, there exists a homeomorph S of the open unit ball in $\mathbb{R}^n$ which cannot be tiled by a family of cubes each of which touches the boundary of S. On the other hand, for $n \geq 2$ every bounded open subset S of $\mathbb{R}^n$ can be tiled with a family of boxes each of which touches the boundary of S.

We denote the interior, boundary, and closure of a subset S of n-dimensional Euclidean space $\mathbb{R}^n$ by Int (S), Bdry (S), and $\bar{S}$, respectively. An *n-box* in $\mathbb{R}^n$ is a subset R of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $b_i > a_i$ for $i = 1, 2, 3, \ldots, n$. An n-box is called an *n-cube* when $b_i - a_i = b_j - a_j$ for all i, j with $1 \leq i < j \leq n$. A 2-box is called a *rectangle* and a 2-cube is called a *square*. Note that the sides of an n-box are required to be parallel to the coordinate axes in $\mathbb{R}^n$.

A family $\mathscr{T}$ of sets is called a *partial tiling* of a subset $S \subseteq \mathbb{R}^n$ when it satisfies the following two properties:

P_1: $R \subseteq \bar{S}$ for every $R \in \mathscr{T}$.

P_2: Int $(R) \cap$ Int $(R') = \varnothing$ for every distinct $R, R' \in \mathscr{T}$.

A partial tiling $\mathscr{T}$ of a subset $S \subseteq \mathbb{R}^n$ is called a *tiling* of S when it also satisfies:

P_3: Int $(S) \subseteq \bigcup\{\bar{R}: R \in \mathscr{T}\}$.

In this paper, we will be concerned with the following problem:

If S is a bounded open subset of $\mathbb{R}^n$, under what conditions does there exist a tiling $\mathscr{T}$ of S so that every set in $\mathscr{T}$ is an n-box (alternately, an n-cube) that touches the boundary of S? More formally, we will require that the tiling also satisfy the following property:

P_4: $\bar{R} \cap$ Bdry $(S) \neq \varnothing$ for every $R \in \mathscr{T}$.

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Every open subset S of $\mathbb{R}^n$ can be tiled with a family of n-cubes, although it takes a little effort to show this. However, when we require that the tiling also satisfy property P_4, some rather surprising results are encountered.

2. Summary of Results

In this section, we summarize the principal results of the paper. We defer the proofs of these theorems, as well as the development of supporting lemmas, until Section 3. We begin with a positive result.

Theorem 1. *If $S \subseteq \mathbb{R}^2$ and S is homeomorphic to the open disk $\{(x, y) \in \mathbb{R}^2: x^2 + y^2 < 1\}$, then S can be tiled with a family of squares each of which touches the boundary of S.*

Theorem 1 is somewhat surprising since several intuitive approaches to defining an algorithm for producing a tiling fail when the boundary of S is irregular. In particular, the obvious scheme of choosing a square of maximum size in the region not already tiled is defective. To see that this is the case, consider the open set S bounded by the curve shown in Figure 1. The first three squares chosen under this scheme will have side lengths of 10, 8, and 6, respectively. However, these three squares will then leave a 3×2 rectangle that cannot be suitably tiled.

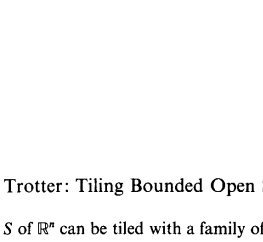

Figure 1

In view of Theorem 1, it is natural to expect that bounded open regions that are not simply connected may present difficulties. However, we will show that the argument given for Theorem 1 can be modified to settle the question in the affirmative when S is an annulus, i.e., an open disk with one hole.

Theorem 2. *If $S \subseteq \mathbb{R}^n$ and S is homeomorphic to the open disk with one hole $\{(x, y) \in \mathbb{R}^2: 1 < x^2 + y^2 < 2\}$, then S can be tiled with a family of squares each of which touches the boundary of S.*

A careful examination of the argument used to prove Theorem 2 would lead one to conjecture that regions with more than one hole may fail to have a tiling with squares each of which touches the boundary.

To show that the conjecture is true and to confirm that Theorem 2 is best possible, we establish the following result.

THEOREM 3. *There exists a homeomorph S of the open disk with two holes* $\{(x, y) \in \mathbb{R}^2\colon\ 1 < x^2 + y^2 < 5$ *and* $(x - 3)^2 + y^2 > 1\}$ *which cannot be tiled with squares each of which touches the boundary of S.*

In view of THEOREM 3, it is reasonable to conjecture that there exist more complicated regions which cannot be tiled with rectangles (each of which touches the boundary) either. In fact, it is not too difficult to construct a bounded open set S in the plane so that for every $\varepsilon > 0$, there exists a partial tiling $\mathscr{T}(\varepsilon)$ of S so that: (1) if R is a rectangle in $\mathscr{T}(\varepsilon)$ and R is properly contained in a rectangle R', then Int (R') is not a subset of S; (2) the sum of the areas of the rectangles in $\mathscr{T}(\varepsilon)$ is less than ε; and (3) Int $(S - \bigcup \mathscr{T}(\varepsilon)) = \varnothing$. In other words, we have covered an arbitrarily small portion of S but cannot augment the partial tiling $\mathscr{T}(\varepsilon)$ to complete the task of tiling S. Despite these apparent difficulties, we will establish the following result.

THEOREM 4. *If S is a bounded open subset of the plane, then S can be tiled with rectangles each of which touches the boundary of S.*

The proof of THEOREM 4 can be extended immediately to higher dimensions.

THEOREM 5. *If* $n \geq 2$ *and S is a bounded open subset of* $\mathbb{R}^n$, *then S can be tiled with n-boxes each of which touches the boundary of S.*

Although THEOREM 4 extends immediately to higher dimensions, we will show that THEOREMS 1 and 2 do not.

THEOREM 6. *If* $n \geq 3$, *there exists a homeomorph S of the open ball* $\{(x_1, x_2, \ldots, x_n)\colon x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$ *which cannot be tiled with n-cubes each of which touches the boundary of S.*

3. DETAILS OF THE ARGUMENTS

Let S be a bounded open subset of $\mathbb{R}^2$ and let $\mathscr{T}$ be a tiling of S with rectangles each of which touches the boundary of S. If $\mathscr{T}'$ is a finite subcollection of $\mathscr{T}$, then $\mathscr{T} - \mathscr{T}'$ is a tiling of the open set $S' = S - \bigcup \mathscr{T}'$ and every rectangle in $\mathscr{T} - \mathscr{T}'$ touches Bdry $(S') \cap$ Bdry (S). It is therefore natural to consider the following problem.

Let S be a bounded open subset of the plane and let Bdry $(S) = B_a(S) \cup B_r(S)$ be a partition of the boundary of S into two sets which we call the *artificial boundary* of S and the *real boundary* of S, respectively. Under what conditions can S be tiled with a family of squares each of which touches the real boundary of S? More formally, we require that the tiling satisfy the following variant of property P_4.

P_4': $\bar{R} \cap B_r(S) \neq \varnothing$ for every $R \in \mathscr{T}$.

EXAMPLE 7. Let S be the interior of the rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$, and let $B_r(S)$ consist of the points on the top and right sides of this rectangle, i.e., $B_r(S) = \{(x, d)\colon a \leq x \leq b\} \cup \{(b, y)\colon c \leq y \leq d\}$. Then S can be tiled with a family of squares each of which touches the real boundary of S.

Proof. Let $S_0 = S$. We then define S_k and Q_k inductively for every $k \geq 0$ by the following rule. If $S_k = (a, b) \times (c, d)$, let $m = \min\{b - a,\ d - c\}$, $Q_k = [a, a + m] \times [c, c + m]$, and $S_{k+1} = S_k - Q_k$. If this definition results in $S_{k+1} = \varnothing$ for some $k \geq 0$, then the finite collection $\mathscr{T} = \{Q_0, Q_1, \ldots, Q_k\}$ is a tiling of S by squares each of which touches the real boundary of S. On the other hand, if $S_{k+1} \neq \varnothing$ for all $k \geq 0$, then the collection $\mathscr{T} = \{Q_k\colon k \geq 0\}$ is the desired tiling. ▮

We have included this elementary example for two reasons. First, the algorithm used in the example will be an essential device in subsequent arguments, and second, we want to emphasize the fact that we allow infinite tilings. Tiling problems for finite families of squares and rectangles have been studied extensively, and we refer the reader to [2–4] for details.

In order to generalize Example 7, we consider an open set $S \subseteq \mathbb{R}^2$ whose artificial boundary forms a pair of perpendicular line segments which are parallel to the coordinate axes and form a right angle that is included in S.

We call S a *quarter-region* and refer to the point $p_0(S)$ where the perpendicular line segments meet as the *corner point* of S. We will require that the real boundary $B_r(S)$ of a quarter-region S be a closed connected subset of Bdry (S). We denote the endpoints of $B_r(S)$ by $p_0'(S)$ and $p_0''(S)$ and call the two line segments determined by $\{p_0(S), p_0'(S)\}$ and $\{p_0(S), p_0''(S)\}$ the *sides* of S. We will assume that these points have been labeled so that if we begin at $p_0'(S)$ and proceed clockwise around Bdry (S), then we will encounter the corner point $p_0(S)$ before $p_0''(S)$. (See FIGURE 2.)

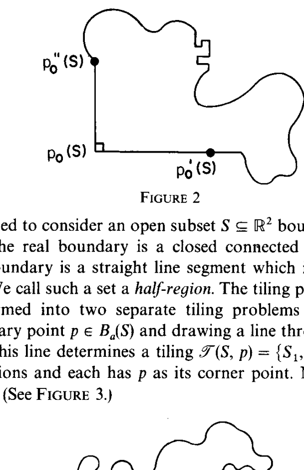

FIGURE 2

We will also need to consider an open subset $S \subseteq \mathbb{R}^2$ bounded by a simple closed curve for which the real boundary is a closed connected subset of Bdry (S) and whose artificial boundary is a straight line segment which is parallel to one of the coordinate axes. We call such a set a *half-region.* The tiling problem for a half-region S can be transformed into two separate tiling problems for quarter-regions by choosing an arbitrary point $p \in B_a(S)$ and drawing a line through p which is perpendicular to $B_a(S)$. This line determines a tiling $\mathscr{T}(S, p) = \{S_1, S_2\}$ of S, where S_1 and S_2 are quarter-regions and each has p as its corner point. Note that $B_r(S_1) \subseteq B_r(S)$ and $B_r(S_2) \subseteq B_r(S)$. (See FIGURE 3.)

FIGURE 3

Next we describe an algorithm for tiling a quarter-region. Let S_0 be an arbitrary quarter-region. Then there is a unique square $Q(S_0)$ of maximum size satisfying the following conditions: $Q(S_0) \subseteq \bar{S}_0$ and $p_0(S_0)$ is one of the corners of $Q(S_0)$. (See FIGURE 4.)

FIGURE 4

Note that $Q(S_0)$ touches the real boundary of S_0. We assume that $Q(S_0) \neq \bar{S}_0$ and observe that Bdry $(Q(S_0)) \cap S_0$ is a nonempty open subset of Bdry $Q(S_0)$ so that Bdry $(Q(S_0)) \cap S_0$ can be uniquely partitioned into a countable (and possibly infinite) collection of pairwise disjoint connected sets each of which is an open subset of Bdry $(Q(S_0))$. Let Bdry $(Q(S_0)) \cap S_0 = \bigcup\{U_\alpha: \alpha \in \mathscr{A}(S_0)\}$ be this partition.

We now describe a process for determining a tiling $\mathscr{F}(S_0)$ of $S - Q(S_0)$ so that each set $S \in \mathscr{F}(S_0)$ is a quarter-region and $B_r(S) \subseteq B_r(S_0)$. Beginning at the corner point $p_0(S_0)$ and proceeding clockwise around $Q(S_0)$, we label the sides s_1, s_2, s_3, and s_4; i.e., s_1 and s_4 meet at $p_0(S_0)$. For $i = 1, 2, 3$, label the corner where s_i and s_{i+1} meet as $p_i(S_0)$. We then observe that the partition Bdry $(Q(S_0)) \cap S_0 = \bigcup\{U_\alpha: \alpha \in \mathscr{A}(S_0)\}$ induces a natural tiling $\mathscr{U}(S_0) = \{S_\alpha: \alpha \in \mathscr{A}(S_0)\}$ of the open set $S_0 - Q(S_0)$ where each S_α is an open subset of S_0 whose artificial boundary contains U_α and whose real boundary is a subset of the real boundary of S_0. Note that each S_α is the interior of a simple closed curve. At this point, we must divide the construction of $\mathscr{F}(S_0)$ into two cases. (See FIGURE 5.)

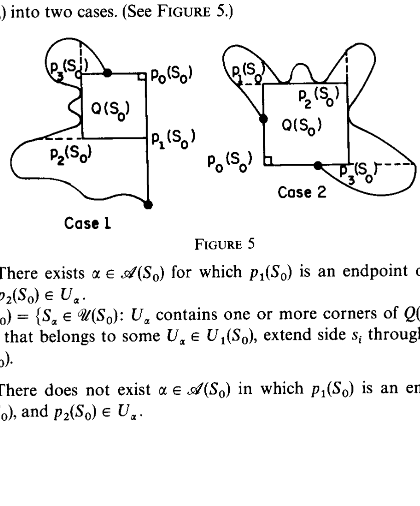

FIGURE 5

Case 1. There exists $\alpha \in \mathscr{A}(S_0)$ for which $p_1(S_0)$ is an endpoint of U_α, $p_1(S_0) \in B_a(S_0)$, and $p_2(S_0) \in U_\alpha$.

Let $\mathscr{U}_1(S_0) = \{S_\alpha \in \mathscr{U}(S_0): U_\alpha$ contains one or more corners of $Q(S_0)\}$. For each corner $p_i(S_0)$ that belongs to some $U_\alpha \in U_1(S_0)$, extend side s_i through $p_i(S_0)$ until it touches $B_r(S_0)$.

Case 2. There does not exist $\alpha \in \mathscr{A}(S_0)$ in which $p_1(S_0)$ is an endpoint of U_α, $p_1(S_0) \in B_a(S_0)$, and $p_2(S_0) \in U_\alpha$.

In this case, for each corner $p_i(S_0)$ that belongs to some $U_\alpha \in \mathscr{U}_1(S_0)$, we extend side s_{i+1} through $p_i(S_0)$ until it touches $B_r(S_0)$.

It is easy to see that this two-part definition determines a tiling $\mathscr{F}'(S_0)$ of $S_0 - Q(S_0)$ where each region in $\mathscr{F}'(S_0)$ is either a quarter-region or a half-region whose real boundary is a subset of $B_r(S_0)$.

Let $\mathscr{F}''(S_0) = \{S \in \mathscr{F}'(S_0): S$ is a half-region$\}$. Then for each $S \in \mathscr{F}'(S_0)$, choose a point $p(S) \in B_a(S) \cap$ Bdry $(Q(S_0))$ and define $\mathscr{F}(S_0) = (\mathscr{F}' - \mathscr{F}''(S_0)) \cup (\bigcup\{\mathscr{T}(p(S), S): S \in \mathscr{F}''(S_0)\})$. It is easy to see that $\mathscr{F}(S_0)$ is a tiling of $S_0 - Q(S_0)$ with quarter-regions and $B_r(S) \leq B_r(S_0)$ for every $S \in \mathscr{F}(S_0)$.

The procedure described above can now be repeated on each quarter-region in $\mathscr{F}(S_0)$. We set $\mathscr{G}_0(S_0) = \{S_0\}$ and $\mathscr{T}_0(S_0) = \{Q(S_0)\}$. If $\mathscr{G}_k(S_0)$ and $\mathscr{T}_k(S_k)$ have been defined for some $k \geq 0$, we define $\mathscr{G}_{k+1}(S_0) = \bigcup\{\mathscr{F}(S): S \in \mathscr{G}_k(S_0)\}$ and $\mathscr{T}_{k+1}(S_0) = \mathscr{T}_k(S_0) \cup \{Q(S): S \in \mathscr{G}_{k+1}(S_0)\}$. We then define $\mathscr{G}(S_0) = \bigcup\{\mathscr{G}_n(S_0): n \geq 0\}$ and $\mathscr{T}(S_0) = \bigcup\{\mathscr{T}_n(S_0): n \geq 0\}$.

Hereafter, we will refer to this recursive construction as *Algorithm* 1 and call $\mathscr{G}(S_0)$ and $\mathscr{T}(S_0)$, respectively, the family of quarter-regions and the family of squares obtained by applying Algorithm 1 to S_0. Strictly speaking, $G(S_0)$ and $\mathscr{T}(S_0)$ are not uniquely determined since each time that a half-region S is encountered in the construction, the choice of the point $p(S) \in B_a(S)$ is arbitrary. We will now proceed to show that regardless of the manner in which the half-regions are tiled by quarter-regions, the collection $\mathscr{T}(S_0)$ is a tiling of the quarter-region S_0 by squares, and every square in $\mathscr{T}(S_0)$ touches the real boundary of S_0.

We begin by observing that it is clear that $\mathscr{T}(S_0)$ satisfies properties P_1, P_2, and P'_4. In order to complete the proof of our claim, we need only show that $\mathscr{T}(S_0)$ satisfies property P_3; i.e., we must show that $S_0 \subseteq \bigcup\mathscr{T}(S_0)$. To accomplish this will require some preliminary work. For a point $x \in \mathbb{R}^2$ and a positive number ε we let $U(x, \varepsilon)$ denote the open ball with center at x and radius ε.

LEMMA 8. *Let S_0 be a quarter-region and let $\mathscr{T}(S_0)$ be the family of squares obtained by applying Algorithm 1 to S_0. Also let $\{Q_\alpha: \alpha \in \mathscr{A}\}$ be an infinite subfamily of $\mathscr{T}(S_0)$ and for each $\alpha \in \mathscr{A}$, let $p_\alpha \in Q_\alpha$. Then every limit point of $\{p_\alpha: \alpha \in \mathscr{A}\}$ is an element of $B_r(S_0)$.*

Proof. Let y_0 be a limit point of $\{p_\alpha: \alpha \in \mathscr{A}\}$. Then choose an infinite sequence $\{\alpha_n: n \geq 1\} \subseteq \mathscr{A}$ so that $\{p_{\alpha_n}: n \geq 1\}$ converges to y_0. Now let ε be an arbitrary positive number. We show that $U(y_0, \varepsilon) \cap B_r(S_0) \neq \varnothing$.

Since $\{p_{\alpha_n}: n \geq 1\}$ converges to y_0 and $\varepsilon/10 > 0$, there exists an integer n so that $p_{\alpha_n} \in U(y_0, \varepsilon/10)$ for all $n \geq n_1$. However, there are at most four squares in $\mathscr{T}(S_0)$ which intersect both $U(y_0, \varepsilon/10)$ and $\bar{S}_0 - U(y_0, \varepsilon)$. Thus there exists an integer n_2 so that $Q_{\alpha_n} \subseteq U(y_0, \varepsilon)$ for all $n \geq n_2$. Since each Q_α touches the real boundary of S_0, it follows that $U(y_0, \varepsilon) \cap B_r(S_0) \neq \varnothing$. Since $B_r(S_0)$ is closed and ε is arbitrary, we conclude that $y_0 \in B_r(S_0)$. And since y_0 was arbitrary, the proof is complete. ▮

LEMMA 9. *Let S_0 be a quarter-region and let $\mathscr{T}(S_0)$ be the family of squares obtained by applying Algorithm 1 to S_0. If $x_0 \in B_a(S_0)$, then there exists an integer $n \geq 0$ and a square $Q \in \mathscr{T}_n(S_0)$ so that $x_0 \in Q$.*

Proof. Suppose to the contrary that $x_0 \in B_a(S_0)$ but that $x_0 \notin \bigcup\mathscr{T}_n(S_0)$ for every $n \geq 0$. Since $p_0(S_0) \in Q(S_0) \in \mathscr{T}_0(S_0)$, we know that $x_0 \neq p_0(S_0)$ and $x_0 \notin Q(S_0)$. Now suppose that for some integer $k \geq 0$, we have $x_0 \in B_a(S_k)$ and $x_0 \notin Q(S_k)$. Then there is a unique $S_{k+1} \in \mathscr{F}(S_k)$ for which $x_0 \in B_a(S_{k+1})$. Hence $x_0 \notin Q(S_{k+1})$. This inductive definition determines an infinite sequence $\{S_n: n \geq 0\}$ of quarter-regions for which the sequence $\{p_0(Q(S_n)): n \geq 0\}$ converges to a point y_0 on the side of $B_a(S_0)$ that contains x_0. Furthermore, y_0 is between $p_0(S_0)$ and x_0 on this side. However, in view of LEMMA 8, we must also have $y_0 \in B_r(S_0)$ which is a contradiction. ▮

COROLLARY 10. *Let S_0 be a quarter-region and let $\mathscr{T}(S_0)$ be the collection of squares obtained by applying Algorithm 1 to S_0. If $x_0 \in B_a(S_0)$, then there exists a positive number ε and a subcollection $\mathscr{T}' \subseteq \mathscr{T}(S_0)$ so that $\mathscr{T}'$ contains at most two squares and $U(x_0, \varepsilon) \cap \bar{S}_0 \subseteq \bigcup \mathscr{T}'$.*

Proof. If $x = p_0(S_0)$, the result is trivial since we may choose $\mathscr{T}' = \{Q(S_0)\}$ and ε as the length of a side of the square $Q(S_0)$. Now suppose that $x_0 \neq p_0(S_0)$. Then it follows from the algorithm given in LEMMA 9 that there is a least integer $n \geq 0$ for which $x_0 \in \bigcup \mathscr{T}_n(S_0)$. Then let S_1 be the unique quarter-region in $\mathscr{G}_n(S_0)$ for which $x_0 \in B_a(S_1)$. Note that $x_0 \in Q(S_1)$ and that $x_0 \neq p_0(Q(S_1))$. If x_0 is not a corner of $Q(S_1)$, the conclusion follows immediately since we may take $\mathscr{T}' = \{Q(S_1)\}$ and ε as the minimum distance from x_0 to a corner of $Q(S_1)$. On the other hand, if x_0 is a corner of $Q(S_1)$, then there exists a unique quarter-region $S_2 \in \mathscr{F}(S_1)$ for which $x_0 = p_0(S_2)$. Then set $\mathscr{T}' = \{Q(S_1), Q(S_2)\}$ and ε as the minimum of the side lengths of these two squares. ▮

LEMMA 11. *Let S_0 be a quarter-region and let $\mathscr{T}(S_0)$ be the family of squares obtained by applying Algorithm 1 to S_0. If $x_0 \in S_0$, $S^* \in \mathscr{G}(S_0)$, and $x_0 \in B_a(S^*)$, then there exists $\varepsilon > 0$ so that $U(x_0, \varepsilon) \subseteq \bigcup \mathscr{T}(S_0)$.*

Proof. From the recursive definitions for $\mathscr{G}_n(S_0)$ and $\mathscr{T}_n(S_0)$, it follows that for each $n \geq 0$, the family $\mathscr{C}_n(S_0) = \mathscr{T}_n(S_0) \cup \{\bar{S}: S \in \mathscr{G}_{n+1}(S_0)\}$ is a tiling of S_0. For each $n \geq 0$, let $\mathscr{C}_n(S_0, x_0) = \{A \in \mathscr{C}_n(S_0): x_0 \in A\}$. If $n \geq 0$ and there exists $A \in \mathscr{C}_n(S_0, x_0)$ for which $x \in \text{Int}(A)$, then $|\mathscr{C}_n(S_0, x_0)| = 1$. On the other hand, if $n \geq 0$ and $x \in \text{Bdry}(A)$ for every $A \in \mathscr{C}_n(S_0, x_0)$, then it follows that $2 \leq |\mathscr{C}_n(S_0, x_0)| \leq 4$.

From the hypotheses we know that $x_0 \in B_a(S^*)$ so we may choose the unique integer n_0 for which $\bar{S}^* \in \mathscr{C}_{n_0}(S_0, x_0)$. If $A \in \mathscr{C}_{n_0}(S_0, x_0)$ and $A \in \mathscr{T}_{n_0}(S_0)$, set $\mathscr{T}_A = \{A\}$ and let ε be the length of a side of the square A. If $A \in \mathscr{C}_{n_0}(S_0 x_0)$ and $A = \bar{S}$ for some $S \in \mathscr{G}_{n+1}(S_0)$, then we know that $x_0 \in B_a(S)$. We may therefore choose a finite subcollection $\mathscr{T}_A \subseteq \mathscr{T}(S_0)$ and a positive number ε_A so that $U(x_0, \varepsilon_A) \cap \bar{S} \subseteq \bigcup \mathscr{T}_A$.

Finally we set $\varepsilon = \min\{\varepsilon_A: A \in \mathscr{C}_n'(S_0, x_0)\}$. Then

$$\begin{aligned} U(x_0, \varepsilon) &= \bigcup\{U(x_0, \varepsilon) \cap A: A \in \mathscr{C}_n'(S_0, x_0)\} \\ &\subseteq \bigcup\{U(x_0, \varepsilon_A) \cap A: A \in \mathscr{C}_n'(S_0, x_0)\} \\ &\subseteq \bigcup\{\bigcup \mathscr{T}_A: A \in \mathscr{C}_n'(S_0, x_0)\} \\ &\subseteq \bigcup \mathscr{T}(S_0). \quad ▮ \end{aligned}$$

Recall that a *path* from a point x to a point y in $\mathbb{R}^2$ is a continuous injection f: $[0, 1] \to \mathbb{R}^2$ so that $f(0) = x$ and $f(1) = y$.

LEMMA 12. *Let S_0 be a quarter-region and let $\mathscr{T}(S_0)$ be the family of squares obtained by applying Algorithm 1 to S_0. Then $B_a(S_0) \cup S_0 \subseteq \bigcup \mathscr{T}(S_0)$.*

Proof. Choose an arbitrary point $x_0 \in B_a(S_0) \cup S_0$. We show that $x_0 \in \bigcup \mathscr{T}(S_0)$. The result follows from LEMMA 9 if $x_0 \in B_a(S_0)$, so we may assume that $x_0 \in S_0$. Choose a path f: $[0, 1] \to \mathbb{R}^2$ from $p_0(S_0)$ to x_0 so that $f(t) \in S_0$ for all $t \in (0, 1]$.

Now suppose that $x_0 \notin \bigcup \mathscr{T}(S_0)$ and let $T = \{t \in [0, 1]: f([0, t]) \subseteq \bigcup \mathscr{T}(S_0)\}$ and let $t_0 = \text{l.u.b. } (T)$. Also, let $y_0 = f(t_0)$. Now suppose that $y_0 \in \bigcup \mathscr{T}(S_0)$. Then $y_0 \neq x_0$ and $t_0 < 1$. Choose a square $Q \in \mathscr{T}(S_0)$ for which $y_0 \in Q$. If $y_0 \in \text{Int}(Q)$, then there exists $\varepsilon_1 > 0$ so that $y_0 \in U(y_0, \varepsilon_1) \subseteq \text{Int}(Q) \subseteq \bigcup \mathscr{T}(S_0)$. On the other hand, if $y_0 \in \text{Bdry}(Q)$, then there exists $S \in G(S_0)$ so that $y_0 \in B_a(S)$. It follows from LEMMA 11 that there exists $\varepsilon_2 > 0$ so that $U(y_0, \varepsilon_2) \subseteq \bigcup \mathscr{T}(S_0)$. In either case, there exists an open set U so that $y_0 \in U \subseteq \bigcup \mathscr{T}(S_0)$. Since f is continuous, there exists $\varepsilon_3 > 0$ so that $t_0 + \varepsilon_3 \leq 1$ and $f([t_0, t_0 + \varepsilon_3]) \subseteq U \subseteq \bigcup \mathscr{T}(S_0)$. Thus $f([0, t_0 + \varepsilon_3]) \subseteq \bigcup \mathscr{T}(S_0)$ and $t_0 + \varepsilon_3 \in T$. This contradicts the assumption that $t_0 = \text{l.u.b. } (T)$. We may therefore assume that $y_0 \notin \bigcup \mathscr{T}(S_0)$. In this case, we note that $t_0 > 0$.

We will now show that $y_0 \in B_r(S_0)$. The argument is essentially the same as the proof of LEMMA 8.

We first show that y_0 is a limit point of $B = \{p_0(Q): Q \in \mathscr{T}(S_0)\}$. To the contrary, suppose that y_0 is not a limit point of B and choose $\varepsilon_1 > 0$ so that $U(y_0, \varepsilon_1) \cap B = \varnothing$ and $U(y_0, \varepsilon_1) \subseteq S_0$.

Since each square in $\mathscr{T}(S_0)$ touches the real boundary of S_0, it follows that $\mathscr{A} \cap (\bar{S}_0 - U(y_0, \varepsilon_1)) \neq \varnothing$ for every $Q \in \mathscr{T}(S_0)$. Now let $\mathscr{S} = \{Q \in \mathscr{T}(S_0): Q \cap U(y_0, \varepsilon_1/10) \neq \varnothing\}$. Then $|S| \leq 4$. Since $y_0 \notin \bigcup \mathscr{T}(S_0)$, we may then choose ε_2 so that $\varepsilon_2 < \varepsilon_1/10$ and $U(y_0, \varepsilon_2) \cap (\bigcup \mathscr{S}) = \varnothing$. Thus $U(y_0, \varepsilon_2) \cap (\bigcup \mathscr{T}(S_0)) = \varnothing$. Since f is continuous, there exists $\varepsilon_3 > 0$ so that $0 \leq t_0 - \varepsilon_3$ and $f((t_0 - \varepsilon_3, t_0]) \subseteq U(y_0, \varepsilon_2)$. This contradicts the assumption that $t_0 = \text{l.u.b. } (T)$.

We may thus conclude that y_0 is a limit point of B, and thus by LEMMA 8, $y_0 \in B_r(S_0)$. The contradiction completes the proof. ▮

COROLLARY 13. *Let S_0 be a half-region. Then S_0 can be tiled with a family of squares each of which touches the real boundary of S_0.*

Proof. Choose an arbitrary point $p \in B_a(S_0)$ and let $\{S_1, S_2\} = \mathscr{T}(S_0, p)$. Then $\mathscr{T} = \mathscr{T}(S_1) \cup \mathscr{T}(S_2)$ is the desired tiling. ▮

We are now ready to present the proof of THEOREM 1.

THEOREM 1. *Let S be a homeomorph of the open disk $\{(x, y) \in \mathbb{R}^2: x^2 + y^2 < 1\}$. Then S can be tiled with squares each of which touches the boundary of S.*

Proof. Choose an arbitrary point $p \in S$ and draw a horizontal line through p. This determines a tiling $\{S_1, S_2\}$ of S where each S_i is a half-region containing p on its artificial boundary. The desired tiling is then found by applying COROLLARY 13 to each of these half-regions. ▮

Before proceeding to the proof of THEOREM 2, we pause to develop some notation and conventions concerning lines and curves. When we speak of a line or line segment, it will be assumed that it is parallel to one of the coordinate axes. If x and y are distinct points on a line (or line segment) L, then we let $L(x, y)$ denote the set of points on L that are (strictly) between x and y. Note that $x, y \notin L(x, y)$ and that $L(x, y) = L(y, x)$. We then define $L[x, y] = L[y, x] = L(x, y) \cup \{x, y\}$, $L(x, x) = \varnothing$ and $L[x, x] = \{x\}$.

On the other hand, when we try to extend this notation to curves, it is necessary to specify the direction in which the curve is to be traversed. For distinct points x, y on a simple closed curve C in $\mathbb{R}^2$, we let $C(x, y)$ be the points on C that are between x and y when we traverse C from x to y in the clockwise direction. Note that $C(x, y) \neq C(y, x)$. We then define $C[x, y] = C(x, y) \cup \{x, y\}$, $C(x, x) = \varnothing$ and $C[x, x] = \{x\}$.

Let π_1 and π_2 denote the projection maps from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Now let x be a point on a horizontal line L. Then we refer to the sets $\{y \in L: \pi_1(y) \geq \pi_1(x)\}$ and $\{y \in L: \pi_1(y) \leq \pi_1(x)\}$ as *half-lines emanating from x*. Half-lines are defined similarly for vertical lines.

Now let S be an open subset of the plane, with the boundary of S being a simple closed curve C containing a closed line segment $L[x, y]$. Now consider the collection $\mathscr{L}$ of all half-lines that are perpendicular to $L[x, y]$ and emanate from a point in $L[x, y]$. There is a natural partition $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2$ of this collection into two subcollections depending on the direction taken by the line segments. For example, if $L[x, y]$ is vertical, let $\mathscr{L}$ denote the half-lines that emanate from a point in $L[x, y]$ and satisfy $\pi_1(z) \leq \pi_1(x)$ for every $z \in M$ and every $M \in \mathscr{L}_1$. For exactly one of these subcollections, the following statement holds: There exists $p \in L[x, y]$, $M \in \mathscr{L}_i$, and $\varepsilon > 0$ so that M emanates from p and $M \cap U(p, \varepsilon) \subseteq \bar{S}$. In this case, we say that the half-lines in $\mathscr{L}_i$ are *interior to S*.

We are now ready to establish THEOREM 2. The argument will involve the construction of a tiling $\mathscr{H}(S)$ of an annulus S by a collection of quarter-regions and half-regions so that $B_r(S') \leq \text{Bdry}(S)$ for every $S' \in \mathscr{H}(S)$. At a crucial stage of the argument, we will need to take advantage of the fact that when we tile a half-region S, we choose an arbitrary point $p \in B_r(S)$ and begin by setting $\{S_1, S_2\} = \mathscr{T}(S, p)$.

THEOREM 2. *Let S_0 be a homeomorph of the open disk with one hole $\{(x, y) \in \mathbb{R}^2$: $1 < x^2 + y^2 < 2\}$. Then S_0 can be tiled with squares each of which touches the boundary of S_0.*

Proof. Let C and C' denote the interior and exterior boundary curves of S_0; i.e., S_0 consists of those points in $\mathbb{R}^2$ that are outside the curve C and inside the curve C'. Let $t_0 = \text{l.u.b. } \{\pi_2(p): p \in C\}$. Since C is compact, there exists a nonempty compact subset $A \subseteq C$ so that $\pi_2(p) = t_0$ for every $p \in A$. Then let $t_1 = \text{l.u.b. } \{\pi_1(p): p \in A\}$. It follows that there exists a unique point $q_0 \in A$ so that $\pi_1(q_0) = t_1$.

Let L_0 be the horizontal line containing q_0. Then there exists a unique point $q_1 \in C'$ with $\pi_1(q_0) < \pi_1(q_1)$ for which $L_0(q_0, q_1) \subseteq S_0$. Similarly there exists a unique point $q'_0 \in L_0 \cap C$ so that $\pi_1(q'_0) = \text{g.l.b. } \{\pi_1(p): p \in A\}$, and a unique point $q'_1 \in L_0 \cap C'$ so that $\pi_1(q'_1) < \pi_1(q'_0)$ and $L_0(q'_0, q'_1) \subseteq S$. Note that $\pi_1(q'_1) < \pi_1(q'_0) \leq \pi_1(q_0) < \pi_1(q_1)$. Then let D_0 denote the half-region whose real boundary is $C'[q'_1, q_1]$ and whose artificial boundary is $L_0(q'_1, q_1)$. (See FIGURE 6.)

If $L_0(q'_0, q_0) \cap S_0 = \varnothing$, we set $\mathscr{E}_0(S_0) = \varnothing$. If $L_0(q'_0, q_0) \neq \varnothing$, we let $L_0(q'_0, q_0) \cap S_0 = \{U_\alpha: \alpha \in \mathscr{A}\}$ be the partition into connected open intervals. Then for each $\alpha \in \mathscr{A}$, we choose $q_\alpha, q'_\alpha \in L_0$ so that $U_\alpha = L_0(q_\alpha, q'_\alpha)$ where $\pi_1(q_\alpha) < \pi_1(q'_\alpha)$. Next, for each $\alpha \in \mathscr{A}$, we let E_α be the half-region whose artificial boundary is $L_0(q_\alpha, q'_\alpha)$ and whose real boundary is $C[q'_\alpha, q_\alpha]$. Then let $\mathscr{E}_0(S_0) = \{E_\alpha: \alpha \in \mathscr{A}\}$.

Let M_0 be the half-line that emanates from q'_1 and contains q'_0, q_0, and q_1. Then let $B_0 = M_0[q_0, q_1]$ and let S_1 be the open set whose artificial boundary is $M_0(q'_1, q'_0) \cup M_0(q_0, q_1)$ and whose real boundary is $C[q_0, q'_0] \cup C'[q_1, q'_1]$. Now consider the following recursive definition. For an integer $k \geq 0$, suppose we have a set of

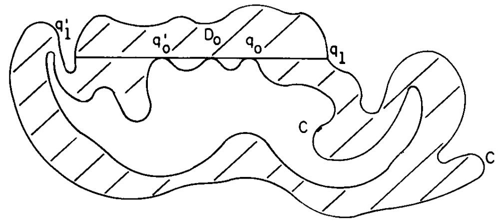

FIGURE 6

points $\{q_i: 0 \leq i \leq 2k + 1\}$, a set of half-lines $\{M_i: 0 \leq i \leq k\}$, a set of closed line segments $\{B_i: 0 \leq i \leq k\}$, a set of open sets $\{S_i: 0 < i \leq k + 1\}$, and a set of points $\{p_i: 0 \leq i \leq 2k + 1\}$ which satisfy the following properties:

1. If $0 \leq i \leq k$, then $q_{2i} \in C \cap M_k$ and $q_{2i+1} \in C' \cap M_k$.
2. If $0 \leq i \leq k$, then $B_i = M_i[q_{2i}, q_{2i+1}]$.
3. If $0 < i \leq k$, then $q_{2i} \in C[q_{2i-2}, q'_0]$ and $q_{2i+1} \in C'[q_{2i-1}, q'_1]$.
4. If $0 < i \leq k + 1$, S_i is the open subset of S_{i-1} whose artificial boundary is $M_0(q'_1, q'_0) \cup M_i(q_{2i}, q_{2i+1})$ and whose real boundary is $C[q_{2i}, q'_0] \cup C'[q_{2i+1}, q'_1]$.
5. If $0 < i \leq k$, then $p_{2i-2} \in B_{i-1}$.
6. If $0 < i \leq k$, then M_i is perpendicular to M_{i-1} and M_i emanates from p_{2i-2} into the interior of S_i.
7. If $0 < i \leq k$, then the component of $M_i \cap \bar{S}_i$ that contains p_{2i-2} is $M_i[p_{2i-2}, p_{2i-1}]$.
8. If $0 < i \leq k$, then $p_{2i-1} \in B_i \subseteq M_i[p_{2i-2}, p_{2i-1}]$.

We now describe a procedure which either will determine q_{2k+2}, q_{2k+3}, M_{k+1}, B_{k+1}, S_{k+2}, p_{2k}, and p_{2k+1} so that these eight properties still hold, or will tell us that the construction is to be halted.

For each point $p \in B_k$, let $M(p)$ be the half-line that is perpendicular to B_k and emanates from p into the interior of S_{k+1}. Then let $N(p)$ be the component of $M(p) \cap \bar{S}_{k+1}$ that contains p. It is easy to see that there exists a unique point $p_{2k} \in B_k$ which satisfies the following property:

$$N(p_{2k}) \cap C \neq \varnothing \neq N(p_{2k}) \cap C'.$$

Then let $M_{k+1} = M(p_{2k})$ and $N(p_{2k}) = M_{k+1}[p_{2k}, p_{2k+1}]$. If $p_{2k+1} \in C'$, we let $q_{2k+3} = p_{2k+1}$ and let q_{2k+2} be the unique point of $N(p_{2k}) \cap C$ which is closest to q_{2k+3}. Dually, if $p_{2k+1} \in C$, we let $q_{2k+2} = p_{2k+1}$ and let q_{2k+3} be the unique point of $N(p_{2k}) \cap C'$ which is closest to q_{2k+2}. Next, we let $B_{k+1} = M_{k+1}[q_{2k+2}, q_{2k+3}]$.

At this step in the construction, there are two cases.

Case 1. Either $k + 1 \leq 2$, or $k + 1 \geq 3$ and $B_{k+1} \cap M_0[q'_1, q'_0] = \varnothing$.

In this case, we note that $q_{2k+2} \in C[q_{2k}, q'_0]$ and $q_{2k+3} \in C'[q_{2k+1}, q'_1]$. We then define S_{k+2} as the open subset of S_{k+1} whose artificial boundary is $M_0(q'_1, q'_0) \cup M_{k+1}(q_{2k+2}, q_{2k+3})$ and whose real boundary is $C[q_{2k+2}, q'_0] \cup C'[q_{2k+3}, q'_1]$. At this point, each of the eight properties listed previously holds.

Case 2. $k + 1 \geq 3$ and $B_{k+1} \cap M_0[q'_1, q'_0] \neq \varnothing$.

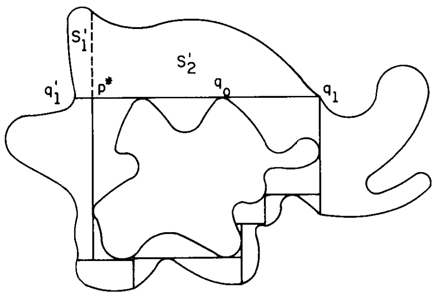

FIGURE 7

In this case, it is clear that B_{k+1} is a vertical line segment. Furthermore, for each i with $1 \leq i \leq k$, it is clear that there is a natural tiling $\mathscr{H}_i$ of $S_i - \bar{S}_{i+1}$ by a family of quarter-regions and half-regions so that $B_r(S) \subseteq \text{Bdry}(S_0)$ and $B_a(S) \subseteq B_{i-1} \cup M_i[p_{2i-2}, p_{2i-1}]$ for every $S \in \mathscr{H}_i$.

Now suppose that $B_{k+1} \cap M_0[q'_1, q'_0] = \{p^*\}$ and that $p^* \neq q'_1$. (See FIGURE 7.) Then let $\{S'_1, S'_2\} = \mathscr{T}(D_0, p^*)$ where $q'_1 \in \bar{S}'_1$. Then it follows that there is a natural tiling $\mathscr{H}_{k+1}$ of the open set $S'_1 \cup S_{k+1} \cup M_0(q'_1, p^*)$ where each set in $\mathscr{H}_{k+1}$ is either a quarter-region or half-region with $B_r(S) \subseteq \text{Bdry}(S_0)$ and $B_a(S) \subseteq B_k \cup M_{k+1}[p_{2k}, p_{2k+1}] \cup M_0(p^*, q'_0)$ for every $S \in \mathscr{H}_{k+1}$. However, we may then conclude that $\mathscr{H}(S_0) = \{S'_2\} \cup \mathscr{E}_0(S_0) \cup \mathscr{H}_1 \cup \mathscr{H}_2 \cup \cdots \cup \mathscr{H}_{k+1}$ is the tiling of S_0 by a collection of quarter-regions and half-regions so that $B_r(S) \subseteq \text{Bdry}(S_0)$ for every $S \in \mathscr{H}(S_0)$. The desired tiling of S_0 by squares each of which touches the boundary of S_0 then follows from LEMMA 12 and COROLLARY 13.

Now suppose that $B_{k+1} \cap M_0[q'_1, q'_0] = \{p^*\}$ but $p^* = q'_1$. Then there is a natural tiling $\mathscr{H}'_{k+1}$ of the open set S_{k+1} by a family of quarter-regions and half-regions so that $B_a(S) \subseteq B_k \cup M_{k+1}[p_{2k}, p_{2k+1}] \cup M_0[q'_1, q'_0]$ and $B_r(S) \subseteq \text{Bdry}(S_0)$ for every $S \in \mathscr{H}'_{k+1}$. We may then conclude that $\mathscr{H}'(S_0) = \{D_0\} \cup \mathscr{E}_0(S_0) \cup \mathscr{H}_1 \cup \mathscr{H}_2 \cup \cdots \cup \mathscr{H}_k \cup \mathscr{H}'_{k+1}$ is a tiling of S_0 by a family of quarter-regions and half-regions with $B_r(S) \subseteq \text{Bdry}(S_0)$ for every $S \in \mathscr{H}'(S_0)$.

We may summarize the argument presented thus far by stating that if the recursive definition ever halts because the criteria for Case 2 are satisfied, then we obtain a tiling of S_0 by a family of squares each of which touches the boundary of S_0. It remains only to consider the case where this recursive construction never terminates. If this occurs, then we obtain an infinite sequence of open subsets $\{S_n: n \geq 1\}$ and an infinite sequence of line segments $\{B_n: n \geq 1\}$ in which $B_n \subseteq \text{Bdry}(S_{n+1})$, $S_{n+1} \subseteq S_n$, and B_n is perpendicular to B_{n+1} for every $n \geq 1$. Now let d denote the usual metric in $\mathbb{R}^2$ and then let $t_2 = \text{g.l.b.}\ \{d(x, y): x \in C, y \in C'\} = d(C, C')$. Since C and C' are compact, $t_2 > 0$. Also let μ denote the usual measure function for subsets of $\mathbb{R}^2$. Then let $x_n = \{q_{2n}, q_{2n+1}\} - \{p_{2n-1}\}$ for every $n \geq 1$. It follows that $\mu(S_n - S_{n+1}) \geq (\pi/4)(t_2^2)$ for every $n \geq 1$. To see that this is true consider the open disk $U(x_n, t_2)$ and observe that one of the quadrants of this disk must be a subset of $S_n - S_{n+1}$.

However, we observe that it is not possible to have $S_{n+1} \subseteq S_n \subseteq S_0$ and $\mu(S_n - S_{n+1}) \geq (\pi/4)(t_2^2)$ for every $n \geq 1$ since S_0 is bounded. We therefore conclude that the recursive construction described previously must halt in a finite number of steps and with this observation, the proof of the theorem is complete. ▮

We now proceed immediately to show that the previous theorem cannot be improved.

THEOREM 3. *There exists a homeomorph S of the open disk with two holes $\{(x, y) \in \mathbb{R}^2: 1 < x^2 + y^2 < 5, (x-3)^2 + y^2 > 1\}$ which cannot be tiled with squares each of which touches the boundary of S.*

Proof. Let $M = 100$, $\varepsilon = 0.01$, and $\delta = 0.0001$. Then consider the open set S consisting of those points which belong to the interior of the large square Q_3 with corner points at $(\pm M, \pm M)$ and are in the exterior of the small squares Q_1 and Q_2 which have side length ε and have their centers respectively at $(-1/2 - \varepsilon/2, 0)$ and $(+1/2 + \varepsilon/2, 0)$.

We suppose that the open set S can be tiled with a family $\mathscr{T}$ of squares each of which touches the boundary of S and proceed to obtain a contradiction.

We refer to a closed line segment B as a *fault line* of $\mathscr{T}$ when there exists a finite subcollection $\mathscr{T}' \subseteq \mathscr{T}$ so that $B \subseteq \bigcup\{\text{Bdry}(Q): Q \in \mathscr{T}'\}$. We now establish the following property of $\mathscr{T}$ which we call the "Tunnel Property."

1. Let $R = [a, b] \times [c, d]$ be a rectangle with $R \subseteq S$. Let B_1 and B_2 be opposite sides of R and let d_1 be the distance from B_1 to B_2. Then let d_2 be the length of the other two sides of R. If B_1 and B_2 are fault lines of $\mathscr{T}$, then $d_2 \leq 2d_1$.

To see that this inequality must hold, we observe that if $d_2 > 2d_1$ and Q is any square in $\mathscr{T}$ that contains the midpoint $p = ((a+c)/2, (b+d)/2)$ of R, then $Q \subseteq R \subseteq S$ and $Q \cap \text{Bdry}(S) = \varnothing$.

We now show that $\mathscr{T}$ satisfies the following property which we call the "Tangent Property."

2. Let $B = L[x, y]$ be a fault line of $\mathscr{T}$ and let Q_i be one of the square holes in S. If $L[z, w]$ is a side of Q_i, $z \in L(x, w)$, and $w \in L(z, y)$, then either $d(x, z) < 5\varepsilon$ or $d(w, y) < 5\varepsilon$.

Before giving the argument for this property, we pause to emphasize that this property holds for each of the four sides of the square holes in S. Now suppose that the Tangent Property fails to hold, i.e., $d(x, z) \geq 5\varepsilon$ and $d(w, y) \geq 5\varepsilon$. If necessary, we rotate the coordinate system so that the configuration formed by B and Q_i is aligned as shown in FIGURE 8.

It follows from the Tunnel Property that there exist squares Q_4 and Q_5 in $\mathscr{T}$ so that (1) Q_4 and Q_5 are tangent to B; (2) Q_4 and Q_5 are above B; (3) Q_4 contains a point p_1 which is δ above B and δ to the right of x; and (4) Q_5 contains a point p_2 which is δ above B and δ to the left of y. Suppose first that $Q_4 \cap Q_i \neq \varnothing \neq Q_5 \cap Q_i$. Then it

FIGURE 8

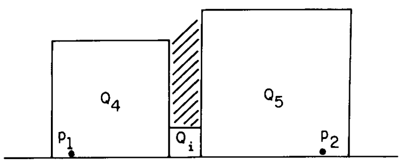

FIGURE 9

follows that Q_4 and Q_5 are tangent to Q_i and each has side length of at least $5\varepsilon - \delta$. However, it is clear that this implies that $\mathscr{T}$ violates the Tunnel Property between Q_4 and Q_5. (See FIGURE 9.)

Now suppose that neither Q_4 nor Q_5 touches Q_i. Then Q_4 and Q_5 are tangent to B, and each has side length of at least $1 - 5\varepsilon$. Again this implies that $\mathscr{T}$ violates the Tunnel Property between Q_4 and Q_5.

We may therefore assume without loss of generality that $Q_4 \cap Q_i \neq \varnothing = Q_5 \cap Q_i$. Then Q_4 has side length at least $5\varepsilon - \delta$ and Q_5 has side length at least $1 - 5\varepsilon$. (See FIGURE 10.)

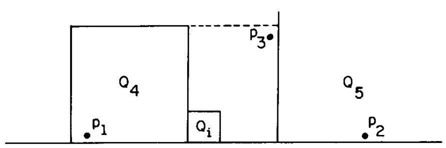

FIGURE 10

Let B_1 be the left boundary of Q_5 and consider the point p_3 which is δ to the left of B_1 and $5\varepsilon - 2\delta$ above B. If $p_3 \in Q_6 \in \mathscr{T}$, then Q_6 has side length at most 5ε and thus $Q_6 \cap Q_i \neq \varnothing$. Thus Q_6 has side length at least 4ε. If Q_6 is not tangent to Q_4, then $\mathscr{T}$ violates the Tunnel Property between Q_4 and Q_6. On the other hand, if Q_6 is not tangent to B, then $\mathscr{T}$ violates the Tunnel Property between Q_6 and B. Since Q_6 cannot be tangent to Q_4 and B, the argument for the Tangent Property is complete.

Next, we consider the square $Q_0 \in \mathscr{T}$ which contains the origin (0, 0). Q_0 must touch the boundary of one or both of the square holes in S. Thus Q_0 has side length of at least $\frac{1}{2}$ and at most 1. It follows from the Tangent Property that $\mathscr{T}$ has a horizontal fault line B_0 (found by either the top or the bottom of Q_0) which is between the square holes and at least $\frac{1}{2} - 6\varepsilon$ away from the origin. Without loss of generality we assume that B_0 is beneath the origin. Now consider the point p which is δ beneath the center of B and let $p \in Q \in \mathscr{T}$. Then it follows that Q does not touch Q_1 or Q_2 so Q must touch Q_3 and hence has side length at least $M - 1$. Let B_1 denote the top side of Q. We may assume without loss of generality that B_1 extends at least $M - 2$ to the right of Q_2. Let $d_1 = d(Q_2, B_1)$. Then $\frac{1}{2} - 6\varepsilon < d_1 < 1$. (See FIGURE 11.)

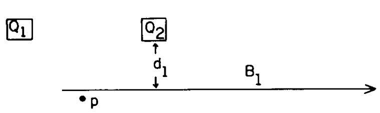

FIGURE 11

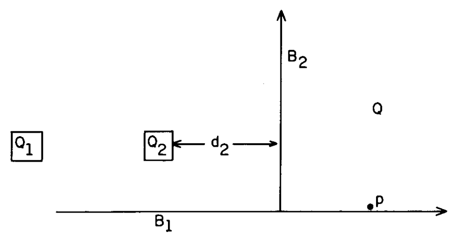

FIGURE 12

Now consider a point p which is δ above B_1 and 2 to the right of Q_2. Choose $Q \in \mathscr{T}$ so that $p \in Q$. Then Q is tangent to B_1. If Q touches Q_2, then $\mathscr{T}$ violates the Tangent Property. Therefore Q touches Q_3 and has side length at least $M - 3$. The left side of Q forms a fault line B_2 which is perpendicular to B_1 and extends at least $M - 2$ above Q_2. Let $d_2 = d(Q_2, B_2)$; then $0 < d_2 \leq 2$. (See FIGURE 12.)

Now consider a point p to the left of B_2 with $d(p, B_2) = \delta$ and $d(p, B_1) = 5$ and choose a square $Q \in \mathscr{T}$ with $p \in Q$. If Q touches Q_2 then it also touches Q_1 and violates the Tangent Property on the top of Q_1. Therefore Q is tangent to B_2, touches Q_3, and has side length at least $M - 5$. Let B_3 be the bottom edge of Q and let $d_3 = d(Q_1, B_3)$. Then $0 < d_3 \leq 5$.

Now consider a point p under B_3 with $d(p, B_3) = \delta$ and $d(p, B_2) = 15$ and choose a square $Q \in \mathscr{T}$ with $p \in Q$. It is clear that Q is tangent to B_2.

At this point, we consider two cases.

Case 1. $Q \cap Q_1 \neq \varnothing$.

We let B_4 be the right side of Q and $d_4 = d(B_4, Q_1) = 0$. Then let p' be a point to the right of B_4 with $d(p', B_4) = \delta$ and $d(p', B_3) = 8$. Choose a square $Q' \in \mathscr{T}$ with $p' \in Q'$; then Q' is tangent to B_4 and Q' touches Q_3. Furthermore, the top side of Q is a fault line which contains B_0.

Case 2. $Q \cap Q_1 = \varnothing$.

We let B_4 be the right side of Q and $d_4 = d(B_4, Q_1)$. Then $0 < d_4 \leq 14$. Let B_4 be the right side of Q_4 and choose a point p' to the right of B_4 with $d(p', B_4) = \delta$ and $d(p', B_3) = 25$. Then choose a square $Q' \in \mathscr{T}$ with $p' \in Q'$. It follows that Q' is tangent to B_4 and that Q' touches Q_3. Furthermore, the top side of Q is a fault line which contains B_0.

It follows that in either case we have determined a rectangle R which contains Q_1 and Q_2 and whose four sides are all fault lines. We denote the bottom, right, top, and left sides of R by T_1, T_2, T_3, and T_4, respectively. Then we may weaken the inequalities obtained previously to the following set:

$$\tfrac{1}{3} < d_1 = d(Q_1, T_1) = d(Q_2, T_1) \leq 1,$$
$$0 \leq d_2 = d(Q_2, T_2) \leq 30,$$
$$0 \leq d_3 = d(Q_1, T_3) = d(Q_2, T_3) \leq 30,$$
$$0 \leq d_4 = d(Q_1, T_4) \leq 30.$$

Now we recall that we chose $Q_0 \in \mathscr{T}$ so that $(0, 0) \in Q_0$. Furthermore, the bottom side of Q_0 is contained in T_1. In view of the symmetric form of the weakened inequalities given above, we may assume without loss of generality that Q_0 touches Q_2. Now consider a point p_4 to the right of Q_0 and above T_1 with $d(p_4, Q_0) = d(p_4, T_1) = \delta$. Then choose a square $Q_4 \in \mathscr{T}$ with $p_4 \in Q$. Since $Q_4 \subseteq R$, Q_4 must be tangent to Q_2 and thus the side length of Q_4 is d_1.

Now suppose that the right side of Q_4 is not contained in T_2. Then consider a point p_5 which is above T_1 and between Q_4 and T_2 with $d(p_5, T_1) = \delta$ and $d(p_5, Q_4) = d(p_5, T_2)$. It follows easily that no square in $\mathscr{T}$ that contains p_5 can touch the boundary of S. The contradiction shows that the right side of Q_4 is contained in T_2, and thus $d_2 = d_1 - \varepsilon$.

Now consider the point p_6 which is above Q_4 and to the left of T_2 with $d(p_6, Q_4) = d(p_6, T_2) = \delta$. Then choose a square $Q_6 \in \mathscr{T}$ with $p_6 \in Q_6$. Then it follows that Q_6 is tangent to Q_2 and has side length $d_1 - \varepsilon$. As before, it follows that the top side of Q_6 is contained in T_3 and thus $d_3 = d_1 - 2\varepsilon$.

Next consider the point p_7 which is below T_3 and to the left of Q_6 with $d(p_7, T_3) = d(p_7, Q_2) = \delta$. Choose a square $Q_7 \in \mathscr{T}$. Then Q_7 is tangent to Q_2 and has side length $d_1 - 2\varepsilon$. In view of the Tunnel Property, we may also conclude that Q_0 has side length $d_1 + \varepsilon$. Since $d_1 + \varepsilon < 1$, we know that $d_1 - 2\varepsilon < 1$ so that Q_7 does not touch Q_1. Next consider the point p_8 which is to the left of Q_7 and under T_3 with $d(p_8, Q_7) = d(p_8, T_3) = \delta$ and choose $Q_8 \in \mathscr{T}$ with $p_8 \in Q_8$. Then Q_8 must touch Q_1 and thus Q_8 has side length $d_1 - 2\varepsilon$. The requirement that Q_8 touch Q_1 implies that $2(d_1 - 2\varepsilon) \geq 1 + \varepsilon$ and thus $d_1 \geq 1/2 + 5\varepsilon/2 > 1/2$. However, in order that Q_8 and Q_1 not violate the Tangent Property, we must have $2(d_1 - 2\varepsilon) \leq \varepsilon + 1 + \varepsilon + 5\varepsilon = 1 + 8\varepsilon$ and thus $d_1 \leq 1/2 + 6\varepsilon$. (Note that if $d_1 > 1/2 + 6\varepsilon$, then the fault line formed by the bottom sides of Q_7 and Q_8 is tangent to Q_1 and extends 5ε to either side.)

Now consider the point p_9 which is above T_1 and to the left of Q_0 with $d(p_9, T_1) = d(p_9, Q_0) = \delta$ and choose $Q_9 \in \mathscr{T}$ with $p_9 \in Q_9$. It follows that Q_9 touches Q_1 and since Q_0 has side length $d_1 > \frac{1}{2}$, we conclude that Q_9 has side length $d_1 - \varepsilon$.

However, this is a contradiction since it implies that the bottom side of Q_8 and the top side of Q_9 violate the Tunnel Property since each extends at least $\frac{1}{2} - 7\varepsilon$ to the right of Q_1. With this observation, the proof of the theorem is complete. ∎

We proceed immediately to the proof of THEOREM 6.

THEOREM 6. *For every $n \geq 3$, there exists a homeomorph S of the open ball $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$ which cannot be tiled with n-cubes each of which touches the boundary of S.*

Proof. We first establish the result when $n = 3$. The extension to the case $n > 3$ will follow easily. Let S_0 denote the open subset of $\mathbb{R}^2$ constructed in the proof of THEOREM 3. Then let $P = 1000 = 10M$.

Next let A be the closed subset of $\mathbb{R}^3$ defined by $A = (\bar{S}_0 \times [0, P]) \cup ([-M, M] \times [-M, M] \times [P, P + 1])$. We may view A as being obtained from the rectangular solid (3-box) $R = [-M, M] \times [-M, M] \times [0, P + 1]$ by drilling two square holes into the square face of R to a depth of P. Since the holes do not go all the way through R, it follows that A is homeomorphic to the closed ball $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$. Thus the open set $S_1 = \text{Int}(A)$ is homeomorphic to the open ball in $\mathbb{R}^3$. (See FIGURE 13.)

FIGURE 13

Now let $I = (P/2, 1 + P/2)$ and for every $z \in I$, let $p(z)$ denote the plane $x_3 = z$; i.e., $p(z)$ is parallel to the plane determined by the x_1-axis and x_2-axis and passes through the point $(x_1, x_2, x_3) = (0, 0, z)$. It follows that $p(z) \cap S_1$ is a copy of S_0 for every $z \in I$.

Now suppose that $\mathscr{T}$ is a tiling of S_1 by cubes so that each cube in $\mathscr{T}$ touches the boundary of S_1. Then $\mathscr{T}$ is countable so we may choose a point $z_0 \in I$ so that no cube in $\mathscr{T}$ has a face contained in the plane $p(z_0)$. Now let $\mathscr{T}(z_0)$ be the subcollection of all cubes in $\mathscr{T}$ that intersect $p(z_0)$. Then $\mathscr{T}' = p(z_0) \cap Q : Q \in \mathscr{T}(z_0)$ is a tiling of $p(z_0) \cap S_1$ by squares. However, since $P/2 < z_0 < P/2 + 1$ and $P/2 > 2M$, no cube in $\mathscr{T}(z_0)$ can have a face in either of the planes $x_3 = 0$ or $x_3 = P + 1$. It follows that each square in $\mathscr{T}'$ touches the boundary of $p(z_0) \cap S_1$ which contradicts THEOREM 3.

As promised, the extension to the case $n > 3$ is immediate since we need only consider the Cartesian product $S_1 \times (0, P + 1) \times (0, P + 1) \times \cdots \times (0, P + 1)$ of S_1 with $n - 3$ copies of $(0, P + 1)$. ∎

The reader may conjecture that convex sets in $\mathbb{R}^n$ can be tiled with n-cubes but this is not true. Although we do not include the details here, the reader may verify that the convex set S which is the interior of the convex hull determined by $\{(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1)\}$ in $\mathbb{R}^3$ cannot be tiled with cubes each of which touches the boundary of S.

Before proving THEOREM 4, we pause to introduce some notation and terminology. We will refer to a rectangle $R = [a, b] \times [c, d]$ as a *strip* and call a strip $R' = [a, b] \times [e, f]$ a *substrip* of R when $c \leq e < f \leq d$.

We then let $\mathscr{S}(R, S)$ denote the set of all substrips of R which satisfy the requirement that $\text{Int}(R) \subseteq S$. Partial order $\mathscr{S}(R, S)$ by set inclusion and let $\mathscr{M}(R, S)$ denote the set of all maximal elements of $\mathscr{S}(R, S)$. Observe that every strip in $\mathscr{M}(R, S)$ touches the boundary of S except possibly for R itself. However, we can ensure that every strip in $\mathscr{M}(R, S)$ touches the boundary of S by further requiring that Int (R) contain a point which does not belong to S. Next, we let $\mathscr{H}_1(R, S)$ consist of those strips in $\mathscr{M}(R, S)$ which satisfy the additional condition that their height is at least as large as their width. Note that $\mathscr{H}_1(R, S)$ is always finite.

Since $\mathscr{H}_1(R, S)$ is finite it follows that there is a finite collection $\mathscr{H}_2(R, S)$ of substrips of R which forms a tiling of minimum cardinality of $R - \bigcup \mathscr{H}_1(R, S)$. Finally, in a strip $R = [a, b] \times [c, d]$, we let $l(R) = [a, (a + b)/2] \times [c, d]$ and $r(R) = [(a + b)/2, b] \times [c, d]$. We consider $l(R)$ and $r(R)$ as the left half and right half of R, respectively.

THEOREM 4. *If S is a bounded open subset of $\mathbb{R}^2$, then S can be tiled with a family of rectangles each of which touches the boundary of S.*

Proof. Without loss of generality, we may assume that $S \subseteq (0, 1) \times (0, 1)$. Then let $R_0 = [0, 2] \times [0, 2]$, $\mathscr{F}_0 = \varnothing$, and $\mathscr{G}_0 = \{R_0\}$. Then suppose that for some k with $k \geq 0$, we have determined $\mathscr{F}_k$ and $\mathscr{G}_k$ so that:

1. $\mathscr{F}_k \cup \mathscr{G}_k$ is a tiling of R_0 by a finite collection of rectangles.
2. $\text{Int}(R) \subseteq S$ and $R \cap \text{Bdry}(S) \neq \varnothing$ for every $R \in \mathscr{F}_k$.
3. $\text{Int}(R) \nsubseteq S$, for every $R \in \mathscr{G}_k$.
4. $\text{Width}(\mathscr{R}) \leq 2^{1-k}$ for every $R \in \mathscr{G}_k$.
5. If $0 < i \leq k$, then $\mathscr{F}_{i-1} \subseteq \mathscr{F}_i$.

We observe that each of these five properties is satisfied when $k = 0$. Note that R_0 was chosen sufficiently large to ensure that the third property would hold. We now describe a procedure for determining $\mathscr{F}_{k+1}$ and $\mathscr{G}_{k+1}$.

Let $R' \in \mathscr{G}_k$ and consider the collections $\mathscr{H}_1(R', S)$ and $\mathscr{H}_2(R', S)$. Since $\text{Int}(R') \nsubseteq S$, we know that $R' \notin \mathscr{H}_1(R', S)$ and therefore, every strip in $\mathscr{H}_1(R', S)$ touches the boundary of S.

Then let $\mathscr{H}_3(R', S) = \{R'' \in \mathscr{H}_2(R', S)\colon \text{Int}(R'') \subseteq S\}$ and $\mathscr{H}_4(R', S) = \mathscr{H}_2(R', S) - \mathscr{H}_3(R', S)$. Note that each strip in $\mathscr{H}_3(R', S)$ touches the boundary of S, and that $\text{Int}(R) \nsubseteq S$ for every $R \in \mathscr{H}_4(R', S)$.

Now let $\mathscr{H}_4(R', S) = \mathscr{H}_5(R', S) \cup \mathscr{H}_6(R', S) \cup \mathscr{H}_7(R', S)$ be the partition defined by $\mathscr{H}_5(R', S) = \{R'' \in \mathscr{H}_4(R', S)\colon \text{Int}(l(R'')) \subseteq S\}$, $\mathscr{H}_6(R', S) = \{R'' \in \mathscr{H}_4(R', S) - \mathscr{H}_5(R', S)\colon \text{Int}(r(R'')) \subseteq S\}$, and $\mathscr{H}_7(R', S) = \mathscr{H}_4(R', S) - \mathscr{H}_5(R', S) - \mathscr{H}_6(R', S)$.

Next, let $R'' = [a, b] \times [c, d] \in \mathscr{H}_5(R', S)$ and then define $f(R'') = \text{l.u.b.}\ \{t\colon \text{Int}([a, t] \times [c, d]) \subseteq S\}$. Then $(a + b)/2 \leq f(R'') < b$. Then let $\mathscr{H}_8(R', S) = \{[a, f(R'')] \times [c, d]\colon R'' = [a, b] \times [c, d] \in \mathscr{H}_5(R', S)\}$. Note that each strip in $\mathscr{H}_8(R', S)$ touches the boundary of S on its right side.

Next define $\mathscr{H}_9(R', S) = \{[f(R''), b] \times [c, d]\colon R'' = [a, b] \times [c, d] \in \mathscr{H}_5(R', S)\}$ and let $\mathscr{H}_{10}(R', S) = \{R'' \in \mathscr{H}_9(R', S)\colon \text{Int}(R'') \subseteq S\}$. Note that each strip in $\mathscr{H}_{10}(R', S)$ touches the boundary of S on its left side. Then let $\mathscr{H}_{11}(R', S) = \mathscr{H}_9(R', S) - \mathscr{H}_{10}(R', S)$. Note that $\text{Int}(R) \nsubseteq S$ for every $R \in \mathscr{H}_{11}(R', S)$.

Next, let $R'' = [a, b] \times [c, d] \in \mathscr{H}_6(R', S)$ and define $g(R'') = \text{g.l.b.}\ \{t\colon \text{Int}([t, b] \times [c, d]) \subseteq S\}$. Then $a < t \leq (a + b)/2$. Then define $\mathscr{H}_{12}(R', S) = \{[g(R''), b] \times [c, d]\colon r'' = [a, b] \times [c, d] \in \mathscr{H}_6(R', S)\}$ and note that each strip in $\mathscr{H}_{12}(R', S)$ touches the boundary of S on its left side. Then let $\mathscr{H}_{13}(R', S) = \{[a, g(R'')] \times [c, d]\colon R' = [a, b] \times [c, d] \in \mathscr{H}_6(R', S)\}$ and $\mathscr{H}_{14}(R', S) = \{R'' \in \mathscr{H}_{13}(R', S)\colon \text{Int}(R'') \subseteq S\}$. Note that each strip in $\mathscr{H}_{14}(R', S)$ touches the boundary of S on its right side. Then let $\mathscr{H}_{15}(R', D) = \mathscr{H}_{13}(R', S) - \mathscr{H}_{14}(R', S)$. Note that $\text{Int}(R) \nsubseteq S$ for every $R \in \mathscr{H}_{15}(R', S)$.

Next, let $\mathscr{H}_{16}(R', S) = \{l(R'')\colon R'' \in \mathscr{H}_7(R', S)\} \cup \{r(R')\colon R'' \in \mathscr{H}_7(R', S)\}$. Note that $\text{Int}(R) \nsubseteq S$ for every $R \in \mathscr{H}_{16}(R', S)$. Now let $\mathscr{H}_{17}(R', S) = \mathscr{H}_1(R', S) \cup \mathscr{H}_3(R', S) \cup \mathscr{H}_8(R', S) \cup \mathscr{H}_{10}(R', S) \cup \mathscr{H}_{12}(R', S) \cup \mathscr{H}_{14}(R', S)$. Then it follows that $\mathscr{H}_{17}(R', S)$ is a finite collection of rectangles so that $\text{Int}(R) \subseteq S$ and $R \cap \text{Bdry}(S) \neq \varnothing$ for every $R \in \mathscr{H}_{17}(R', S)$. Next, let $\mathscr{H}_{18}(R', S) = \mathscr{H}_{11}(R', S) \cup \mathscr{H}_{15}(R', S) \cup \mathscr{H}_{16}(R', S)$. Note that $\mathscr{H}_{18}(R', S)$ is finite, and $\text{Int}(R) \nsubseteq S$ and $\text{width}(R) \leq \text{width}(R')/2 \leq 2^{1-k}/2 = 2^{1-(k+1)}$ for every $R \in \mathscr{H}_{18}(R', S)$. Furthermore, note that $\mathscr{H}_{17}(R', S) \cup \mathscr{H}_{18}(R', S)$ is a tiling of the rectangle R'.

Finally, we set $\mathscr{F}_{k+1} = \mathscr{F}_k \cup (\bigcup\{\mathscr{H}_{17}(R', S)\colon R' \in \mathscr{G}_k\})$ and $\mathscr{G}_{k+1} = \bigcup\{\mathscr{H}_{18}(R', S)\colon R' \in \mathscr{G}_k\}$. We then define $\mathscr{F} = \bigcup_{k=0}^{\infty} \mathscr{F}_k$. In the same manner as used in the proof of THEOREM 1, we refer to this recursive procedure as Algorithm 2 and refer to $\mathscr{F}$ as the family of rectangles obtained by applying Algorithm 2 to S. As was the case in the

proof of THEOREM 1, we have an algorithm which produces a family of rectangles which is easily seen to satisfy P_1, P_2, and P_4. The question is whether $\mathscr{F}$ also satisfies P_3; i.e., Is it true that $S \subseteq \bigcup\mathscr{F}$? We now proceed to show that the answer is yes.

Choose an arbitrary point $x_0 \in S$. Since S is open, there exists $\varepsilon > 0$ so that the square $Q = [\pi_1(x_0) - \varepsilon,\ \pi_1(x_0) + \varepsilon] \times [\pi_2(x_0) - \varepsilon,\ \pi_2(x_0) + \varepsilon] \subseteq S$. Now choose $k \geq 0$ so that $2^{1-k} \leq \varepsilon$. Since $\mathscr{F}_k \cup \mathscr{G}_k$ is a tiling of R_0 and $x_0 \in R_0$, there exists a strip $R' \in \mathscr{G}_k$ with $x_0 \in R'$. Since $\text{Int}(R') \nsubseteq S$, it follows that $\text{Int}(R') \nsubseteq Q$. Thus the strip R' contains the center x_0 of the square Q and must either extend out the top of Q or out the bottom of Q. However, this implies that there exists a substrip $R'' \in \mathscr{H}_1(R', S)$ so that $x_0 \in R''$. Since $\mathscr{H}_1(R'', S) \subseteq \mathscr{F}$, we conclude that $x_0 \in \bigcup\mathscr{F}$. Since x_0 is arbitrary, we conclude that $S \subseteq \bigcup\mathscr{F}$ and the proof of the theorem is complete. ∎

In the interest of brevity, we do not include the details of the proof of THEOREM 5 since it is a straightforward extension of the previous result.

4. CONCLUDING REMARKS

The problems discussed in this paper were proposed by Bennett and Sharpley [1] who tried to extend the elementary result that every bounded open subset S of $\mathbb{R}^1$ can be tiled by closed intervals each of which touches the boundary of S. It is interesting to note that the theorems in this paper were obtained in the following order. The first result established was the result for rectangles—THEOREM 4. THEOREM 5 came as an immediate corollary. The next result was THEOREM 1 which was extended to the case where S is a simply connected open set, but we do not include the argument for the result here. The next result was to produce an open set in the plane which could not be suitably tiled with squares. The first counterexample had a "large—but finite" number of holes. Later, an example with six holes was found, and then one with four. Bernhard Ganter and Bill Sands (as communicated by George McNulty) produced an example with two holes; the proof given here for THEOREM 3 is a simplification of their argument. Only after this result was obtained was the problem involving the annulus investigated. Of course, the negative result for tiling homeomorphs of the unit ball in $\mathbb{R}^n$ with n-cubes was obtained concurrently with the first counterexample for squares.

It would be of interest to investigate the following tiling problems:

1. For $n \geq 3$, which bounded open sets can be tiled with n-cubes each of which touches the boundary?
2. For $n \geq 2$, which bounded open sets are tiled by squares each of which touches the boundary by applying the "Greedy Algorithm" of always choosing a square of maximum size in the region not already tiled? The reader may note that this algorithm fails for certain convex sets in $\mathbb{R}^2$.

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Part VI. Computational Aspects

REMARKS ON THE SPHERE OF INFLUENCE GRAPH

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1. Introduction

Let S be a finite set of points in the plane. For each point $x \in S$, let r_x be the closest distance to any other point in the set, and let C_x be the circle of radius r_x centered at x. Toussaint has defined the sphere of influence graph as a graph on S with an edge between points x and y if and only if the circles C_x and C_y intersect in at least two places. It is shown that:

(i) The sphere of influence graph has at most $29n$ edges ($n = |S|$).

(ii) Every decision tree algorithm for computing the sphere of influence graph requires at least $\Omega(n \log n)$ steps in the worst case.

As an application of (i), ElGindy observed that an algorithm of Bentley and Ottman can be used to find the sphere of influence graph in $O(n \log n)$ time.

Several recent papers in the pattern recognition literature have investigated various graph structures that can be defined on a set of points in the plane. The goal is to find a graph that preserves the perceptual relevance of the pattern. Such graphs include the minimum spanning tree, gabriel graph, relative neighborhood graph, and Delaunay triangulation. For a general discussion of these structures and their applications to pattern recognition, the reader is referred to the excellent survey papers by Toussaint [5, 6].

Let $S = \{x_1, \ldots, x_n\}$ be a set of n points in the plane and let

$$r_i = \min_{k \neq i} d(x_i, x_k)$$

where d represents Euclidean distance. Finally, let C_i be the circle of radius r_i centered at x_i. Then the *closed sphere of influence graph* $\overline{G}(S)$ can be defined on S by joining points x_i and x_j of S by an edge whenever $C_i \cap C_j \neq \varnothing$.

The open sphere of influence graph $G(S)$ is a subgraph of $\overline{G}(S)$: Points x_i and x_j are joined by an edge whenever C_i and C_j intersect in precisely two points. Figure 1 gives examples of these graphs.

The sphere of influence graph is not simply related to any of the above-mentioned graphs derived from a set of points in the plane. As Toussaint has shown [5], the minimum spanning tree is a subgraph of the relative neighborhood graph,

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