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GRAPHS AND ORDERS IN RAMSEY THEORY AND IN DIMENSION THEORY

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ABSTRACT

The purpose of this paper is to present a concise and relatively self contained treatment of recent results linking partially ordered sets with topics more traditionally associated with graph theory and combinatorics: Ramsey theory and chromatic graph theory. In particular, we will present the major theorems of Nesetřil and Rödl ([31], [33], and [34]) concerning Ramsey theory for partially ordered sets in a new setting which will allow nonspecialists to appreciate the power and beauty of these results. Other Ramsey theoretic results for partially ordered sets will be discussed briefly and some directions for future research will be indicated.

We will also present a concise treatment of the constructions of Ross and Trotter ([53], [54]) for irreducible partially ordered sets utilizing familiar concepts from chromatic graph (and hypergraph) theory. When combined with the complexity theorems of Yannakakis [57], these constructions show that partially ordered sets can simultaneously exhibit both mathematical elegance and awkward pathology.

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1. Notation, Terminology, and Preliminary Results.

Throughout this paper, we will use the notation and terminology of [50] for partially ordered sets (posets, ordered sets, and dimension). For the sake of completeness we give here the central definitions, notations, and conventions. Formally a poset $\underset{\sim}{X}$ consists of a pair (X,P) where X is a nonempty set and P is a reflexive, antisymmetric, and transitive relation on X. P is called a partial order on X. The notations $(x,y) \in P$, xPy, $x \leq y$ in P, and $x \leq y$ in $\underset{\sim}{X}$ are used interchangeably. Similarly, we write $x < y$ in P or $x < y$ in $\underset{\sim}{X}$ when $x \leq y$ in P and $x \neq y$. When neither (x,y) nor (y,x) is in P, we say x and y are incomparable and write xIy in P or xIy in $\underset{\sim}{X}$. We denote the set of all incomparable pairs by I_P or $I_{\underset{\sim}{X}}$. A poset (X,P) is called an antichain if xIy in P for every $x,y \in X$ with $x \neq y$.

If (X,P) and (Y,Q) are posets, then we say (X,P) is isomorphic to (Y,Q) when there exists a function $f\colon X \xrightarrow[\text{onto}]{1-1}$ so that $(x_1,x_2) \in P$ if and only if $(f(x_1),f(x_2)) \in Q$. In this paper, we do not distinguish between isomorphic posets and write $\underset{\sim}{X} = \underset{\sim}{Y}$ when $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are isomorphic. Similarly, we say $\underset{\sim}{Y}$ is contained in $\underset{\sim}{X}$ (or $\underset{\sim}{X}$ contains $\underset{\sim}{Y}$) and write $\underset{\sim}{Y} \subseteq \underset{\sim}{X}$ when Y is isomorphic to a subposet of $\underset{\sim}{X}$.

A partial order L on a set X is called a linear order (also total order or simple order) when $I_L = \phi$, and the poset (X,L) is called a chain. We denote an n-element chain by $\underset{\sim}{n}$. We denote the chain consisting of the positive integers by $\underset{\sim}{N}$ and the chain consisting of the real numbers by $\underset{\sim}{R}$. We denote the free sum (also called disjoint sum) of poset $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ by $\underset{\sim}{X} + \underset{\sim}{Y}$ and the cartesian product by $\underset{\sim}{X} \times \underset{\sim}{Y}$. The notation $\underset{\sim}{X}^t$ means the product of t copies of $\underset{\sim}{X}$.

If P and Q are partial orders on a set X and $P \subseteq Q$, then Q is called an extension of P. A linear order L which is an extension of P is called a linear extension of P. A theorem of Szpilrajn [43] asserts that for any partial order P, the collection C of all linear extensions of P is nonempty and $\cap C = P$. The dimension [7] of a poset (X,P), denoted $\dim(X,P)$ or $\dim(\underset{\sim}{X})$, is the smallest positive integer t for which there exist linear extensions $L_1, L_2, \ldots, L_t$ of P whose intersection is P. A poset has dimension one if and only if it is a chain.

Alternately, the dimension [35] of (X,P) is the smallest positive integer t for which there exists a function f which assigns to each $x \in X$ a sequence $f(x)(1), f(x)(2), \ldots, f(x)(t)$ of real numbers so that $x \leq y$ in P if and only if $f(x)(i) \leq f(y)(i)$ in $\underset{\sim}{R}$ for $i = 1,2,\ldots,t$. The function f is called an embedding of (X,P) in $\underset{\sim}{R}^t$.

If (X,P) is a poset and Y is a nonempty subset of X, then the restriction of P to Y, denoted $P(Y)$, is defined by

$P(Y) = P \cap (Y \times Y)$. It is clear that $P(Y)$ is a partial order on Y, and we say $(Y,P(Y))$ is a subposet of (X,P). Obviously $\mathrm{Dim}(Y,P(Y)) \leq \mathrm{Dim}(X,P)$ whenever $\phi \neq Y \subseteq X$. The subposet $(Y,P(Y))$ is called a proper subposet of (X,P) when $\phi \neq Y \neq X$.

Part I: RAMSEY THEORY FOR POSETS

2. Ramsey Theory - Background Material

Intuitively speaking, Ramsey theory is the formal study of generalizations of the elementary result known as the "Pigeon Hole Principle" which asserts that if $(m-1)\,n + 1$ pigeons are placed in n distinct holes, then there is (at least) one hole into which (at least) m pigeons have been placed. In order to simplify the formal statements of the Ramsey theoretic results which follow, we pause to introduce some notation and terminology.

Throughout this section, we will be discussing functions of a set X to the set $\{1,2,\ldots,r\}$ of the first r positive integers. The integers $\{1,2,\ldots,r\}$ are called colors, and the function is called a coloring of X. We will write $\psi\colon X \to \{1,2,\ldots,r\}$ and we call ψ an r-coloring if we also want to specify the number r of colors.

For a set S and a positive integer m, we let $\binom{S}{m}$ denote the set of all m-element subsets of S.

For an r-coloring $\psi\colon \binom{S}{m} \to \{1,2,\ldots,r\}$, we call a subset $H \subseteq S$ monochromatic (or homogeneous) if ψ assigns every $A \in \binom{H}{m}$ to the same color. If this color is the integer i, we will say $\binom{H}{m}$ is monochromatic in color i. The fundamental question is to determine when such monochromatic sets necessarily exist. The following, now classic, theorem of F.P. Ramsey [38] is the starting point for this section.

Theorem 1: Let m and r be positive integers and let $n_1,n_2,\ldots,n_r$ be integers with $n_i \geq m$ for $i = 1,2,\ldots,r$. Then there exists an integer n_0 (whose value depends on $m,r,n_1,n_2,\ldots,n_r$) so that for every set S with $|S| \geq n_0$ and for every r-coloring $\psi\colon \binom{S}{m} \to \{1,2,\ldots,r\}$ of the set of m-element subsets of S, there exists a color i and an n_i-element monochromatic subset $H_i \subseteq S$ so that ψ assigns every m-element subset of H_i to color i, i.e., H_i is monochromatic in color i. □

We will find it convenient to utilize Ramsey theoretic results stated in terms of infinite sets. The infinite version of the preceding result is:

Theorem 1': Let m and r be positive integers and let $\psi\colon \binom{\underset{\sim}{N}}{m} \to \{1,2,\ldots,r\}$ be an r-coloring. Then there exists a color i and an infinite subset $H \subseteq \mathbb{N}$ so that H is monochromatic in color i. □

Well known generalizations of this theorem have been developed by Schur [41], Rado [36], Van de Waerden [55], Folkman [9], and Hales, Jewett [16]. We refer the reader to the survey articles by Graham [11], [17] and the book by Graham, Rothschild and Spencer [15] for additional background material on the subject. For a concise discussion of the latest developments on the frontiers of Ramsey theory, we refer the reader to [13].

In this paper, we will require two well known Ramsey theoretic results. The first of these is a special case of the so called Product Ramsey theorem [15]. Let k be a positive integer and let $S_1, S_2, \ldots, S_k$ be sets. For a positive integer m, we refer to the elements of $\binom{S_1}{m} \times \binom{S_2}{m} \times \ldots \times \binom{S_k}{m}$ as <u>grids</u>.

<u>Theorem 2</u>: Let k and m be positive integers and let $S_1, S_2, \ldots, S_k$ be infinite subsets of N, the set of positive integers. Then let $\psi: \binom{S_1}{m} \times \binom{S_1}{m} \times \ldots \times \binom{S_1}{m} \to \{1,2,\ldots,r\}$ be an r-coloring of grids. Then for each $j = 1,2,\ldots,k$, there exists an infinite subset $H_j \subseteq S_j$ and a color i so that ψ assigns every grid in $\binom{H_1}{m} \times \binom{H_2}{m} \times \ldots \times \binom{H_k}{m}$ to color i. □

Let S be a set and let s be a positive integer. Then we denote by $\Pi(S,s)$ the set of all partitions of the form $S = S_1 \cup S_2 \cup \ldots \cup S_s$ with $S_i \neq \phi$ for $i = 1,2,\ldots,s$. We write $P:[S = S_1 \cup S_2 \cup \ldots \cup S_s]$ to indicate the precise form of partition $P \in \Pi(S,s)$. The sets $S_1, S_2, \ldots, S_s$ are called the <u>parts</u> of the partition P. Also, we say P is a <u>partition of S into s parts</u>. A partition differs from a coloring in that the parts of the partition are not ordered.

When $|S| = n$, we let $\Pi_n(S) = \bigcup_{s=1}^{n} \Pi(S,s)$. A natural partial order on $\Pi_n(S)$ is defined by setting $P_1 \leq P_2$ when every part of P_2 is the union of one or more of the parts of the partition P_1.

The next theorem is a very natural extension of Ramsey's theorem to partitions and was first proved by Rothschild [40]. We encourage the reader to investigate the important paper of Graham and Rothschild [14], in which the concept of an n-parameter set is introduced. Many of the well known Ramsey theoretic results follow as easy corollaries to the principal result of [14].

<u>Theorem 3</u>: Let s and t be positive integers with $s \leq t$. Then there exists an integer k_0 so that if $|S| = k \geq k_0$ and $\psi: \Pi(S,s) \to \{1,2,\ldots,r\}$ is any r-coloring of the partitions of S into s parts, then there exists a partition $P_0 \in \Pi(S,t)$ and a color i so that ψ assigns every partition $P \in \Pi(S,s)$ with $P_0 \leq P$ to color i. □

Many readers may be more familiar with the "finite unions" theorem [9] which follows as an easy corollary to the preceding Ramsey theorem for partitions. For a set S, let $P(S)$ denote the set of nonempty subsets. If $F = \{F_1, F_2, \ldots, F_t\}$ is a family of t pairwise disjoint sets of $P(S)$, we call a set $A \in P(S)$ a <u>finite union</u> of F if there is a nonempty subset $B \subseteq \{1,2,\ldots,t\}$ so that $A = \bigcup_{i \in B} F_i$.

<u>Theorem 4 [9]</u>: For every pair r,t of positive integers, there exists an integer so that if S is any set with $|S| \geq k$ and $\psi: P(S) \to \{1,2,\ldots,r\}$ is any r-coloring of $P(S)$, then there exists a family $F = \{F_1, F_2, \ldots, F_t\}$ of t pairwise disjoint sets in $P(S)$ and a color i so that ψ assigns all finite unions of F to color i. □

3. <u>Ramsey Theory for Partially Ordered Sets</u>.

The <u>width</u> of a poset is the maximum size of an antichain. The classical theorem of R.P. Dilworth [6] asserts that if the width is finite, there exists a partition of the poset into width number of chains.

<u>Theorem 5</u>: If (X,P) is a poset of finite width n, then (X,P) can be partitioned into n chains. □

An immediate consequence of this theorem is the following Ramsey theoretic result (It also follows from the dual version of Dilworth's theorem - an elementary result.)

<u>Theorem 6</u>: If (X,P) is a poset and $|X| \geq (n_1-1)(n_2-1) + 1$, then (X,P) contains an n_1-element antichain or an n_2-element chain. □

It follows immediately from Theorem 1 that for every pair n_1, n_2 of positive integers there exists a number n_0 so that if (X,P) is any poset with $|X| \geq n_0$, then (X,P) contains an n_1-element chain or an n_2-element antichain. To see this we need only observe that the partial order P determines a 2-coloring of $\binom{X}{2}$: Assign a pair (x,y) to color i if (x,y) is a comparable pair and to color 2 when it is an incomparable pair. Of course, Theorem 6 allows the surprising conclusion that $n_0 = (n_1-1)(n_2-1) + 1$ works. This value is considerably smaller than what would be guaranteed by appealing only to Theorem 1.

Another topic which yields Ramsey theoretic results for partially ordered sets is the study of regressions. Let (X,P) be a poset. Then a map $f: X \to X$ is called a <u>regression</u> if $f(x) \leq x$ in P for every $x \in X$. Note that a regression need not be order preserving. On the other hand, it is of interest to investigate conditions which force a regression to be order preserving on some chain of specified size. A chain $C = \{x_1 < x_2 < \ldots < x_k\}$ is called a monotonic k-chain if $f(x_1) \leq f(x_2) \leq \ldots \leq f(x_k)$ in P.

It is straightforward to conclude from Theorem 1 that for every

w, k, there exists a number n_0 so that if (X,P) is a poset of width at most w and $|X| \geq n_0$, then for every regression f on X, there exists a monotonic k-chain. As was the case with Theorem 5, the combinatorial elegance of partially ordered sets allows a precise determination of n_0 and its value is considerably smaller than the one guaranteed by appealing only to Ramsey theory. The result is due to Peck, Schor, Trotter, and West [56].

Theorem 7: Let w and k be positive integers and let (X,P) be any poset with $|X| \geq (w+1)^{k-1}$ and width at most w. Then every regression on (X,P) has a monotonic k-chain. □

In order to force long monotonic chains, it is of course necessary that a poset have long chains. Furthermore, either the width must be bounded or we must impose some additional conditions on the structure of the poset such as is the case in the following theorems.

Theorem 8: (Rado [37] Harzheim [17], [18]): For every k, there exists an integer n_0 so that $n \geq n_0$ and if f is any regression on the Boolean algebra $\underline{2}^n$, then f has a monotonic k-chain □

The following result was announced by K. Leeb at the 1982 Conference on Ordered Sets in Banff, Canada.

Theorem 9 (Leeb [26]): For every k, there exists an integer n_0 so that if $n \geq n_0$, $S = \{1,2,\ldots,n\}$, and f is any regression on the partition lattice $\Pi_n(S)$, then f has a monotonic k-chain. □

In another direction, it is of interest to investigate Ramsey theoretic theorems where there are close analogies between the result for graphs and partially ordered sets. For example, consider the following two theorems of Rival and Sands [39].

Theorem 10: Every infinite graph $G = (V,E)$ contains an infinite subset $S \subseteq V$ so that every vertex of G is adjacent to precisely none, one, or infinitely many of the elements of S. Furthermore, every vertex of S is adjacent to none or infinitely many of the vertices of S. □

Theorem 11: Every infinite poset (X,P) of finite width contains an infinite chain C so that every point of P is comparable with none or infinitely many of the points in C. Moreover, if X is countable, then C can be chosen so that every point in X is either comparable with none of points in C or is comparable with all but finitely many elements of C.

In comparing these two results, the reader should note the slightly stronger form of Theorem 11 which requires the special structure of posets.

4. The Nešetřil-Rödl Theorems

In this section, we present the elegant theory developed by Nešetřil and Rödl in a series of important papers [31], [32], [33],

and [34]. For nonspecialists, these papers can be overwhelming and our primary goal for this section will be to present straightforward combinatorial proofs of the major results of this theory. Some notation will be required.

Let $\underset{\sim}{X} = (X,P)$ and $\underset{\sim}{Y} = (Y,Q)$ be posets. We denote by $\binom{\underset{\sim}{X}}{\underset{\sim}{Y}}$ the set of all subposets of $\underset{\sim}{X}$ which are isomorphic to $\underset{\sim}{Y}$. A poset $\underset{\sim}{Y}$ is called a Ramsey poset if for every $r \geq 2$ and every poset $\underset{\sim}{X}$, there exists a poset $\underset{\sim}{Z}$ so that for every r-coloring ψ of $\binom{\underset{\sim}{Z}}{\underset{\sim}{Y}}$, there exists a subposet $\underset{\sim}{X}' \subseteq \underset{\sim}{Z}$ with $\underset{\sim}{X}'$ isomorphic to $\underset{\sim}{X}$ so that ψ sends every poset in $\binom{\underset{\sim}{X}'}{\underset{\sim}{Y}}$ to the same color.

A finite poset $\underset{\sim}{Y} = (Y,Q)$ is called a weak order [42] if there exists a function $f: Y \to \underset{\sim}{R}$ so that $y_1 < y_2$ in Q if and only if $f(y_1) < f(y_2)$ in $\underset{\sim}{R}$. Equivalently [42], $\underset{\sim}{Y}$ is a weak order if and only if $\underset{\sim}{Y}$ does not contain a subposet isomorphic to $\underline{2} + \underline{1}$. Also, a weak order is the ordinal sum of antichains.

Let $\underset{\sim}{Y} = (Y,Q)$ and let $\mathcal{Q}$ denote the set of all linear extensions of Q. Define an equivalence relation $\sim$ on $\mathcal{Q}$ by setting $M_1 \sim M_2$ exactly when there exists an isomorphism $g: \underset{\sim}{Y} \to \underset{\sim}{Y}$ so so that $y_1 < y_2$ in M_1 if and only if $g(y_1) < g(y_2)$ in M_2. It is straightforward to verify that $\sim$ is an equivalence relation on $\mathcal{Q}$. The following result is an easy exercise.

Lemma 12: A poset $\underset{\sim}{Y} = (Y,Q)$ is a weak order if and only if $M_1 \sim M_2$ for every pair of linear extensions of Q. □

Let $\underset{\sim}{Y} = (Y,Q)$ and $\underset{\sim}{Y}' = (Y',Q')$ be isomorphic posets. If L and L' are linear extensions of Q and Q' respectively, we extend the preceding definition by defining $L \sim L'$ when there exists an isomorphism $f: Y \to Y'$ so that $x \leq y$ in L if and only if $f(x) \leq f(y)$ in L'. With this terminology behind us, we can now state the principal result of the Nešetřil-Rödl theory.

Theorem 13: A poset $\underset{\sim}{Y}$ is a Ramsey poset if and only if it is a weak order. □

Before giving the proof of Theorem 13, we must develop some additional background material. First, let $\underset{\sim}{X} = (X,P)$ be a poset and let $\mathcal{P}$ denote the set of all linear extensions of P. If $L \in \mathcal{P}$ and $L: [x_1 < x_2 < \ldots < x_n]$ then we can use L to linearly order $\mathcal{P}$ lexicographically. To be more precise, if $|\mathcal{P}| = t$, we can label the linear extensions in $\mathcal{P}$ as $L_1, L_2, \ldots, L_t$ with $L_i: [x_{i_1} < x_{i_2} < \ldots < x_{i_n}]$ so that whenever $1 \leq i < j \leq t$ and α is the least integer for which $x_{i\alpha} \neq x_{j\alpha}$, we always have $x_{i\alpha} < x_{j\alpha}$ in L. Note that this rule implies that $L = L_1$.

If $\underset{\sim}{Y} = (Y,Q)$ is a subposet of $\underset{\sim}{X}$ and $\mathcal{Q}$ denotes the set of all linear extensions of Q, we can use $L = L_1$ to lexicographically order Q as $M_1 < M_2 < \ldots < M_s$. Note that $M_1 = L_1(Y)$, the restriction of L_1 to Y, and the ordering on $\mathcal{Q}$ is the same as would be produced if we used M_1 to lexicographically order $\mathcal{Q}$.

Lemma 14: Let $\underset{\sim}{X} = (X,P)$ be a poset and let $\underset{\sim}{Y} = (Y,Q)$ be a subposet. Then let $\mathcal{P}$ and $\mathcal{Q}$ denote the sets of linear extensions of P and Q respectively. Let L be an arbitrary linear extension from $\mathcal{P}$ and use L to lexicographically order $\mathcal{P}$ and $\mathcal{Q}$ as $L_1 < L_2 < \ldots < L_t$ and $M_1 < M_2 < \ldots < M_s$ respectively. Define a function $f: \{1,2,\ldots,s\} \to \{1,2,\ldots,t\}$ by $f(i)$ = least j for which $L_j(Y) = M_i$. Then f is strictly increasing.

Proof: It is obvious that f is one to one, so it remains to show that f is order-preserving. To prove this, we will consider an algorithm which computes f, or more precisely computes $L_{f(i)}$. Let $M_i = [y_1 < \ldots < y_m]$ be a member of $\mathcal{Q}$, and suppose that $L_{f(i)} = [x_1 < \ldots < x_n]$. That is, $[x_1 < \ldots < x_n]$ is the first linear extension of P which restricts to $[y_1 < \ldots < y_m]$.

The key fact about lexicographic order is that to find the first sequence satisfying certain conditions, one can use a "greedy" approach. That is, first make the first entry as small as possible, then make the second entry as small as possible, and so on. Therefore we can inductively compute x_1, then x_2, then x_3, and so on.

The statement that $[x_1 < \ldots < x_n]$ is a linear extension of P is equivalent to the statement that x_k is minimal in the poset $\underset{\sim}{X} - \{x_1,x_2,\ldots,x_{k-1}\}$ for each $k = 1,2,\ldots,n$. Saying that $[x_1 < \ldots < x_n]$ has $[y_1 < \ldots < y_m]$ as a restriction is the same as saying that each x_k either does not belong to Y, or else x_k is the first element of $[y_1 < \ldots < y_m]$ that does not belong to $\{x_1,x_2,\ldots,x_{k-1}\}$.

Without further delay, here is the algorithm that computes $L_{f(i)}$, written in pidgin Pascal. More discussion will follow.

```
r: = 1;  (*y_r is the first element of M_i not yet used. *)
for k: = 1 to n do
begin (*Compute x_k. Either x_k = y_r or else x_k is not in Y.*)
      (*First find z, the best we can do with something not in Y.*)
A: = set of minimal elements of the poset X - {x_1,x_2,...,x_{k-1}};
if  A - Y ≠ φ then
    z: = least element of (A-Y, L_1(A-Y));
if  A - Y ≠ φ and (r > m or (z,y_r) ∈ L_1)
  then  x_k: = z
  else begin
         x_k: = y_r;
         r: = r + 1
       end
end
```


A few words of explanation are in order for the else clause. If $A - Y = \phi$, then $A \subseteq Y$. Since A is not empty, it follows that y_r exists, i.e. $r \leq m$. Furthermore, $y_r \in A$. For otherwise there would be an element y_i of A such that $y_i < y_r$ in Q. Since M_i is a linear extension of Q, it follows that $i < r$. But then $y_i \in \{x_1,x_2,\ldots,x_{k-1}\}$, which contradicts the statement $y_i \in A$. Thus, since $y_r \in A$, it is permissible to let $x_k := y_r$.

The remaining case is that $A - Y$ is nonempty and $(y_r,z) \in L_1$. As argued above, there is no element y_i of Y such that $y_i \in A$ and $y_i < y_r$ in Q. Consequently, if $y_r \notin A$, then there is some element w in $A - Y$ such that $w < y_r$ in P. Since L_1 is a linear extension of P, $w < y_r$ in L_1. But this contradicts the choice of z. Therefore $y_r \in A$.

Now let us use the algorithm above to show that f is order-preserving. Suppose that $M, M' \in \mathcal{Q}$ and M precedes M' lexicographically. Let j be the first place in which M and M' differ. Thus $M = [y_1 < y_2 < \ldots < y_m]$, $M' = [y_1 < \ldots < y_{j-1} < y'_j < \ldots < y'_m]$, and $y_j < y'_j$ in L_1. Imagine two instances of the algorithm above computing $[x_1 < \ldots < x_n]$, the first extension containing M, and $[x'_1 < \ldots < x'_n]$, the first extension of P containing M', in parallel. So long as $r < j$, the two computations will clearly proceed identically. Consider what happens when $r = j$. As long as $A - Y \neq \phi$ and $(z,y_j) \in L_1$, z will be used as both x_k and x'_k. But this can happen only finitely many times. If $A - Y$ is empty, then we have $x_k = y_j$ and $x'_k = y'_j$, so $x_k < x'_k$ in L_1. If $A - Y$ is nonempty and $(y_j,z) \in L_1$, then $x_k = y_j$ and x'_k is either z or y'_j, so again $x_k < x'_k$ in L_1. Thus, if k is the first place in which $[x_1 < \ldots < x_n]$ and $[x'_1 < \ldots < x'_n]$ differ, then $x_k < x'_k$ in L_1. This proves that f is order-preserving. □

At first glance, the preceding lemma may seem hopelessly technical, but it will play a vital role in the proof of Theorem 13, and we trust that its value will then be clear. We now present the proof of the positive part of Theorem 13.

Lemma 15: If $\underset{\sim}{Y} = (Y,Q)$ is a weak order, then $\underset{\sim}{Y}$ is a Ramsey poset.

Proof: Let $\underset{\sim}{X} = (X,P)$ be an arbitrary poset and let r be a positive integer. We show that there exists a poset $\underset{\sim}{Z} = (Z,R)$ so that for every r-coloring of the subposets of $\underset{\sim}{Z}$ which are isomorphic to $\underset{\sim}{Y}$, there exists a subposet $\underset{\sim}{X}'$ of $\underset{\sim}{Z}$ so that $\underset{\sim}{X}' \cong \underset{\sim}{X}$ and all copies of $\underset{\sim}{Y}$ in $\underset{\sim}{X}'$ are assigned to the same color.

Let $\mathcal{P}$ and $\mathcal{Q}$ denote the sets of linear extensions of P and Q respectively. Then let $L \in \mathcal{P}$ and $M \in \mathcal{Q}$. Use L and M to linearly order $\mathcal{P}$ and $\mathcal{Q}$ as $L_1 < L_2 < \ldots < L_t$ and $M_1 < M_2 < \ldots < M_s$ respectively. Let $|X| = n$ and label the points in X so that $L_1: [x_1 < x_2 < \ldots < x_n]$. Similarly, let $m = |Y|$ and label the points in Y so that $M_1: [y_1 < y_2 < \ldots < y_m]$. For each $j = 1,2,\ldots,s$, let $M_j: [y_{j1} < y_{j2} < \ldots < y_{jm}]$. Note that $y_{1\gamma} = y_\gamma$ for each $\gamma = 1,2,\ldots,m$.

Next, we apply the Ramsey theorem for partitions to choose a positive integer $k = k_0$ so that if S is any set with $|S| = k$ and $\psi': \Pi(S,s) \to \{1,2,\ldots,r\}$ is any r-coloring of the partitions of S into s parts, then there exists a partition $P_0 \in \Pi(S,t)$ and a color δ so that ψ' assigns every $P \in \Pi(S,s)$ with $P_0 \le P$ to color δ.

In order to complete the argument, it suffices to show that we may choose $Z = N^k$. (To simplify the presentation of the argument, we take Z to be an infinite poset. Of course, we can actually choose Z as $\underline{p}^k$ where p is a sufficiently large integer.) Let $\psi: \binom{Z}{Y} \to \{1,2,\ldots r\}$ be any r-coloring of the copies of Y in Z. We now proceed to show that there exists a subposet $X' \subseteq Z$ and a color γ so that $X' \cong X$ and every copy of Y in X' is assigned color γ by ψ.

Let $S = \{1,2,\ldots,k\}$ and let $\ell = |\Pi(S,s)|$. Label $\Pi(S,s)$ arbitrarily as $P_1, P_2, \ldots, P_\ell$. Then set $Z_1 = Z = N^k$ and suppose that for some α with $1 \le \alpha \le \ell$ we have defined Z_α with Z_α isomorphic to N^k. We use the partition P_α to determine a subposet $Z_{\alpha+1}$ of Z_α with $Z_{\alpha+1}$ also isomorphic to N^k. At the same time, we will inductively define an r-coloring $\psi': \Pi(S,s) \to \{1,2,\ldots,r\}$.

Let $S_1 = S_2 = \ldots = S_k = N$ so that $Z_\alpha = S_1 \times S_2 \times \ldots \times S_k$. For each $\beta = 1,2,\ldots,k$, let $T_\beta = \{t_{1\beta} < t_{2\beta} < \ldots < t_{m\beta}\}$ be an m-element subset of S_β. With the grid $G = T_1 \times T_2 \times \ldots \times T_k$ and the partition P_α, we associate a subposet $Y(G,\alpha)$ of Z_α with $Y(G,\alpha)$ isomorphic to Y. This is accomplished as follows. Since P_α is a partition of $\{1,2,\ldots,k\}$ into s parts, we can label these parts as $B_{\alpha 1}, B_{\alpha 2}, \ldots, B_{\alpha s}$ so that the least integer in $B_{\alpha i}$ is less than the least integer in $B_{\alpha j}$ when $1 \le i < j \le s$. Label the points in $Y(G,\alpha)$ as $\{(y'_{i1}, y'_{i2}, \ldots, y'_{ik}): 1 \le i \le m\}$. For each $\beta = 1,2,\ldots,k$ and each $i = 1,2,\ldots,m$, $y'_{i\beta}$ is given by: $y'_{i\beta} = t_{\gamma\beta}$ if and only if $\beta \in B_{\alpha j}$ and $y_i = y_{j\gamma}$. With this definition, it follows easily that $Y(G,\alpha)$ is isomorphic to Y by the map $y_i \to (y'_{i1}, y'_{i2}, \ldots, y'_{ik})$. To see this, we simply observe that for each $\beta = 1,2,\ldots,k$, the βth coordinates linearly order the points in $Y(G,\alpha)$ in the same order as M_j orders the corresponding points in Y where $B_{\alpha j}$ is the part in the partition P_α containing the integer β.

We can now define an r-coloring $\psi_\alpha: \binom{S_1}{m} \times \binom{S_2}{m} \times \binom{S_k}{m} \to \{1,2,\ldots,r\}$ of the grids in $\binom{S_1}{m} \times \binom{S_2}{m} \times \ldots \times \binom{S_k}{m}$ by the rule $\psi_\alpha(G) = \psi(Y,\alpha))$. Next, we apply the Product Ramsey theorem. For each $i = 1,2,\ldots,s$, choose an infinite subset $H_i \subseteq S_i$ and a color γ, so that ψ_α assigns every grid in $\binom{H_1}{m} \times \binom{H_2}{m} \times \ldots$

$\times \binom{H_k}{m}$ to color γ. We set $\psi'(P_\alpha) = \gamma$ and $Z_{\alpha+1} = H_1 \times H_2 \times \ldots \times H_k$. Observe that $Z_{\alpha+1} \cong N^k$ as was desired.

We repeat this process inductively until the poset $Z_{\ell+1}$ and the r-coloring ψ' of $\Pi(S,s)$ have been obtained. Next, we apply the Ramsey theorem for partitions and conclude that there exists a partition $P_0 \in \Pi(S,t)$ and a color δ_0 so that ψ' assigns every partition $P \in \Pi(S,s)$ with $P_0 \le P$ to color δ_0. We label the parts of P_0 as $A_1, A_2, \ldots, A_t$ so that whenever $1 \le i < j \le t$, the least integer in A_i is less than the least integer in A_j.

Let $Z_{\alpha+1} = S_1 \times S_2 \times \ldots \times S_k$. For each $\beta = 1,2,\ldots,k$, choose an n-element subset $H_i^2 = \{h_{1\beta} < h_{2\beta} < \ldots < h_{n\beta}\} \subseteq S_i$. Define a subposet $X' \subseteq H_1 \times H_2 \times \ldots \times H_k$ with $X' = \{(x'_{i1}, x'_{i2}, \ldots, x'_{ik}): 1 \le i \le n\}$ by the rule: $x'_{i\beta} = h_{\gamma\beta}$ if and only if $\beta \in A_j$ and $x_i = x_{j\gamma}$. As was the case before, it follows easily that the mapping $x_i \to (x'_{i1}, x'_{i2}, \ldots, x'_{ik})$ is an isomorphism from X to X'. This statement follows from the observation that for each $\beta = 1,2,\ldots,k$, the βth coordinates linearly order the points in X' in the same order as L_j orders the corresponding points in X, where A_j is the part in the partition P_0 containing the integer β.

To complete the argument, it remains only to show that every copy of Y in X' is assigned to color δ_0 by the r-coloring ψ. Let Y'' be any subposet of X' with Y' isomorphic to Y. Label the points in Y'' as $y''_1, y''_2, \ldots, y''_m$ so that the map $y_i \to y''_i$ is an isomorphism. Order the s linear extensions of Y'' lexicographically as $M''_1, M''_2, \ldots, M''_s$, using $M''_1: [y''_1 < y''_2 < \ldots < y''_m]$. Note that for all i_1, i_2, j with $1 \le i_1 < i_2 \le m$ and $1 \le j \le s$, we have $y_{i_1} < y_{i_2}$ in M_j if and only if $y''_{i_1} < y''_{i_2}$ in M''_j.

We next determine a partition P of $s = \{1,2,\ldots,k\}$ into s parts (actually, P depends on Y'') by the rule: $\beta \in B_j$ if the βth coordinates order the points in Y'' in the same order as M''_j. Of course, we have $P_0 \le P$ so that ψ' assigns color δ_0 to the partition P. Now we appeal to the technical lemma preceding this theorem to conclude that if $1 \le j_1 < j_2 \le s$, the least integer in B_{j_1} is less than the least integer in B_{j_2}. To see this, we choose an arbitrary pair j_1, j_2 with $1 \le j_1 < j_2 < s$ and let β_1 and β_2 be the least integer in B_{j_1} and B_{j_2} respectively. In the partition P_0, let β_1 and β_2 belong to A_{γ_1} and A_{γ_2} respectively. Then β_1 is the least integer in A_{γ_1} and β_2 is the least integer in A_{γ_2}. Note that A_{γ_1} consists of those β for which the β^{th} coordinates order the points in X' in the same order as L_{γ_1} orders the corresponding points in X. It follows from Lemma 14 that $\gamma_1 < \gamma_2$ and thus the least integer in A_{γ_1} is less than the least integer in A_{γ_2}. But clearly for each $i = 1,2$, the least integer in A_{γ_i} is the least integer in B_{j_i}

and the desired conclusion has been reached.

Now choose the unique integer α with $1 \le \alpha \le \ell$ for which $P = P_\alpha$. It follows that $\mathbf{Y}'' = \mathbf{Y}(G,\alpha)$ for the grid $G = T_1 \times T_2 \times \ldots \times T_k$ where for each $\beta = 1,2,\ldots,k$, T_β is the m-element subset of H_β formed by the βth coordinates of the points in $\mathbf{Y}''$. It follows that $\psi(\mathbf{Y}'') = \psi(\mathbf{Y}(G,\alpha)) = \psi_\alpha(G) = \psi'(P_\alpha) = \delta_0$, i.e. ψ assigns the subposet $\mathbf{Y}''$ to color δ_0. Since $\mathbf{Y}''$ was arbitrary, the proof of the claim is complete. □

To show that a poset $\mathbf{Y}$ which is not a weak order is not a Ramsey poset requires the development of an intermediate theorem of independent interest. However, the proof technique is amazingly similar to the preceding result.

<u>Theorem 16</u>: Let $\mathbf{Y} = (Y,Q)$ be a poset and let M be an arbitrary linear extension of Q. Then there exists a poset $\mathbf{X} = (X,P)$ so that if L is an arbitrary linear extension of P, then there exists a subposet $\mathbf{Y}' = (Y',Q')$ of (X,P) and an isomorphism $f: \mathbf{Y} \cong \mathbf{Y}'$ so that $y_1 < y_2$ in M if and only if $f(y_1) < f(y_2)$ in $L(Y')$.

<u>Proof</u>: We assume without loss of generality that $\dim(\mathbf{Y}) = s \ge 3$. Otherwise embed $\mathbf{Y}$ as a subposet of a poset of larger dimension. Let $s = \dim(\mathbf{Y})$ and then let $\{M_1, M_2, \ldots, M_s\}$ be a realizer of Q with $M = M_1$. (Note: this collection need only be a realizer of Q. It is not necessarily a list of all linear extensions of Q.) We label $\mathbf{Y}$ so that $M_1: [y_1 < y_2 < \ldots < y_m]$. For each $i = 1,2,\ldots,s$, we also let $M_i: [y_{i1} < y_{i2} < \ldots < y_{im}]$.

Next, we apply the Ramsey theorem for partitions with $t=s$, $s=2$, and $r=2$, i.e., we choose an integer k so that if S is any set with $|S| \ge k$ and $\psi: \Pi(S,2) \to \{1,2\}$ is a 2-coloring of the partitions of S into two parts, then there exists a partition $P_0 \in \Pi(S,s)$ and a color δ so that if $P \in \Pi(S,2)$ and $P_0 \le P$, then P is assigned by ψ to color δ.

We will now proceed to show that we may take $\mathbf{X} = \mathbf{N}^k$. (As was the case in the previous proof, we find it convenient to use infinite posets. We can actually take $\mathbf{X} = p^k$ where p is a sufficiently large integer.) Let L be an arbitrary linear extension of $\mathbf{X}$.

Let $S = \{1,2,\ldots,k\}$ and let $\ell = |\Pi(S,2)|$. Then let $P_1, P_2, \ldots, P_\ell$ be any listing of the partitions in $\Pi(S,2)$. Set $\mathbf{X}_1 = \mathbf{X}$. Suppose that for some α with $1 \le \alpha \le \ell$, we have defined a subposet $\mathbf{X}_\alpha$ of $\mathbf{X}$ with $\mathbf{X}_\alpha$ also isomorphic to $\mathbf{N}^k$. We now proceed to determine $\mathbf{X}_{\alpha+1}$. At the same time, we inductively define a 2-coloring $\psi: \Pi(S,2) \to \{1,2\}$ of the partition of S into two parts.

For each $\beta = 1,2,\ldots,k$, let $\mathbf{S}_\beta = \mathbf{N}$ so that $\mathbf{X}_\alpha = \mathbf{S}_1 \times \mathbf{S}_2 \times \ldots \times \mathbf{S}_k$. Define a 2-coloring $\psi_\alpha: \binom{S_1}{2} \times \binom{S_1}{2} \times \ldots \times \binom{S_n}{2} \to \{1,2\}$ of grids as follows.

For each $\beta = 1,2,\ldots,k$, choose a 2-element subset $T_\beta = \{t_{1\beta} < t_{2\beta}\}$ of S_β and let G be the grid $T_1 \times T_2 \times \ldots \times T_k$. We associate with the grid G and the partition P_α a two element antichain $\{u_1, u_2\}$ of $\mathbf{X}_\alpha$ by the following rules: Label the parts of P_α as A_1 and A_2 so that the least integer in A_1 is less than the least integer in A_2. Then let $u_1 = (u_{11}, u_{12}, \ldots, u_{1k})$ and $u_2 = (u_{21}, u_{22}, \ldots, u_{2k})$ where

$$u_{1\beta} = t_{1\beta} \quad \text{if } \beta \in A_1$$
$$u_{1\beta} = t_2 \quad \text{if } \beta \in A_2$$
$$u_{2\beta} = t_2 \quad \text{if } \beta \in A_1$$
$$u_{2\beta} = t_1 \quad \text{if } \beta \in A_2$$

It is clear that $\{u_1, u_2\}$ is an antichain since the β^{th} coordinate of u_1 is less than the β^{th} coordinate of u_2 when $\beta \in A_1$ while the reverse statement holds when $\beta \in A_2$.

Now set $\psi_\alpha(G) = 1$ if $u_1 < u_2$ in L and $\psi_\alpha(G) = 2$ if $u_2 < u_1$ in L. Next, we apply the product Ramsey theorem to choose an infinite subset $H_\beta \subseteq S_\beta$ for each $\beta = 1,2,\ldots,k$ and a color γ so that ψ_α assigns each grid in $\binom{H_1}{2} \times \binom{H_2}{2} \times \ldots \times \binom{H_k}{2}$ to color γ. Set $\mathbf{X}_{\alpha+1} = \mathbf{H}_1 \times \mathbf{H}_2 \times \ldots \times \mathbf{H}_k$ and $\psi(P_\alpha) = \gamma$. Note that $\mathbf{X}_{\alpha+1}$ is isomorphic to $\mathbf{N}^k$ as was desired.

Repeat this process until the poset $\mathbf{X}_{\ell+1}$ has been obtained and the 2-coloring ψ has been completely determined. We now apply the Ramsey theorem for partitions to obtain a partition P_0 of S into s parts and a color δ_0 so that ψ assigns every partition $P \in \Pi(S,2)$ with $P_0 \le P$ to color δ_0. From Ramsey theory alone, we only know that either $\delta_0 = 1$ or $\delta_0 = 2$. However, using special properties of posets, we will conclude that δ_0 must be 1.

We label the parts of P_0 as $A_1, A_2, \ldots, A_s$ so that whenever $1 \le i < j \le s$, the least integer in A_i is less than the least integer in A_j.

Next, let $\mathbf{X}_{\ell+1} = \mathbf{S}_1 \times \mathbf{S}_2 \times \ldots \times \mathbf{S}_k$ and for each $\beta = 1,2,\ldots,k$, let $H_\beta = \{h_{1\beta} < h_{2\beta} < \ldots < h_{m\beta}\}$ be an m-element subset of S_β. Define a subposet $\mathbf{Y}'$ of $\mathbf{X}_{\ell+1}$ so that the point set of $\mathbf{Y}'$ is $\{y_i': 1 \le i \le m\}$ with $y_i' = (y_{i1}', y_{i2}', \ldots, y_{ik}')$ where for each $i = 1,2,\ldots,m$ and each $\beta = 1, 2, \ldots, k$, we have $y_{i\beta}' = h_{\gamma\beta}$ if $\beta \in A_i$ and $y_i = y_{i\gamma}$, i.e., the β^{th} coordinate of y_i' occupies the same position in the m-element chain H_β that y_i occupies in the m-element chain M_i where A_i is the part of the partition P_0 containing the integer β. It follows easily that the map $y_i \to y_i'$ is an isomorphism from $\mathbf{Y}$ to $\mathbf{Y}'$. Note that $1 \in A_1$ so that $y_i' = h_{\gamma 1}$ if and only if $y_i = y_{1\gamma}$. It follows that the linear ordering M_1' of $\mathbf{Y}'$ by first coordinates is $M_1': [y_1' < y_2' < \ldots < y_m']$. Recall that $M = M_1$ and that $M_1: [y_1 < y_2 < \ldots < y_m]$.

Now let $\{y_{i_1}', y_{i_2}'\}$ be a 2-element antichain in $\mathbf{Y}'$ where $i_1 < i_2$. Then we may associate with this antichain a partition P of S into two parts B_1 and B_2 where B_1 consists of those integers β for which the βth coordinates place y_{i_1}' less than