Journal Title: Graph Theory, Combinatorics and Applications

Volume: 2
Issue: 
Month/Year: 1991
Pages: 739 - 742

Article Author: Kierstead and Trotter

Article Title: A Note on Removable Pairs

Call #: QA166 .I55 1988

Location: 3E - 7/9

William Trotter (wt48)
School of Mathematics
Georgia Tech
Atlanta, GA 30332

Faculty
Math

COPYRIGHT NOTICE:
This material may be protected by copyright law (Title 17 U.S. Code).
A Note on Removable Pairs

H.A. Kierstead
W.T. Trotter

Department of Mathematics
Arizona State University

ABSTRACT

A long standing conjecture in the dimension theory for finite ordered sets asserts that every ordered set (of at least three points) contains a pair whose removal decreases the dimension at most one. Two stronger conjectures have been made:

1) If $(x, y)$ is a critical pair, then $\dim(P) \leq 1 + \dim(P - \{x, y\})$.

2) For every $x \in P$, there exists $y \in P - \{x\}$ so that $\dim(P - \{x, y\}) = \dim(P) - 1 + \dim(P - \{x\})$.

K. Reuter has disproved conjecture 1 by constructing a four-dimensional poset $P$ containing a critical pair $(x, y)$ so that $\dim(P - \{x, y\}) = 2$. In this note, we construct for every $n \geq 5$ an $n$-dimensional poset $P_n$ containing a critical pair $(x, y)$ so that $\dim(P_n - \{x, y\}) = n - 2$. Point $y$ is a maximal point of $P_n$.

1. Preliminaries

Recall that the dimension of a finite ordered set $P$ in the least positive integer $t$ so that there exist $t$ linear extensions $L_1, L_2, \ldots, L_t$ so that $P = L_1 \cap L_2 \cap \ldots \cap L_t$. An incomparable pair $(x, y)$ is called a critical pair if any point less than $x$ is less than $y$ and any point greater than $y$ is greater than $x$. The dimension of $P$ is the least $t$ for which there exist $t$ linear extensions of $P$ so that for every critical pair $(x, y)$, there is at least one $i$ for which $y < x$ in $L_i$. We refer the reader to the survey article [3] by D. Kelly and W.T. Trotter and the chapters [6], [7] by Trotter for additional background information on dimension theory.

1. Research supported in part by the Office of Naval Research.
2. Research supported in part by the National Science Foundation.
2. Removable Pairs

The following conjecture is one of the best known open problems in dimension theory and is a featured problem in ORDER. We believe the first reference to the conjecture is [1].

Conjecture 0 If $P$ is an ordered set having at least three points, then $P$ contains a distinct pair $(x, y)$ so that $\text{dim}(P) \leq 1 + \text{dim}(P - \{x, y\})$.

A pair $x, y \in P$ for which $\text{dim}(P) \leq 1 + \text{dim}(P - \{x, y\})$ is called a 1-removable pair, so that Conjecture 0 asserts that every poset contains a 1-removable pair.

The first reference to the following conjecture is apparently [5].

Conjecture 1 Every critical pair is 1-removable.

In [2], D. Kelly made the following conjecture which is also stronger than Conjecture 0.

Conjecture 2 For every $x \in P$, there is a point $y \in P - \{x\}$ so that $x, y$ is a 1-removable pair.

K. Reuter [4] has disproved Conjecture 1 by constructing the ordered set shown in Figure 1. This ordered set $P$ has dimension 4, $(x, y)$ is a critical pair, and $\text{dim}(P - \{x, y\}) = 2$. Note that $y$ is a maximal point.

![Figure 1](image)

The purpose of this note is to show that Reuter's example is not an isolated phenomenon. To accomplish this, we will establish the following result.

Theorem For every $n \geq 4$, there exists a critical pair $(x, y)$ in a 1-removable, i.e., $\text{dim}(P)$, $P$.

Proof For $n = 4$, we have the ordered set of $P_4$ contains $4n - 2$ vertices, $\{c_i : 1 \leq i \leq n - 2\} \cup \{d_i : 1 \leq j \leq n - 2 \text{ and } j \neq i\}$, where $d_i$ is the point at which $1 \leq i \leq n - 2$, we have $d_i < c_i$. We also have $w < z < d_i$. We also have $w < z < d_i$.

We first show that $d_1 < \ldots < d_{n-1}$ and $L_1, L_2, \ldots, L_{n-1}$ be linearly independent. Without loss of generality, we may assume that $x > y$. We have $x > y$ in $L_{n-1}$ and $y < x \bigcup 2, \ldots, n - 2$, there exists a critical pair $(x, y)$. 


Theorem For every $n \geq 4$, there exists an $n$-dimensional ordered set $P_n$ containing a critical pair $(x, y)$ so that $y$ is a maximal element in $P_n$, but $(x, y)$ is not 1-removable, i.e., \( \dim(P - \{x, y\}) = n - 2 \).

Proof For $n = 4$, we have Reuter's example shown in Figure 1. For $n \geq 5$, the point set of $P_n$ contains $4n - 4$ points labelled $\{a_i : 1 \leq i \leq n - 2\} \cup \{b_i : 1 \leq i \leq n - 2\} \cup \{c_i : 1 \leq i \leq n - 2\} \cup \{d_i : 1 \leq i \leq n - 2\} \cup \{x, y, z, w\}$. For all $i, j$ with $1 \leq i, j \leq n - 2$ and $i \neq j$, we have the cover relations $a_i < b_j$ and $c_i < d_j$. For each $i$ with $1 \leq i \leq n - 2$, we have $a_i < y, c_i < y, c_i < x, z < b_i, w < b_i, w < d_i, x < b_i$, and $z < d_i$. We also have $w < y$. We illustrate this definition with a diagram for $P_n$ when $n = 5$.  

![Figure 2](image-url)

Figure 2

We first show that $\dim(P_n) \geq n$. To the contrary, suppose $\dim(P_n) \leq n - 1$, and let $L_1, L_2, ..., L_{n-1}$ be linear extensions whose intersection is $P_n$. Without loss of generality, we may assume that $b_i < a_i$ in $L_i$ for $i = 1, 2, ..., n - 2$. Thus we must have $x > y$ in $L_{n-1}$ and $z > y$ in $L_{n-1}$. However, this implies that for each $i = 1, 2, ..., n - 2$, there exists a unique $j_i \in \{1, 2, ..., n - 2\}$ so that $c_i > d_{j_i}$ in $L_{j_i}$. Hence
w > x in L_{n-1} also. But this implies that w > x > y in L_{n-1} which is impossible since w < y in P_n. The contradiction completes the proof that \( \text{dim}(P_n) \geq n \).

We observe that y is maximal element of P_n and that (x, y) is a critical pair. We now show that \( \text{dim}(P_n - (x, y)) \leq n - 2 \). To accomplish this, consider the poset \( Q_n = P_n - (x, y) \). In Q_n, we observe that z and w have duplicated holdings so that \( \text{dim}(Q_n - (z)) = \text{dim}(Q_n) \). Let \( Q'_n = Q_n - (z) \). We show that \( Q'_n \) has \( n - 2 \) linear extensions which intersect to give \( Q'_n \). Let \( A = \{a_1, a_2, ..., a_{n-2}\}, B = \{b_1, b_2, ..., b_{n-2}\}, C = \{c_1, c_2, ..., c_{n-2}\} \) and \( D = \{d_1, d_2, ..., d_{n-2}\} \).

For \( i = 1, 2, L_i \) is any linear extension of \( Q'_n \) so that \( A - \{a_i\} < C - \{c_i\} < w < d_i < c_i < b_i < a_i < B - \{b_i\} < D - \{d_i\} \). For \( i = 3, 4, ..., n - 2, L_i \) is any linear extension of \( Q'_n \) so that \( w < C - \{c_i\} < d_i < c_i < D - \{d_i\} < A - \{a_i\} < b_i < B - \{b_i\} \). It is easy to see that any family constructed by these rules forms a realizer. With this observation, our proof is complete. \( \Box \)

We pause to note that the construction given in the preceding family for \( \{P_n : n \geq 5\} \) does not work for \( n = 4 \). In this case, \( \text{dim} P_n = 4 \), but \( \text{dim}(P_n - (x, y)) = 3 \). Thus to handle the case \( n = 4 \), we need a special example, and Reuter's construction suffices.

3. Concluding Remarks

We view the results of this note as providing additional evidence as to the difficulty of Conjecture 0, but we are unable to decide whether our theorem argues for or against the conjecture. It is easy to see that the examples satisfy Conjecture 2, so at least this stronger form of the original conjecture remains open.

REFERENCES


On the Number \( V(G) \) of a \( (V, E) \)-graph

Let \( V(G) \) be the set of vertices of a \( (V, E) \)-graph \( G \) which is not joined by and edge of \( G \) and is independent and is not a subset of \( V(G) \). Let \( m(G) \) be the number of m.i.s. of \( G \). It is known that for any \( (V, E) \)-graph \( G \), there are some subrange of the vertices and some edge of \( G \) which are described. The result of this note is to give an upper bound for \( m(G) \) and a lower bound for \( V(G) \) and \( E(G) \).

1. Introduction

The definitions of the graph of \( G \) are given in the introduction of this paper. Let \( G \) be an undirected labelled graph with \( V(G) = V, E(G) = E \), and \( m(G) \) is denoted by \( |A| \). We use the notation \( (v, e) \)-graph. Let \( \{v; y\} \) be a set of vertices and \( \{e; y\} \) be a set of edges. The intersection of \( G \) obtained from \( G \) by removing \( \{v; y\} \) and \( \{e; y\} \) (or \( G \) without the corresponding edge), the completely disconnected \( G \) is \( V(G) \) and \( E(G) \) are the set of vertices, and the complete graph \( G + H \) is \( V(G) \cup V(H), E(G) \cup E(H) \), and \( E(G,H) \). By \( G + H \) we mean the graph obtained from \( G \) by adjoining \( E(G,H) \). By \( N_G(x) \) we denote the set of neighbors of \( x \) in \( G \).

1 This paper is based on a paper of H. S. Wilf at the University of Pennsylvania.