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Colorings of diagrams of interval orders and α-sequences of sets

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Abstract

We show that a proper coloring of the diagram of an interval order I may require $1 + \lceil \log_2 \text{height}(I) \rceil$ colors and that $2 + \lceil \log_2 \text{height}(I) \rceil$ colors always suffice. For the proof of the upper bound we use the following fact: A sequence $C_1, \ldots, C_h$ of sets (of colors) with the property

(α) $\quad C_j \not\subseteq C_{i-1} \cup C_i \quad$ for all $1 < i < j \leqslant h$

can be used to color the diagram of an interval order with the colors of the C_i. We construct α-sequences of length $2^{n-2} + \lfloor (n-1)/2) \rfloor$ using n colors. The length of α-sequences is bounded by $2^{n-1} + \lfloor (n-1)/2) \rfloor$ and sequences of this length have some nice properties. Finally we use α-sequences for the construction of long cycles between two consecutive levels of the Boolean lattice. The best construction known until now could guarantee cycles of length $\Omega(N^c)$ where N is the number of vertices and $c \approx 0.85$. We exhibit cycles of length $\geqslant \frac{1}{4}N$.

Keywords: Interval order; Diagram; Chromatic number; Hamiltonian path; Boolean lattice

1. Introduction and overview

For a nonnegative integer k, let I_k be the interval order defined by the open intervals with endpoints in $\{1, \ldots, 2^k\}$. It has height $2^k - 1$ and is isomorphic to the *canonical* interval order of this height (see [1] for canonical interval orders).

Two vertices v and w in I_k are a *cover*, denoted by $v \prec w$, exactly if the right endpoint of the interval of v equals the left endpoint of the interval of w. The diagram D_{I_k} of I_k is

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thus recognized as the *shift graph* $\mathscr{G}(2^k, 2)$ (see [1] for shift graphs). In general we denote by D_I the diagram of an interval order I, and we denote the chromatic number of the diagram by $\chi(D_I)$.

We include the (well-known) proof of the next lemma since we will need similar methods in later arguments.

Lemma 1.1.

$$\chi(D_{I_k}) = \lceil \log_2 \mathrm{height}(I_k) \rceil = k.$$

Proof. Suppose we have a proper coloring of D_{I_k} with colors $\{1, \ldots, c\}$. With each point i associate the set C_i of colors used for the intervals having their right endpoint at i. Note that $C_1 = \emptyset$. For $1 \leqslant i < j \leqslant 2^k$, we have $C_j \nsubseteq C_i$; otherwise the interval (i,j) would have the same color as some interval (l, i). This proves that all of the 2^k subsets C_i of $\{1, \ldots, c\}$ are distinct; therefore $2^c \geqslant 2^k$ and $c \geqslant k$.

A coloring of D_{I_k} using k colors can be obtained by the following construction. Take a linear extension of the Boolean lattice $\mathscr{B}_k$ and let C_i be the ith set in this list. Assign to the interval (i, j) any color from C_j / C_i. A coloring obtained in this way is easily seen to be proper. □

We derive a result for later use and a theorem from this construction.

Result 1.2. *In a coloring of D_{I_k} which uses exactly k colors, every point $i \in \{1, \ldots, 2^k\}$ is incident with an interval of each color.*

Proof. The crucial fact here is that every subset of $\{1, \ldots, k\}$ is the C_i for some i. Now choose any $i \in \{1, \ldots, 2^k\}$ and a color $c \in \{1, \ldots, k\}$. We have to show that an interval of color c is incident with i.

If $c \in C_i$, then this is immediate from the definition of C_i. Otherwise, i.e., if $c \notin C_i$, then there is a $j_c > i$ such that $C_{j_c} = C_i \cup \{c\}$ and the interval (i, j_c) is colored c. □

With the next lemma we improve the lower bound: There are interval orders I with $\chi(D_I) \geqslant 1 + \log_2(\mathrm{height}(I))$. Compared with Lemma 1.1, this is a minor improvement, but we feel it worth stating, since later we will prove an upper bound of $2 + \log_2(\mathrm{height}(I))$ on the chromatic number of the diagram of I.

Lemma 1.3. *For each k there is an interval order I_k^* such that*

$$\chi(D_{I_k^*}) \geqslant 1 + \lceil \log_2 \mathrm{height}(I_k^*) \rceil = k.$$

Proof. Take I_k^* as the order obtained from I_k (see Lemma 1.1) by removing the intervals of odd length, i.e., the interval order defined by the open intervals (i, j) with $i, j \in \{1, \ldots, 2^k\}$ and $j - i \equiv 0 \pmod 2$. The height of I_k^* is $2^{k-1} - 1$ which is the height of I_{k-1}; however, as we are now going to prove, a proper coloring of I_k^* requires at least k colors. Note that two intervals (i_1, j_1) and (i_2, j_2) with $j_1 \leqslant i_2$ induce an edge in the diagram of I_k^* if either $j_1 = i_2$ or $j_1 = i_2 - 1$.

In I_k^* we find an isomorphic copy of I_{k-1} consisting of the intervals (i, j) with both i and j odd. Call this the odd I_{k-1}. The even I_{k-1} is defined by the interval (i, j) with i and j even. Let C_i be the set of colors used for intervals with right end-point $2i - 1$, and let D_i be the set of colors used for intervals with right end-point $2i$. From Lemma 1.1, we know that if both the odd and the even copy only need $k - 1$ colors, then the C_i and the D_i have to form linear extensions of the Boolean lattice $\mathscr{B}_{k-1}$. Now define $\bar{C}_i$ as the set of colors used for intervals with left-endpoint $2i - 1$. From Result 1.2, we know that $\bar{C}_i$ is exactly the complement of C_i. With the corresponding definition, $\bar{D}_i$ and D_i are seen to be complementary sets as well. Note that a proper coloring requiring $C_i \cap \bar{D}_i = \emptyset$. We therefore have $C_i \subseteq D_i$. A similar argument gives $D_i \subseteq C_{i+1}$. Altogether we find that the C_i have to be a linear extension of $\mathscr{B}_{k-1}$ with $C_i \subseteq C_{i+1}$ for all i. This is impossible. The contradiction shows that at least k colors are required. □

Now we turn to the upper bound which we view as the more interesting aspect of the problem.

Theorem 1.4. *If I is an interval order, then*

$$\chi(D_I) \leqslant 2 + \log_2 \text{height}(I).$$

Proof. In this first part of the proof, we convert the problem into a purely combinatorial one. The next section will then deal with the derived problem.

Let $I = (V, <)$ be an interval order of height h, given together with an interval representation. For $v \in V$, let $(l_v, r_v]$ (left open, right closed) be the corresponding interval. With respect to this representation, we distinguish the 'leftmost' h-chain in I. This chain consists of the elements $x_1, \ldots, x_h$ where x_i has the leftmost right-endpoint r_v among all elements of height i. It is easily checked that $x_1, \ldots, x_h$ is indeed a chain. Now let $r_i = r_{x_i}$ be the right endpoint of x_i's interval and define a partition of the real axis into blocks. The ith block is

$$B(i) = [r_i, r_{i+1}).$$

This definition is made for $i = 0, \ldots, h$ with the convention that $B(0)$ extends to minus infinity and $B(h)$ to plus infinity.

In some sense these blocks capture a relevant part of the structure of I. This is exemplified by two properties.

- The elements v with $r_v \in B(i)$ are an antichain for each i. This gives a minimal antichain partition of I.
- If $r_v \in B(j)$, then $l_v \in B(i)$ for some i less than j.

Suppose we are given a sequence $C_1, \dots, C_h$ of sets (of colors) with the following property:

(α) $C_j \not\subseteq C_{i-1} \cup C_i$ for all $1 < i < j \leqslant h$.

A sequence with this property will henceforth be called an *α-sequence*. The α-sequence $C_1, \dots, C_h$ may be used to color the diagram D_I with the colors occurring in the C_i. The rule is: to an element $v \in V$ with $l_v \in B(i)$ and $r_v \in B(j)$ assign any color from $C_j \backslash (C_{i-1} \cup C_i)$. This set of colors is nonempty by the (α) property of the sequence C_i, since $i < j$. We claim that a coloring obtained this way is proper. Assume to the contrary that there is a covering pair $w \prec v$ such that w and v obtain the same color. Let $r_w \in B(k)$ and $l_v \in B(i)$. Since $w \prec v$, we know that $k \leqslant i$. Due to our coloring rule, we know that the color of w is an element of C_k and the color of v is not contained in $C_{i-1} \cup C_i$; hence $k < i - 1$. This, however, contradicts our assumption that $w \prec v$, since $l_{x_i} \in B(i-1)$ and $l_v \geqslant r_{x_i} = r_i$ gives $w < x_i < v$.

We have thus reduced the original problem to the determination of the minimal number of colors which admits a α-sequence of length h. We will demonstrate in next section, Lemmas 2.1 and 2.3, how to construct a α-sequence of length $2^{n-2} + \lfloor (n+1)/2 \rfloor$ using n colors. This will complete the proof of the theorem. □

In Section 3 we give an upper bound of $2^{n-1} + \lfloor (n+1)/2 \rfloor$ for the maximal length of a α-sequence. From the proof, we derive some further properties α-sequences of this length necessarily satisfy. Finally we apply the construction of long α-sequences to the problem of finding long cycles between two consecutive levels of the Boolean lattice. A famous instance of this problem is the question whether there is a Hamiltonian cycle between the middle two levels of the Boolean lattice (see e.g., [2] or [3]). The best constructions known until now could guarantee cycles of length $\Omega(N^c)$ where N is the number of vertices and $c \approx 0.85$. We exhibit cycles of length $\geqslant \frac{1}{4}N$.

2. A construction of long α-sequences

Let $t(n,k)$ denote the maximal length of a sequence C_i of sets satisfying:

(1) $C_i \subseteq \{1, \dots, n\}$,

(2) $|C_i| = k$ and

(α) if $i < j$ then $C_j \not\subseteq C_{i-1} \cup C_i$.

Lemma 2.1.

$$t(n,k) \geqslant \binom{n-1}{k} + 1.$$

Proof. The sequences actually constructed will have the additional property

(4) $|C_{i-1} \cup C_i| = k + 1$ for all $i \geqslant 2$.

The proof is by induction. For all n and $k = 1$ or $k = n$ the claim is obviously true.

Now suppose that two α-sequences as specified have been constructed on $\{1,\dots,n-1\}$: first a sequence of k-sets $\mathscr{A} = A_1,\dots,A_s$ of length $s = \binom{n-2}{k} + 1$, and second a sequence of $(k-1)$-sets $\mathscr{B} = B_1,\dots,B_t$ of length $t = \binom{n-2}{k-1} + 1$.

Property (4) guarantees that there is a permutation π of the colors such that $A_S = B_1^\pi \cup B_2^\pi$. Now let

$$C_i = \begin{cases} A_i & \text{if } 1 \leqslant i \leqslant s, \\ B_{i-s+1}^\pi \cup \{n\} & \text{if } s+1 \leqslant i \leqslant s+t-1. \end{cases}$$

The length of the new sequence is $s + t - 1 = \binom{n-1}{k} + 1$. Properties (1) and (2) are obviously true for the sequence C_i and property (4) is true for both the $\mathscr{A}$ and the $\mathscr{B}$ sequence. These observations and the choice of π give property (4) for the $\mathscr{C}$ sequence. It remains to verify property (α). If $i < j < s+1$, this property is inherited from the $\mathscr{A}$ sequence. If $s+1 < i < j$, it is inherited from the $\mathscr{B}$ sequence. In case $i < s+1 \leqslant j$, we have $n \in C_j$ and $n \notin C_{i-1} \cup C_i$. The remaining case is $s+1 = i < j$. Here the choice of π and the sacrifice of B_1 show that $C_s \cup C_{s+1} = A_s \cup B_2^\pi \cup \{n\} = B_1^\pi \cup B_2^\pi \cup \{n\}$. Again the property (α) can be concluded from this property for the $\mathscr{B}$ sequence. □

For $k = 2$ and $k = n-1$, we can prove that the inequality of Lemma 2.1 is tight, but in general the value of $t(n,k)$ is open.

Problem 2.2. Determine the true value of $t(n,k)$.

Let $T(n)$ denote the maximal length of a sequence C_i of sets satisfying:
(1) $C_i \subseteq \{1,\dots,n\}$ and
(α) if $i < j$ then $C_j \not\subseteq C_{i-1} \cup C_i$.

Lemma 2.3.

$$T(n) \geqslant \sum_{\substack{k \leqslant n \\ k \text{ odd}}} \left(\binom{n-1}{k} + 1 \right) = 2^{n-2} + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Proof. Let $\mathscr{L}(n,k)$ be the (n,k)-sequence constructed in the preceding lemma. We claim that $\mathscr{L} = \mathscr{L}^{\pi_1}(n,1) \oplus \mathscr{L}^{\pi_3}(n,3) \oplus \mathscr{L}^{\pi_5}(n,5) \oplus \cdots$ with appropriate permutations π_j is a α-sequence of subsets of $\{1,\dots,n\}$. The π_k's can be found recursively. $\pi_1 = id$ and if π_{k-2} has been determined, then π_k is chosen as a permutation such that the last set of the sequence $\mathscr{L}^{\pi_{k-2}}(n,k-2)$ is a subset of the first set of $\mathscr{L}^{\pi_k}(n,k)$. Let C_i be the ith set in the sequence $\mathscr{L}$. We now check property (α). If the three sets C_{i-1}, C_i and C_j are in the same subsequence $\mathscr{L}^{\pi_k}(n,k)$, then the property is inherited from this subsequence. If $C_i \in \mathscr{L}^{\pi_k}(n,k)$ and $C_j \in \mathscr{L}^{\pi_{k'}}(n,k')$ with $k \leqslant k'-2$, then $|C_{i-1} \cup C_i| < |C_j|$ is a consequence of property (4) for the subsequence $\mathscr{L}^{\pi_k}(n,k)$, and gives the claim in this case. There remains the situation where C_{i-1} is the last set of its subsequence. The choice of the π_k gives $C_{i-1} \subset C_i$ and the property reduces to $C_j \nsubseteq C_i$, which is obvious.

The length of $\mathscr{L}$ is the sum over the length of the $\mathscr{L}^{\pi_k}(n,k)$ used in $\mathscr{L}$. This is the sum over $\binom{n-1}{k} + 1$ with k odd, which is $2^{n-2} + \lfloor (n+1)/2 \rfloor$. □

3. The structure of very long α-sequences

Theorem 3.1. *Let $\mathscr{C} = C_1, \ldots, C_t$ be a α-sequence of subsets of $\{1, \ldots, n\}$. Then $t \leqslant 2^{n-1} + \lfloor (n+1)/2 \rfloor$.*

Proof. We start with some definitions. For $1 \leqslant i \leqslant t-1$, let

$$S_i = \{S \colon C_{i+1} \subset S \subseteq C_i \cup C_{i+1}\} \tag{1}$$

and $s_i = |S_i|$. Observe that with $r_i = |C_i \backslash C_{i+1}|$ we have the equation

$$s_i = 2^{r_i} - 1. \tag{2}$$

We now prove two important properties of the sets S_i

• $S_i \cap S_j = \emptyset$ *if* $i \neq j$. Assume to the contrary that $S \in S_i \cap S_j$ and let $i < j$. From the definition of the S_i, we obtain $C_{j+1} \subset S \subseteq C_i \cup C_{i+1}$ which contradicts the (α) property of the sequence $\mathscr{C}$.

• $\mathscr{C} \cap S_i = \emptyset$ *for all* i. Assume $C_j \in S_i$. If $j \leqslant i$, then $C_{i+1} \subset C_j$ gives a contradiction. If $j = i+1$, note that $C_{i+1} \notin S_i$ from the definition. If $j > i+1$, the contradiction comes from $C_j \subseteq C_i \cup C_{i+1}$.

Therefore, $\mathscr{C}$ and the S_i are pairwise disjoint subsets of $\mathscr{B}_n$. This gives the inequality

$$2^n \geqslant t + \sum_{i=1}^{t-1} s_i \tag{3}$$

We now partition the indices $\{1, \ldots, t-1\}$ into three classes

• $I_1 = \{i \colon |C_i| = |C_{i+1}|\}$; note that $i \in I_1$ implies $s_i \geqslant 1$.

• $I_2 = \{i \colon |C_i| < |C_{i+1}|\}$; trivially $s_i \geqslant 0$ for $i \in I_2$.

• $I_3 = \{i \colon |C_i| > |C_{i+1}|\}$; note that if $i \in I_3$, then the corresponding s_i is relatively large, i.e., $s_i \geqslant 2^{|C_i| - |C_{i+1}| + 1} - 1$. This estimate is a consequence of Eq. (2) and the fact that C_{i+1} has to contain an element not contained in C_i.

First we investigate the case $I_3 = \emptyset$. This condition guarantees that the sizes of the sets in $\mathscr{C}$ is a nondecreasing sequence. Since $\mathscr{B}_n$ has $n+1$ levels, the size of the sets in $\mathscr{C}$ can increase at most n times, i.e., $|I_2| \leqslant n$ and $|I_1| \geqslant t - 1 - n$. It follows that

$$2^n \geqslant t + \sum_{i \in I_1} s_i + \sum_{i \in I_2} s_i \geqslant t + |I_1| \geqslant t + (t - 1 - n).$$

This gives $2t \leqslant 2^n + (n+1)$; hence $t \leqslant 2^{n-1} + \lfloor (n+1)/2 \rfloor$ in this case.

The case $I_3 \neq \emptyset$ is somewhat more complicated. Let the number of descending steps be d and $I_3 = \{i_1, \ldots, i_d\}$. Let m_{i_j} denote the number of levels the sequence is decreasing when going from C_{i_j} to C_{i_j+1}, i.e., $m_{i_j} = |C_{i_j}| - |C_{i_j+1}|$ and $s_{i_j} \geqslant 2^{m_{ij}+1} - 1$. Again we can estimate the size of I_2, namely $|I_2| \leqslant n + \sum_{j=1}^{d} m_{i_j}$. It follows that

$$2^n \geqslant t + \sum_{i \in I_1} s_i + \sum_{i \in I_2} s_i + \sum_{i \in I_3} s_i \geqslant t + |I_1| + \sum_{j=1}^{d} (2^{m_{ij}+1} - 1)$$

$$\geqslant t + ((t-1) - |I_2| - |I_3|) + \sum_{j=1}^{d} 2^{m_{ij}+1} - d$$

$$\geqslant t + \left(t - 1 - n - \sum_{j=1}^{d} m_{i_j} - d\right) + \sum_{j=1}^{d} 2^{m_{ij}+1} - d.$$

Comparing this with the calculations made for the case $I_3 = \emptyset$, we find that $t \geqslant 2^{n-1} + \lfloor (n+1)/2 \rfloor$ would require $-\sum_{j=1}^{d} m_{i_j} - 2d + \sum_{j=1}^{d} 2^{m_{ij}+1} \leqslant 0$. For each j, we have $2^{m_{ij}} > m_{i_j} - 2$; hence the above inequality can never hold. □

Remark. Let $T^*(n) = 2^{n-1} + \lfloor (n+1)/2 \rfloor$ be the upper bound from the theorem. We have seen that a α-sequence $\mathscr{C}$ of length $T^*(n)$ can only exist if $I_3 = \emptyset$. Moreover, the following conditions follow from the argument given for Theorem 3.1.

(1) There are exactly n increasing steps, i.e., $|I_2| = n$.

(2) If $i \in I_1$, then $s_i = 1$, i.e., any two consecutive sets of equal size have to be a *shift*: $C_{i+1} = (C_i \backslash \{x\}) \cup \{y\}$ with $x \in C_i$ and $y \notin C_i$. If $i \in I_2$ then $s_i = 0$, i.e., if $|C_i| < |C_{i+1}|$, then there is a containment $C_i \subset C_{i+1}$.

(3) Every element of $\mathscr{B}_n$ is either an element of $\mathscr{C}$ or appears as the unique element of some S_i, i.e., as $C_i \cup C_{i+1}$.

From these observations, we obtain an alternate interpretation for a sequence $\mathscr{C}$ of length $T^*(n)$ in $\mathscr{B}_n$. In the diagram of $\mathscr{B}_n$, i.e., the n-hypercube, consider the edges (C_i, C_{i+1}) and for $i \in I_2$ for $i \in I_1$ the edges (C_i, T_i) and (T_i, C_{i+1}) where T_i is the unique member of S_i, i.e., $T_i = C_i \cup C_{i+1}$. This set of edges is a Hamiltonian path in the hypercube and respects a strong condition of being level accurate. *After having reached the* kth *level for the first time the path will never come back to level* $k - 2$ (see Fig. 1 for an example, the bullets are the elements of a very long α-sequence).

Problem 3.2. Do sequences of length $T^*(n)$ exist for all n?

We are hopeful that such sequences exist. Our optimism stems in part from computational results. The number of sequences starting with $\emptyset, \{1\}, \{2\}, \ldots, \{n\}$ is 1 for $n \leqslant 4$, 10 for $n = 5$, 123 for $n = 6$ and there are thousands of solutions for $n = 7$. The next case $n = 8$ could not be handled by our program, but Markus Fulmek (personal communication) wrote a program which also resolved this case affirmatively.

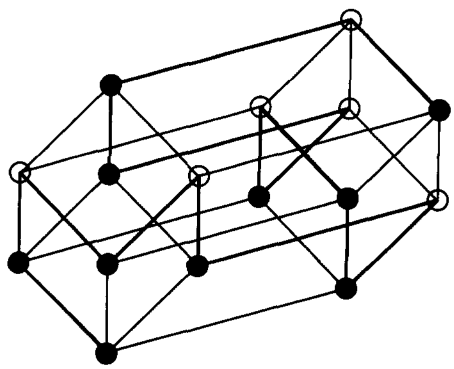

Fig. 1. A level accurate path in $\mathscr{B}_4$.

4. Long cycles between consecutive levels in $\mathscr{B}_n$

Let $B(n,k)$ denote the bipartite graph consisting of all elements from levels k and $k+1$ of the Boolean lattice $\mathscr{B}_n$. A well-known problem on this class of graphs is the following: *Is $B(2k+1,k)$ Hamiltonian for all k?* Until now it was known that this is the case for $k \leqslant 9$. Since the problem seems to be very hard, some authors have attempted to construct long cycles. The best result (see [3]) lead to cycles of length $\Omega(N^c)$ where $N = 2\binom{2k+1}{k}$ is the number of vertices of $B(2k+1,k)$ and $c \approx 0.85$.

Theorem 4.1. *In $B(n,k)$, there is a cycle of length*

$$4\max\left\{\binom{n-3}{k-1}+1,\ \binom{n-3}{n-k-2}+1\right\}.$$

Proof. Note that the graphs $B(n,k)$ and $B(n,n-k-1)$ are isomorphic, it thus suffices to exhibit a cycle of length $4\binom{n-3}{k-1}+4$ in $B(n,k)$. To this end, take a α-sequence $C_1,\ldots,C_t$ of $(k-1)$-sets on $\{1,\ldots,n-2\}$. From Lemma 2.1, we know that $t \geqslant \binom{n-3}{k-1}+1$ can be achieved. Now consider the following set of edges in $B(n,k)$

- $(C_i \cup \{n\},\ C_i \cup C_{i+1} \cup \{n\})$ for $1 \leqslant i < t$,
- $(C_i \cup C_{i+1} \cup \{n\},\ C_{i+1} \cup \{n\})$ for $1 \leqslant i < t$,
- $(C_t \cup \{n\},\ C_t \cup \{n-1,n\})$ and $(C_t \cup \{n-1,n\},\ C_t \cup \{n-1\})$,
- $(C_i \cup \{n-1\},\ C_i \cup C_{i+1} \cup \{n-1\})$ for $1 \leqslant i < t$,
- $(C_i \cup C_{i+1} \cup \{n-1\},\ C_{i+1} \cup \{n-1\})$ for $1 \leqslant i < t$,
- $(C_1 \cup \{n-1\},\ C_1 \cup \{n-1,n\})$ and $(C_1 \cup \{n-1,n\},\ C_1 \cup \{n\})$.

The proof that this set of edges in fact determines a cycle of length $4t$ in $B(n,k)$ is straightforward. □

With a simple calculation on binomial coefficients, we obtain a final theorem.

Theorem 4.2. *There are cycles in $B(2k+1,k)$ of length at least $\frac{1}{4}N$.*

References

[1] Z. Füredi, P. Hajnal, V. Rödl and W.T. Trotter, Interval Orders and shift graphs, Proc. the Hajnal/Sös Conf. on Combinatorics, Budapest, 1991, to appear.

[2] H.A. Kierstead and W.T. Trotter, Explicit matchings in the middle two levels of the Boolean lattice, Order 5 (1988) 163–171.

[3] C. Savage, Long cycles in the middle two levels of the Boolean lattice, Preprint, 1990.