NOTE

POSET BOXICITY OF GRAPHS

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A t-box representation of a graph encodes each vertex as a box in t-space determined by the (integer) coordinates of its lower and upper corner, such that vertices are adjacent if and only if the corresponding boxes intersect. The boxicity of a graph G is the minimum t for which this can be done; equivalently, it is the minimum t such that G can be expressed as the intersection graph of intervals in the t-dimensional poset that is the product of t chains. Scheinerman defined the poset boxicity of a graph G to be the minimum t such that G is the intersection graph of intervals in some t-dimensional poset. In this paper, a special class of posets is used to show that the poset boxicity of a graph on n points is at most O(log log n). Furthermore, Ramsey's theorem is used to show the existence of graphs with arbitrarily large poset boxicity.

1. Introduction

"Boxicity" is a representation parameter of graphs introduced by Roberts [2] and Cohen [1]. It is the minimum dimension in which the graph can be represented as an intersection graph of boxes with sides parallel to the axes. More precisely, a t-box representation of a graph encodes each vertex as a box in t-space determined by the (integer) coordinates of its lower and upper corner, such that vertices are adjacent if and only if the corresponding boxes intersect. The boxicity of a graph G is the minimum t for which this can be done. Since it can be assumed that the upper and lower coordinates are all integers, a t-box representation expresses G as an intersection graph of intervals in the t-dimensional poset that is the product of t chains. Scheinerman [3] defined the poset boxicity of a graph G to be the minimum t such that G is the intersection graph of intervals in a t-dimensional poset. (A general discussion of representation parameters of graphs, included the results mentioned here, appears in [6].)

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In this paper, we consider how large the poset boxicity can be for a graph on \( n \) points. The best possible upper bound for boxicity is \( \lfloor \frac{1}{2} n \rfloor \) [2], with the extremal graphs characterized in [5]. The only graph achieving boxicity \( \frac{1}{2} n \) is \( K_{2 \ldots 2} \), but the poset boxicity of this graph is always at most 4. We will construct a family of graphs whose poset boxicity cannot be bounded by any constant, which we show by repeated application of Ramsey's Theorem. First, we use a special class of posets to show that the poset boxicity of a graph on \( n \) points is always at most \( O(\log \log n) \).

2. The upper bound

**Theorem 1.** The poset boxicity of a graph on \( n \) vertices is at most \( O(\log \log n) \).

**Proof.** Given \( G \) on \( n \) vertices, we define a poset \( p(G) \) of height 2. \( p(G) \) has a maximal element \( a_i \) and a minimal element \( b_i \) for each vertex \( v_i \) in \( G \). \( p(G) \) has a middle element \( c_e \) for each edge \( e \) in \( G \), and the relations are defined by \( a_i > c_e \) and \( b_i < c_e \) if and only if \( i \in e \). For simplicity, we also have \( a_i > b_j \) for all \( i, j \).

Clearly \( G \) is the intersection graph of the intervals \( \{ (a_i, b_i) \} \) in \( p(G) \); the intervals intersect if and only if \( G \) has the edge \( v_i v_j \).

The dimension of \( p(G) \) is at most twice the dimension of the poset \( Q \) induced by its middle and bottom levels, because any realizer \( L \) for \( Q \) can be extended to a realizer for \( P \) by taking two copies \( L_1 \) and \( L_2 \), upside-down, replacing each appearance of \( b_i \) in \( L_2 \) by \( a_i \), adding \( a_1, \ldots, a_n \) at the top of each chain of \( L_1 \), and adding \( b_1, \ldots, b_n \) at the bottom of each chain of the modified \( L_2 \). Hence we consider \( Q \). For any \( G \), the resulting \( Q \) is a subposet of the poset induced by the sets of size 1 and 2 among the lattice of all subsets of an \( n \)-set. Hence its dimension is at most the dimension of that poset. Spencer [4] showed that the dimension of that poset is \( O(\log \log n) \). \( \square \)

3. The lower bound

**Theorem 2.** For any integer \( t \), there exists a graph whose poset boxicity exceeds \( t \).

**Proof.** Suppose that every graph can be represented in a \( t \)-dimensional poset. Consider a graph \( G_n \) defined on the 2-element subsets of \( \{1, \ldots, n\} \) by creating an edge between \( \{i, j\} \) and \( \{j, k\} \) for each triple \( i < j < k \). Let \( P \) be a poset of dimension at most \( t \) in which \( G \) has an interval representation, and let \( I(i, j) \) be the interval of \( P \) assigned to the vertex \( \{i, j\} \) by the representation. Let \( a(i, j) \) and \( b(i, j) \) be the top and bottom elements of \( I(i, j) \). For each triple \( i < j < k \), choose an element \( p(i, j, k) \in I(i, j) \cap I(j, k) \).
Now we define a 2-coloring on the 5-subsets of \{1, \ldots, n\}. Given a 5-set \(i_1 < i_2 < i_3 < i_4 < i_5\), note that \(p(i_1, i_3, i_5)\) cannot belong to \(I(i_2, i_4)\), since there is no edge from \(\{i_2, i_4\}\) to \(\{i_1, i_3\}\) or \(\{i_3, i_5\}\) in \(G_n\). Hence \(p(i_1, i_3, i_5)\) is not greater than \(b(i_2, i_4)\) or is not less than \(a(i_2, i_4)\). Color the 5-set “bottom” if \(p(i_1, i_3, i_5)\) is not greater than \(b(i_2, i_4)\); otherwise, color it “top”. If \(n\) is sufficiently large, we can guarantee as large a set \(H\) as we desire all of whose 5-sets get the same color. By symmetry, we may suppose this color is “bottom”.

Now we \(t\)-color the 5-sets of \(H\). For each \(\{i_1 < i_2 < i_3 < i_4 < i_5\}\) we know \(p(i_1, i_3, i_5)\) is not greater than \(b(i_2, i_4)\), so there is some extension \(L_j\) in the \(t\)-realizer for \(P\) such that \(b(i_2, i_4)\) lies above \(p(i_1, i_3, i_5)\) in \(L_j\); give the 5-set a color corresponding to such an extension. If \(H\) is sufficiently large, then it has some 6-set \(\{i_1 < i_2 < i_3 < i_4 < i_5 < i_6\}\) whose 5-sets all get the same color \(j\). Applying the defining condition for color \(j\) to the 5-sets \(\{i_1 < i_2 < i_3 < i_4 < i_5\}\) and \(\{i_2 < i_3 < i_4 < i_5 < i_6\}\) yields \(b(i_2, i_4) > p(i_1, i_3, i_5) > b(i_3, i_5) > p(i_2, i_4, i_6) > b(i_2, i_4)\) in \(L_j\). This contradiction means that \(G_n\) cannot have an interval representation in a \(t\)-dimensional poset if \(n\) is sufficiently large. \(\square\)

Let \(R_s(k, \ldots, k)\) denote the Ramsey number for \(t\)-coloring \(s\)-sets to force a set of size \(k\) whose \(s\)-sets all get the same color. We have shown that if \(n > R_5(M, M)\), where \(M = R_5(6, \ldots, 6)\) \((t\) colors), then the poset boxicity of \(G_n\), a graph on \(\binom{2}{s}\) vertices, exceeds \(t\). This lower bound for worst-case poset boxicity of a graph on \(N\) vertices grows unimaginably slowly.

References