Journal Title: Graph Theory, Combinatorics and Applications

Volume: 2
Issue:
Month/Year: 1991
Pages: 739 - 742

Article Author: Kierstead and Trotter

Article Title: A Note on Removable Pairs

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A Note on Removable Pairs

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ABSTRACT

A long standing conjecture in the dimension theory for finite ordered sets asserts that every ordered set (of at least three points) contains a pair whose removal decreases the dimension at most one. Two stronger conjectures have been made:

(1) If (x, y) is a critical pair, then $dim(P) \leq 1 + dim(P - \{x, y\})$.

(2) For every $x \in P$, there exists $y \in P - \{x\}$ so that $dim(P) \leq 1 + dim(P - \{x, y\})$.

K. Reuter has disproved conjecture 1 by constructing a four–dimensional poset P containing a critical pair (x, y) so that $dim(P - \{x, y\}) = 2$. In this note, we construct for every $n \geq 5$ an n–dimensional poset P_n containing a critical pair (x, y) so that $dim(P_n - \{x, y\}) = n - 2$. Point y is a maximal point of P_n.

1. Preliminaries

Recall that the *dimension* of a finite ordered set P in the least positive integer t so that there exist t linear extensions $L_1, L_2, \ldots, L_t$ so that $P = L_1 \cap L_2 \cap \ldots \cap L_t$. An incomparable pair (x, y) is called a *critical* pair if any point less than x is less than y and any point greater than y is greater than x. The dimension of P is the least t for which there exist t linear extensions of P so that for every critical pair (x, y), there is at least one i for which $y < x$ in L_i. We refer the reader to the survey article [3] by D. Kelly and W.T. Trotter and the chapters [6], [7] by Trotter for additional background information on dimension theory.

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1. Research supported in part by the Office of Naval Research.
2. Research supported in part by the National Science Foundation.

2. Removable Pairs

The following conjecture is one of the best known open problems in dimension theory and is a featured problem in **ORDER**. We believe the first reference to the conjecture is [1].

Conjecture 0 *If P is an ordered set having at least three points, then P contains a distinct pair (x, y) so that $\dim(P) \le 1 + \dim(P - \{x, y\})$.*

A pair $x, y \in P$ for which $\dim(P) \le 1 + \dim(P - \{x, y\})$ is called a *1–removable* pair, so that Conjecture 0 asserts that every poset contains a 1–removable pair.

The first reference to the following conjecture is apparently [5].

Conjecture 1 *Every critical pair is 1–removable.*

In [2], D. Kelly made the following conjecture which is also stronger than Conjecture 0.

Conjecture 2 *For every $x \in P$, there is a point $y \in P - \{x\}$ so that x, y is a 1-removable pair.*

K. Reuter [4] has disproved Conjecture 1 by constructing the ordered set shown in Figure 1. This ordered set P has dimension 4, (x, y) is a critical pair, and $\dim(P - \{x, y\}) = 2$. Note that y is a maximal point.

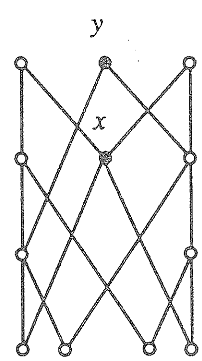

Figure 1

The purpose of this note is to show that Reuter's example is not an isolated phenomenon. To accomplish this, we will establish the following result.

Theorem *For every $n \geq 4$, there exists an n–dimensional ordered set P_n containing a critical pair (x, y) so that y is a maximal element in P_n, but (x, y) is not 1-removable, i.e., $dim(P - \{x, y\}) = n - 2$.*

Proof For $n = 4$, we have Reuter's example shown in Figure 1. For $n \geq 5$, the point set of P_n contains $4n - 4$ points labelled $\{a_i : 1 \leq i \leq n-2\} \cup \{b_i : 1 \leq i \leq n-2\} \cup \{c_i : 1 \leq i \leq n-2\} \cup \{d_i : 1 \leq i \leq n-2\} \cup \{x, y, z, w\}$. For all i, j with $1 \leq i, j \leq n-2$ and $i \neq j$, we have the cover relations $a_i <: b_j$ and $c_i <: d_j$. For each i with $1 \leq i \leq n-2$, we have $a_i <: y$, $c_i <: y$, $c_i < x$, $z <: b_i$, $w <: b_i$, $w < d_i$, $x <: b_i$, and $z < d_i$. We also have $w <: y$. We illustrate this definition with a diagram for P_n when $n = 5$.

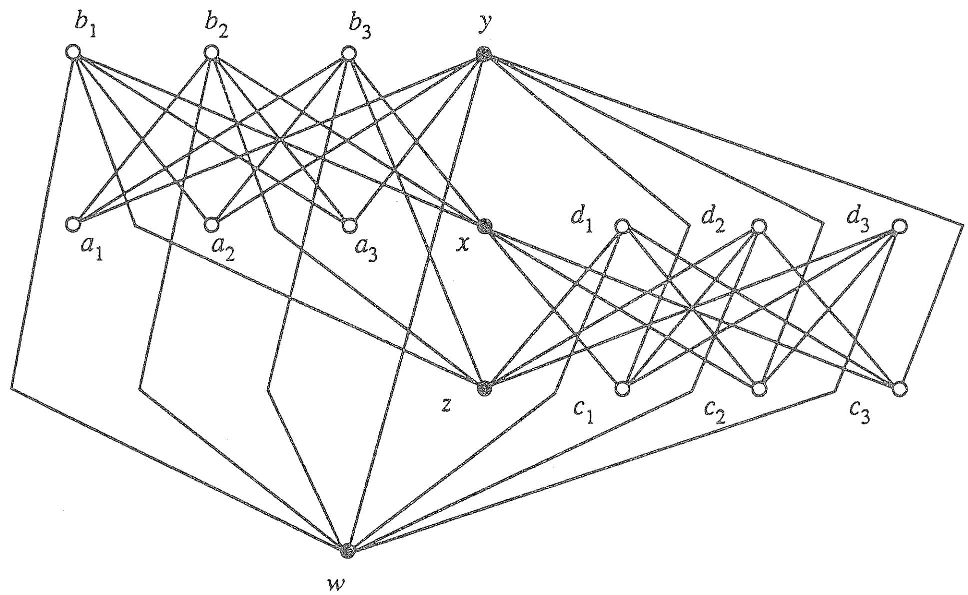

Figure 2

We first show that $\dim(P_n) \geq n$. To the contrary, suppose $\dim(P_n) \leq n - 1$, and let $L_1, L_2, \ldots, L_{n-1}$ be linear extensions whose intersection is P_n. Without loss of generality, we may assume that $b_i < a_i$ in L_i for $i = 1, 2, \ldots, n-2$. Thus we must have $x > y$ in L_{n-1} and $z > y$ in L_{n-1}. However, this implies that for each $i = 1, 2, \ldots, n-2$, there exists a unique $j_i \in \{1, 2, \ldots, n-2\}$ so that $c_i > d_i$ in L_{j_i}. Hence