Induced Matchings in Cubic Graphs

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ABSTRACT

In this paper, we show that the edge set of a cubic graph can always be partitioned into 10 subsets, each of which induces a matching in the graph. This result is a special case of a general conjecture made by Erdös and Nešetřil: For each $d \geq 3$, the edge set of a graph of maximum degree d can always be partitioned into $\lfloor 5d^2/4 \rfloor$ subsets each of which induces a matching. © 1993 John Wiley & Sons, Inc.

1. INTRODUCTION

Throughout this paper, we consider colorings of the edges of a graph with positive integers. Formally, a *t-coloring* of a graph $G = (V, E)$ is a map ψ: $\rightarrow \{1, 2, \ldots, t\}$. A t-coloring is *proper* if $\psi(e) = \psi(f)$ and $e \neq f$ imply that the edges e and f have no common endpoints. Of course, the *chromatic index* of a graph G is the least t for which G has a proper t-coloring. Note that whenever ψ is a proper t-coloring of a graph $G = (V, E)$ and $\alpha \in \{1, 2, \ldots, t\}$, then the edges in $\mathcal{M} = \{e \in E : \psi(e) = \alpha\}$ form a matching in G.

An *induced matching* $\mathcal{M}$ in a graph $G = (V, E)$ is a matching such that no two edges of $\mathcal{M}$ are joined by an edge of G. In other words, an induced matching is an induced subgraph in which every vertex has degree one. A

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strong t-*coloring* of G is a proper t-coloring such that edges with the same color form an induced matching of G. The *strong chromatic index,* $\mathrm{sq}(G)$, is the least t for which G has a strong t-coloring. For convenience, two edges of a graph G will be called *neighbors* in G if they do not form an induced matching, i.e., if either they are incident (share an end point), or they are joined by an edge. Also, we will use the abbreviation $[t]$ for $\{1, 2, \dots, t\}$.

At a seminar in Prague at the end of 1985, Erdös and Nešetřil formulated the following Vising-type problem: Given an upper bound for $\mathrm{sq}(G)$ in terms of $\Delta(G)$, the maximum degree of G. They also conjectured (see [4]) that

$$\mathrm{sq}(G) \leq \frac{5}{4}\Delta^2(G). \qquad (EN)$$

When $\Delta(G)$ is even, this conjecture, if true, is best possible. When $\Delta(G)$ is odd, it may be possible to improve it by some term that is linear in $\Delta(G)$. In any case, this conjecture appears to be quite difficult. It is easy to see that (EN) is true when $\Delta(G) \leq 2$. In [8], it is shown that $\mathrm{sq}(G) \leq 23$ for any graph G with $\Delta(G) = 4$. The trivial upper bound $\mathrm{sq}(G) \leq 2\Delta^2(G) - 2\Delta(G) + 1$ follows from the observations that (1) the color of an edge of G is affected only by the color of its neighbors, and (2) the number of neighbors of any edge of G does not exceed $2\Delta^2(G) - 2\Delta(G)$. But even to improve this inequality to $\mathrm{sq}(G) \leq (2 - \epsilon)\Delta^2(G)$, for some absolute constant $\epsilon > 0$, seems to be very hard.

It is possible that these difficulties are connected with a result of K. Cameron who proved [2] that the problem of determining whether there is an induced matching of size at least k in G is NP-complete even when G is a bipartite graph. In [3], the authors solved a problem posed by J. Bond and independently by Erdös and Nešetřil: What is the maximum number of edges in a graph in which any two edges are neighbors? They showed that such a graph has at most $\frac{5}{4}\Delta^2(G)$ edges, with a linear term improvement when $\Delta(G)$ is odd. It is reasonable to view this result as providing some evidence that conjecture (EN) is true.

In this paper, we prove conjecture (EN) when $\Delta(G) \leq 3$. In fact, we show

Theorem. If G is a graph with $\Delta(G) \leq 3$, then $\mathrm{sq}(G) = 10$. ∎

This theorem answers a specific question posed to us by A. Gyárfás (see also [5] and [6], where many interesting results and problems on the strong chromatic index are stated). The inequality given in our theorem is best possible as there exist graphs with $\Delta(G) \leq 3$ and $\mathrm{sq}(G) = 10$. Two such graphs are (1) an 8-gon with all four diagonals, and (2) a 5-gon in which two consecutive vertices have been multiplied by 2. However, it is not clear whether there are infinitely many cubic graphs with this property.

When preparing this paper, we learned that L. Andersen [1] has also obtained the same theorem. Andersen's proof uses different methods and emphasizes algorithmic aspects.

2. PROOF OF THE THEOREM

The proof requires two lemmas. The arguments for these lemmas and for our theorem involve the construction of a strong 10-coloring ψ of a graph $G = (V, E)$ in an inductive manner. In most cases, ψ will be an extension of a strong 10-coloring ψ_0 of a subgraph (or suitably modified subgraph) of G. Furthermore, we will take care to ensure that it never happens that two edges of G belong to the subgraph, are not neighbors in the subgraph, but are neighbors in G. We will let F denote the edges of G not already assigned colors by ψ_0. For each $e \in F$, we let $S(e)$ denote the set of colors that have not been assigned by ψ_0 to edges that are neighbors of e in the graph G.

In some situations, we will argue that ψ can be constructed by appealing to Hall's theorem [7]. Recall that if $\mathcal{A} = \{S(e): e \in F\}$ is a family of sets, then a set $\{\alpha_e: e \in F\}$ is a *system of distinct representatives* (abbreviated **SDR**) for $\mathcal{A}$ provided

(1) $\alpha_e \neq \alpha_f$, for all $e, f \in F$ with $e \neq f$, and
(2) $\alpha_e \in S(e)$, for all $e \in F$.

Hall's theorem asserts that $\mathcal{A}$ has an **SDR** if and only if $|\cup\{S(E): e \in F'\}| \geq |F'|$, for every $F' \subseteq F$. Whenever $\mathcal{A}$ has an **SDR**, the task of extending ψ_0 to a strong 10-coloring ψ is easy. We just take $\psi(e) = \alpha_e$, for every $e \in F$.

In other cases, for a graph $G = (V, E)$, we will define a family $\mathcal{A} = \{S_e: e \in E\}$ and argue directly that there exists a strong 10-coloring ψ with $\psi(e) \in S(e)$, for all $e \in E$. Sometimes, this will be accomplished by describing a linear order on F so that the *First Fit* algorithm may be applied. This algorithm chooses $\psi(e)$ to be the least (first) integer in $S(e)$ not previously assigned to a neighbor of e. Still other cases will require some ad hoc reasoning.

Lemma 1. If $G = (V, E)$ is a connected graph with $\Delta(G) \leq 3$, and G has a vertex v with $1 \leq \deg(v) \leq 2$, then $\mathrm{sq}(G) \leq 10$.

Proof. We proceed by induction on $|E|$, the number of edges in G. The result is trivially true when $|E| \leq 10$. Now consider a graph G, and assume that the conclusion of the lemma holds for any graph with fewer edges. Each nontrivial component of $G - v$ satisfies the inductive hypothesis, so we may choose a strong 10-coloring ψ_0 of $G - v$. Let F denote the set of edges in G that are incident with v. Each $e \in F$ has at most 8 neighbors in $G - v$, so $|S(e)| \geq 2$. As $|F| = \deg(v) \leq 2$, the family $\mathcal{A} = \{S(e): e \in F\}$ has an **SDR**. It follows that ψ_0 can be extended to a strong 10-coloring ψ of G. ∎

For each $n \geq 3$, let T_n be the tree (actually, a caterpillar) with vertex set $\{v_0, v_1, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and edge set $\{e_i = v_{i-1}v_i : 1 \leq i \leq n\} \cup \{f_i = v_i u_i : 1 \leq i \leq n\}$

Lemma 2. Let $n \geq 3$ and suppose that every edge e of the tree T_n is assigned a set of colors $S(e)$, so that

(1) $|S(e_1)| \geq 1$ and $|S(e_2)| \geq 3$;
(2) $|S(e_i)| \geq 5$, for $i = 3, 4, \ldots, n$;
(3) $|S(f_1)| \geq 2$;
(4) $|S(f_i)|4$, for $i = 2, 3, \ldots, n - 1$; and either
(5a) $|S(f_n)| \geq 4$, or
(5b) $|S(f_n)| \geq 3$ and $|S(e_1) \cup S(f_1)| \geq 3$.

Then there is such a strong coloring ψ of T_n so that $\psi(e) \in S(e)$, for every edge e in T_n.

Proof. In case (5a) holds, the required strong coloring can be obtained by applying First Fit to color the edges of T_n in the following order:

$$e_1, f_1, e_2, f_2, e_3, \ldots, e_{n-1}, f_{n-1}, e_n, f_n.$$

Note that each edge e in the tree T_n has at most at most $|S(e)| - 1$ neighbors preceding it in this list.

In case (5b) holds, First Fit works until the very end, but might possibly fail when coloring f_n. Apparently, the three edges e_{n-1}, f_{n-1}, and e_n could be assigned three (necessarily distinct) colors from $S(f_n)$ leaving no satisfactory choice for f_n. So to complete the proof, we argue by induction on n. Suppose first that $n = 3$.

Consider any distinct pair α, β with $\alpha \in S(e_1)$ and $\beta \in S(f_1)$. Then consider the remaining four sets $S'(e_2) = S(e_2) - \{\alpha, \beta\}$, $S'(f_2) = S(f_2) - \{\alpha, \beta\}$, $S'(e_3) = S(e_3) - \{\alpha, \beta\}$ and $S'(f_3) = S(f_3)$. These four sets have an **SDR** unless $S(f_3) = S'(e_3)$. We may therefore assume that $|S(e_3)| = 5$, $\alpha, \beta \in S(e_3)$, and $\alpha, \beta \notin S(f_3)$. That $S(e_1) \cup S(f_1)$ contains at least three elements allows us to repeat this argument for three distinct pairs $\{(\alpha_i, \beta_i) : i = 1, 2, 3\}$ and conclude that each of the (at least three) elements in the union of these pairs belongs to $S(e_3) - S(f_3)$. But this requires $|S(e_3)| \geq 6$. The contradiction completes the proof of the case $n = 3$.

For larger values of n, note that the condition $|S(e_1) \cup S(f_1)| \geq 3$ allows us to choose $\alpha = \psi(e_1) \in S(e_1)$ and $\beta = \psi(f_1) \in S(f_1)$ so that (5b) holds for the remaining edges when α and β are removed from $S(e_2)$, $S(f_2)$, and $S(e_3)$. Since the remaining edges form a copy of the tree T_{n-1}, the proof is complete. ∎

With the two lemmas in hand, we are now ready to present the central part of the proof of the theorem. We proceed by induction on the number of vertices in the graph. Consider a graph $G = (V, E)$ with $\Delta(G) \leq 3$ and assume that the theorem holds for any graph having fewer vertices than G. We show that $\text{sq}(G) \leq 10$. In view of Lemma 1, we may assume G is connected and 3-regular. Now let n denote the *girth* of G, i.e., the minimum number of vertices in a cycle in G. Choose a minimum length cycle C in G and label the vertices of C as $v_1, v_2, \ldots, v_n$ so that $e_i = v_{i-1}v_i \in E$, for each $i = 2, 3, \ldots, n$, and $e_1 = v_n v_1 \in E$. In what follows, we interpret subscripts cyclically, so that $v_{n+1} = v_1$, etc.

For each $i = 1, 2, \ldots, n$, let u_i be the unique vertex of $V - C$ adjacent to v_i. Also, let $U = \{u_1, u_2, \ldots, u_n\}$. Then let $H = (C \cup U, F)$ be the subgraph of G induced by $C \cup U$. Note that the vertices on the cycle C are distinct, but this may not be the case for the vertices in U. We may have $u_i = u_j$ when $i \neq j$. In any case, the edges in $\{e_1, e_2, \ldots, e_n\} \cup \{f_1, f_2, \ldots, f_n\}$ are distinct.

According to Lemma 1, there exists a strong 10-coloring ψ_0 of $G - H$. For each edge $e \in F$, let $S(e) = \{\alpha \in [10]$: there is no neighbor of e in G so that e is an edge of $G - H$ mapped by ψ_0 to $\alpha\}$. Observe that for each $i = 1, 2, \ldots, n$, the edge $e_i \in F$ has at most 4 neighbors in $G - H$, the edges incident with u_{i-1} and u_i. Similarly, for each $i = 1, 2, \ldots, n$, the edge $f_i \in F$ has at most 6 neighbors in $G - H$. Thus, $|S(e_i)| \geq 6$ and $|S(f_i)| \geq 4$, for $i = 1, 2, \ldots, n$. Note that $|S(e_i) \cap S(f_i)| \geq 2$ and $|S(e_i) \cap S(f_{i-1})| \geq 2$, for each $i = 1, 2, \ldots, n$.

It is easy to see that two edges of $G - H$ are neighbors in $G - H$ if and only if they are neighbors in G. However, the subgraph H does not have this property in general. The remainder of the argument is divided into 5 cases according to the relative size of the girth of G. The basic idea is that we will extend the strong 10-coloring to a strong 10-coloring ψ ψ_0 of G.

Case 1. $n = 3$.

As $|S(e_i)| \geq 6$ and $|S(f_i)| \geq 4$, for each $i = 1, 2, 3$, the family $\mathcal{A} = \{S(e) : e \in F\}$ has an **SDR**, which we may label as $\{\psi(e) : e \in F\}$. The map ψ is then extended to all of E by setting $\psi(e) = \psi_0(e)$ when $e \in E - F$.

Case 2. $n = 4$.

Note that the edges f_1 and f_3 are neighbors in G if and only if $u_1 u_3 \in E$ or $u_1 = u_3$. An analogous statement holds for f_2 and f_4. All the other pairs of edges of H are neighbors in H and have to be colored by distinct colors. To obtain the desired extension ψ, we observe that

(a) If f_i and f_{i+2} are neighbors in G, then $|S(e_{i+1}) \cup S(e_{i+2})| \geq 7$;
(b) If f_1 and f_3 are neighbors, and f_2 and f_4 are neighbors in G, then $|\bigcup_{i=1}^{4} S(e_i)| \geq 8$; and
(c) If f_i and f_{i+2} are not neighbors, then either $S(f_i) \cap S(f_{1+2}) \neq \emptyset$, or $|S(f_i) \cup S(f_{i+2})| \geq 8$.

If $\mathcal{A} = \{S(e) : e \in F\}$ has an **SDR**, then we are done, so we may suppose that it does not. It follows easily that for each $i = 1, 2$, f_i and f_{i+2} are not neighbors and that $S(f_i) \cap S(f_{i+2}) \neq \emptyset$. Choose $\alpha \in S(f_1) \cap S(f_3)$. If $(S(f_2) \cap S(f_4)) - \{\alpha\} = \emptyset$, then $\mathcal{A}' = \{S(e) - \{\alpha\} : e \in F - \{f_1, f_3\}\}$ has an **SDR**. So we may assume that there exists $\beta \neq \alpha$ with $\beta \in S(f_2) \cap S(f_4)$. Then $\mathcal{A}'' = \{S(e_i) - \{\alpha, \beta\} : i = 1,2,3,4\}$ has an **SDR**.

Case 3. $n = 5$.

In this case, we know that the set $U = \{u_1, u_2, \ldots, u_n\}$ is a collection of n distinct vertices. Otherwise, the girth of G would be less than 5. Also, for each $i = 1,2,\ldots,5$, the edges f_i and e_{i+2} are not neighbors in G.

Consider the family $\mathcal{B} = \{B_1, B_2, \ldots, B_5\}$, where $B_i = S(f_i) \cap S(e_{i+2})$, for $i = 1,2,\ldots,5$. If $\mathcal{B}$ has an **SDR**, we obtain the desired strong coloring of H. If not, choose a maximum size subset $J \subseteq [5]$ for which the subfamily $\mathcal{B}' = \{B_j : j \in J\}$ has an **SDR**. Let $|J| = w$, and let $W = \{\alpha_j : j \in J\}\emptyset$ be an **SDR** of $\mathcal{B}'$. Now let $I = [5] - J$, $F' = \cup\{\{f_j, e_{j+2}\} : j \in J\}$, $F'' = F - F'$ and $\mathcal{B}'' = \{S(e) - W : e \in F''\}$. Again, if $\mathcal{B}''$ has an **SDR**, we get the desired strong 10-coloring of H.

Suppose therefore that $\mathcal{B}''$ does not have an **SDR**. Observe that $\mathcal{B}''$ has a total of $10 - 2w$ sets. Of these, there are $5 - w$ sets of the form $S(f_i) - W$. Each of these sets contains at least $4 - w$ elements. Similarly, $\mathcal{B}''$ contains $5 - w$ sets of the form $S(e_{i+2}) - W$, and each of these sets contains at least $6 - w$ elements. From the maximality of J, $B_i - W = \emptyset$, for every $i \in I$, and therefore

$$|(S(f_i) - W) \cup (S(e_{i+2}) - W)| \geq 10 - 2w \quad \text{for all } i \in I. \tag{1}$$

Thus, the fact that $\mathcal{B}''$ does not have an **SDR** forces $|\cup\{S(f_i) - W : i \in I\}| = |I| - 1 = 4 - w$. This implies

(1) $|S(f_i)| = 4$, for each $i \in I$,
(2) $S(f_{i_1}) - W = S(f_{i_2}) - W$, for all $i_1, i_2 \in I$, and
(3) $W \subseteq S(f_i)$, for each $i \in I$.

Furthermore, for each $i = 1,2,\ldots,5$, the edges f_i and f_{i+2} are neighbors in G if and only if $u_i u_{i+2} \in E$. But $u_i u_{i+2} \in E$ implies that f_i has at most 5 neighbors in $G - H$, which contradicts (1). Hence, for each $i \in I$, f_i and f_{i+2} are not neighbors in G. Similarly, for each $i \in I$, f_i and f_{i-2} are not neighbors in G.

Note that $w > 0$, for $w = 0$ requires $S(f_i) \cap S(e_{i+2}) = \emptyset$, for each $i = 1,2,\ldots,5$. Clearly, this would imply that $\mathcal{A} = \{S(e) : e \in F\}$ has an **SDR**. Since $w < 5$, we may assume without loss of generality that $1 \in J$ and $3 \in I$. For $j \in J - \{1\}$, set $\psi(f_j) = \psi(e_{j+2}) = \alpha_j$. Noting that the edges f_1 and f_3 are not neighbors, we set $\psi(f_1) = \psi(f_3) = \alpha_1$ and $F''' = (F'' - \{f_3\}) \cup \{e_3\}$. The family $\mathcal{B}''' = \{S(e) - W : e \in F'''\}$ has cardinality $10 - 2s$ with $4 - s$ sets of cardinality at least $4 - s$ and

$6 - s$ sets of cardinality at least $6 - s$. It is easy to see that $\mathcal{B}'''$ has an **SDR**, and the existence of ψ follows.

Case 4. $n \geq 7$.

In this case, $\boldsymbol{H}$ is an induced subgraph of $\boldsymbol{G}$. This means two edges of $\boldsymbol{H}$ are neighbors in $\boldsymbol{H}$ if and only if they are neighbors in $\boldsymbol{G}$. Thus, $g_1 = u_1u_3 \in E$ and $g_2 = u_2u_4 \notin E$. We now start with a strong 10-coloring ψ_0' of the graph $\boldsymbol{G}'$ formed by adding the edges g_1 and g_2 to $\boldsymbol{G} - \boldsymbol{H}$. Note that the strong 10-coloring ψ_0' exists by Lemma 1. Let F' denote the set of those edges of $\boldsymbol{G}$ not assigned colors by ψ_0', and for each $e \in F'$, let $S'(e)$ denote the set of colors which have not been assigned by ψ_0' to edges of $\boldsymbol{G}$ that are neighbors of e in $\boldsymbol{G}$. For each $i = 1, 2, \ldots, n$, we denote by α_i and β_i the colors of edges of $\boldsymbol{G} - \boldsymbol{H}$ incident with u_i. Thus, for example, $S'(e_2) = [10] - \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$. Note that $\alpha_i, \beta_i \notin S'(f_i)$, for each $i = 1, 2, \ldots, n$. Also note that the presence of the edges g_1 and g_2 in $\boldsymbol{G}'$ imply $\{\alpha_i, \beta_i\} \cap \{\alpha_{i+2}, \beta_{i+2}\} = \emptyset$ for $i = 1, 2$. Furthermore, $S'(f_i) \cap S'(f_{i+2}) \neq \emptyset$ for $i = 1, 2$, because (at least) $\psi_0'(g_i) \in S'(f_i) \cap S'(f_{i+2})$.

Now we show how to obtain a strong 10–coloring ψ of $\boldsymbol{G}$. We consider three subcase depending on the value of $t = |\{\alpha_1, \beta_1\} \cap \{\alpha_2, \beta_2\}|$.

Subcase 4a. $t = 2$. Set $\psi(f_1) = \psi(f_3) = \alpha = \psi_0'(g_1)$. For each edge $e \in F' - \{f_1, f_3\}$, set $S''(e) = S'(e)$, if e is not a neighbor of either f_1 or f_3; otherwise, set $S''(e) = S'(e) - \{\alpha\}$. Observe that

$$
\begin{aligned}
&|S''(f_2)| \geq 3;\ |S''(f_4)| \geq 3;\\
&|S''(f_i)| \geq 4 \text{ for } i = 5, 6, \ldots, n-1;\\
&|S''(e_i)| \geq 5; \text{ for } i = 1, 3, 4, 5, n; \text{ and}\\
&|S''(e_i)| \geq 6 \text{ for } i = 6, 7, \ldots, n-1.
\end{aligned}
$$

Then we remove all elements of $S(f_2)$ from $S''(e_4)$. The edges in $\{e_4, e_5, \ldots, e_n, f_4, f_5, \ldots, f_n\}$ form a copy of T_{n-3} satisfying the hypothesis of Lemma 2, case (5b). Thus, there is a strong 10-coloring ψ_0 of this copy of T_{n-3}. To define our strong 10-coloring ψ, we will set $\psi(e) = \psi_0'(e)$ if e is an edge of $\boldsymbol{G}$ assigned a color by ψ_0', and we will set $\psi(e) = \psi_0(e)$ if e is an edge in the copy of T_{n-3} colored by ψ_0. It remains only to color e_1, e_2, e_3, and f_2.

For each $e \in F = \{e_1, e_2, e_3, f_2\}$, let $S(e)$ be the set of colors not already assigned to a neighbor of e. Since $\{\alpha_1, \beta_1\} = \{\alpha_2, \beta_2\}$, we know that $|S''(e_2)| \geq 8$. So, $|S(e_2)| \geq 4$. Similarly, $|S(e_2)| \geq 2$ and $|S(e_n)| \geq 1$. Finally, $|S(f_2)| \geq 3$, because f_1 and f_3 have been assigned the same color, and the color assigned to e_4 does not belong to $S(f_4)$. Therefore, $\mathcal{A} = \{S(e) : e \in F\}$ has an **SDR**, and the map ψ exists.

Subcase 4b. $t = 1$. Let $\alpha_1 = \alpha_2$. Because the edges g_1 and g_2 belong to G', we know $\alpha_1 \notin \{\alpha_i, \beta_i\}$ for $i = 3,4$. So, we set $\psi(e_4) = \alpha_1$. Furthermore, we set $\psi(f_2) = \psi(f_4) = \psi_0'(g_2) \neq \alpha_1$. As in the preceding case, we let $S''(e)$ denote the set of colors available to color an edge e not already colored. If $\beta_1 \in S''(e_5)$, we remove it. There are at least three other colors in $S''(e_5)$. Hence the copy of T_{n-3} formed by $\{e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1\}$ admits a strong 10-coloring ψ_0 by Lemma 2, case (5b). It remains only to color the edges in $F = \{e_2, e_3, f_3\}$. Now, for each $e \in F$, we let $S(e)$ denote the set of available colors. Note that $|S(e_2)| \geq 2$, $|S(e_3)| \geq 2$ and $|S(f_3)| \geq 1$. It follows that the family $\mathcal{A} = \{S(e) : e \in F\}$ has an **SDR**, and the definition of ψ can be completed, unless $S(e_2)$ and $S(e_3)$ are identical 2-element sets.

If $\alpha = \beta_1$, then $S(e_2) \geq 3$. So we may assume that $\alpha \neq \beta_1$. Then, since $t = 1$, it follows that $\beta_1 \in S(e_3) - S(e_2)$. In either situation, the map ψ can be defined.

Subcase 4c. $t = 0$. By duality, we also assume that $\{\alpha_3, \beta_3\} \cap \{\alpha_4, \beta_4\} = \emptyset$.

(i) Suppose that $\{\alpha_1, \beta_1\} \cap S(f_2) \neq \emptyset\}$, say $\alpha_1 \in S(f_2)$. Set $\psi(f_2) = \alpha_1$ and $\psi(e_3) = \beta_1$. Note that this choice of color for e_3 is permissible, since $t = 0$. Also, set $\psi(f_1) = \psi(f_3) = \alpha = \psi_0'(g_1)$, and note that $\alpha \notin \{\alpha_1, \beta_1\}$. Now use Lemma 2 to color the copy of T_{n-3} formed by the edges $\{e_4, e_5, \ldots, e_n, f_4, f_5, \ldots, f_n\}$. It remains only to color the edges in $F = \{e_1, e_2\}$. However, $|S(e_1)| \geq 1$ and $|S(e_2)| \geq 2$, so this can be done. It follows that we may assume that $\{\alpha_1, \beta_1\} \cap \{S(f_2)\} = \emptyset$.

(ii) Suppose next that $\{\alpha_1, \beta_1\} \cap S(f_3) \neq \emptyset$, say $\alpha_1 \in S(f_3)$. Set $\psi(f_3) = \alpha_1$ and $\psi(e_3) = \beta_1$. Use Lemma 2, case (5b) to color the copy of the tree T_{n-2} formed by the edges $\{e_4, e_5, \ldots, e_1, f_4, f_5, \ldots, f_1\}$. It remains to color the edges in the set $F = \{e_2, f_2\}$. Observe that $|S''(e_2)| \geq 1$ and $|S''(f_2)| \geq 2$, since $\alpha_1, \beta_1 \notin S''(f_2)$. It follows that the definition of ψ can be completed. So, we may now assume that $\{\alpha_1, \beta_1\} \cap \{S(f_3)\} = \emptyset$.

(iii) Suppose that $\{\alpha_2, \beta_2\} = \{\alpha_3, \beta_3\}$. Set $\psi(f_1) = \psi(f_3) = \psi_0'(g_1) = \alpha$ and remove all elements of $\{\alpha\} \cup S''(f_2)$ from $S''(e_4)$. Apply Lemma 2, case (5b) to color the copy of the tree T_{n-3} formed by the edges $\{e_4, e_5, \ldots, e_n, f_4, f_5, \ldots, f_n\}$. Now it remains to color the edges in $F = \{e_1, e_2, e_3, f_2\}$. Observe that $|S(e_1)| \geq 1$ and $|S(e_2)| \geq 2$, $|S(e_3)| \geq 4$ and $|S(f_2)| \geq 3$. It follows that the definition of ψ can be completed. Therefore, we may assume that $\{\alpha_2, \beta_2\} \neq \{\alpha_3, \beta_3\}$. In what follows, we label these elements so that $\alpha_2 \notin \{\alpha_3, \beta_3\}$.

(iv) Suppose that $\{\alpha_1, \beta_1\} \cap \{\alpha_4, \beta_4\} \neq \emptyset$, say $\beta_1 = \beta_4$. Set $\psi(e_3) = \beta_1$ and $\psi(e_4) = \alpha_2$. Note that the choice of the color for e_4 makes use of the convention given at the end of the preceding paragraph. Now choose $\psi(f_3) = \alpha \in S''(f_3) - \{\alpha_2, \beta_1\}$ and

$\psi(f_4) = \beta \in S''(f_4) - \{\alpha, \alpha_2, \beta_1\}$. Note that there are still three colors available for e_5, because $\beta_1 = \beta_4 \notin S''(e_5)$. It follows that we may color the edges in the copy of T_{n-2} formed by the edges in $\{e_5, e_6, \ldots, e_2, f_5, f_6, t, \ldots, f_2\}$. When this is done, all edges in $\boldsymbol{G}$ have been colored, and the definition of ψ may be completed. We may therefore assume that $\{\alpha_1, \beta_1\} \cap \{\alpha_4, \beta_4\} = \emptyset$.

(v) Finally, suppose that $\{\alpha_1, \beta_1\} \cap S(f_2) = \{\alpha_1, \beta_1\} \cap S(f_3) = \{\alpha_1, \beta_1\} \cap \{\alpha_4, \beta_4\} = \emptyset$. Set $\psi(f_2) = \psi(f_4) = \psi_0'(g_2) = \alpha$, $\psi(e_3) = \alpha_1$ and $\psi(e_4) = \beta_1$. We then color the edges in the copy of T_{n-3} formed by $\{e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1\}$ using Lemma 2, case (5b). It remains to color the edges in $F = \{e_2, e_3\}$. Observe that $|S(e_2)| \geq 1$ and $|S(f_3)| \geq 2$, so the definition of ψ can be completed.

Case 5. $n = 6$.

First, observe that whenever $u_i u_{i+3} \in E$, the edges f_i and f_{i+3} are neighbors and have to be colored by distinct colors. But, when this happens, f_i and f_{i+3} have at most 5 neighbors in $\boldsymbol{G}$, and thus $|S(f_i)| \geq 5$ and $|S(f_{i+3})| \geq 5$.

When $u_i u_{i+3} \notin E$, for $i = 1,2,3$, $\boldsymbol{H}$ is an induced subgraph of $\boldsymbol{G}$. Furthermore, the proof for the case $n \geq 7$ applies to this case as well. Note that in the construction of the copies of trees, the smallest tree considered is T_{n-3}. Since $n = 6$, Lemma 2 can still be applied.

Now suppose that $\{u_i u_{i+3} : i \in [3]\} \cap E \neq \emptyset$ and consider the individual steps in the proof of the case $n \geq 7$. Except for the subcase $t = 0$, for each $i = 1,2,3$, the edges f_i and f_{i+3} are not colored at the same stage of the definition of ψ. For example, in the subcase $t = 2$, at the first stage, we color f_1 and f_3; at the second, we color T_{n-3}; and at the third stage, we color e_1, e_2, e_3, and f_2. So, when $u_i u_{i+3} \in E$ and f_i is colored before f_{i+3}, we remove the color assigned to f_i from $S(f_{i+3})$. The slack provided by the inequality $|S(f_{i+3})| \geq 5$ makes this possible.

Finally, in the subcase $t = 0$, we observe that there is a unique value of i for which f_i and f_{i+2} belong to the copy of T_{n-2} that is colored by appeal to Lemma 2. After relabeling, we may assume that the edges in this copy belong to $F = \{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4\}$. If the edge $u_1 u_4 \notin E$, the original argument works. If $u_1 u_4 \in E$, then we observe that the following inequalities hold: $|S(e_1)| \geq 3$, $|S(e_2)| \geq 4$, $|S(e_3)| \geq 6$, $|S(e_4)| \geq 5$, $|S(f_1)| \geq 3$, $|S(f_2)| \geq 4$, $|S(f_3)| \geq 4$, and $|S(f_4)| \geq 4$. It is an easy exercise to show that we can define the strong 10-coloring ψ with $\psi(e) \in S(e)$, for each $e \in F$, given these inequalities. The proof of our theorem is now complete. ∎

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