Correlation and Sorting

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September 20, 2001
Research Interests

- Graph theory
- Discrete geometry
- On-line algorithms
- Adversarial algorithms
- Ramsey theory
- Extremal Problems
- Probabilistic methods
- Partially ordered sets
**Linear Extensions**

**Notation.** Given a poset $P$, $\mathcal{E}(P)$ denotes the set of all linear extensions of $P$. 

![Diagram showing a poset with elements $x$, $y$, $a$, $b$, and $c$.]

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Interval Orders

Intervals from a linear order

The associated interval order
Semiorders

Constant Length Intervals

The associated semiorder
Problem 1. Determine an unknown linear order $L$ of a set $X$ of $n$ elements by asking a series of questions of the form:

Is $x < y$ in $L$?
Classical Sorting Problem

**Problem 1.** Determine an unknown linear order $L$ of a set $X$ of $n$ elements by asking a series of questions of the form:

$$\text{Is } x < y \text{ in } L?$$

**Remark.** There are $n!$ possible linear orders, so at least $\lg n! \sim n \log n$ questions are required. Of course, several well known algorithms sort in $O(n \log n)$ rounds.
Problem 2. Determine an unknown linear extension $L$ of a poset $P$ of $n$ elements by asking a series of questions of the form:

Is $x < y$ in $L$?
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Is $x < y$ in $L$?

Question. If $P$ has $t$ linear extensions, can we always determine $L$ by asking $O(\log t)$ questions?
Notation. Given a poset $P$, $\mathcal{E}(P)$ denotes the set of all linear extensions of $P$. 

```
L_1  L_2  L_3  L_4  L_5  L_6  L_7  L_8  L_9  L_{10}  L_{11}
  d   d   d   b   d   d   d   b   d   d   b
  c   c   b   d   c   c   b   d   c   b   d
  y   b   c   c   y   b   c   c   b   c   c
  b   y   y   y   b   y   y   y   x   x   x
  a   a   a   a   x   x   x   x   y   y   y
  x   x   x   x   a   a   a   a   a   a   a
```
An Elementary Probability Space

**Notation.** Consider \( \mathcal{E}(P) \) as a probability space with each \( L \in \mathcal{E}(P) \) an equally likely outcome.

**Notation.** For distinct elements \( x \) and \( y \), the event \([x < y]\) is then the subset of \( \mathcal{E}(P) \) consisting of those \( L \) with \( x < y \) in \( L \).

**Notation.**

\[
\text{Prob}[x < y] = \frac{[x < y]}{|\mathcal{E}(P)|}
\]
Example

\[ \text{Prob}[x < y] = \frac{8}{11} \]
Conjecture. [Kislytsin, 1966] If $P$ is a finite poset and is not a chain, then there exist distinct $x, y$ in $P$ with

\[
\frac{1}{3} \leq \text{Prob}[x < y] \leq \frac{2}{3}
\]
**Conjecture. [Kislytsin, 1966]** If $P$ is a finite poset and is not a chain, then there exist distinct $x, y$ in $P$ with

$$\frac{1}{3} \leq \text{Prob}[x < y] \leq \frac{2}{3}$$

**Remark.** If true, the inequality is best possible.
Observation

Remark. It is not at all clear that there is any $\delta > 0$ so that for any $P$, there exists a pair $x, y$ with

$$\delta \leq \text{Prob}[x < y] \leq 1 - \delta$$
Observation

Remark. It is not at all clear that there is any \( \delta > 0 \) so that for any \( P \), there exists a pair \( x, y \) with

\[
\delta \leq \text{Prob}[x < y] \leq 1 - \delta
\]

Remark. If there exists such a \( \delta \), then we can determine an unknown linear extension of a poset \( P \) with \( t \) linear extensions in \( O(\log t) \) rounds.
Theorem. [Kahn and Saks] If $P$ is a finite poset and is not a chain, then there exist distinct $x, y$ in $P$ with

$$\frac{3}{11} < \text{Prob}[x < y] < \frac{8}{11}$$
The Kahn/Saks Theorem

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**Remark.** Over the next few years, several other papers on balancing pairs appeared, all with weaker results but somewhat shorter proofs.
A Basic Pigeon-hole

Notation. Given $x \in P$ and $L \in \mathcal{E}(P)$, let $h_L(x)$ denote the height of $x$ in $L$. Also, let $h(x)$ denote the average height of $x$. 
A Basic Pigeon-hole

**Notation.** Given $x \in P$ and $L \in \mathcal{E}(P)$, let $h_L(x)$ denote the **height** of $x$ in $L$. Also, let $h(x)$ denote the average height of $x$.

**Remark.** When $P$ is not a chain, there exists a pair $x, y$ with

$$0 \leq h(y) - h(x) < 1$$
**A Basic Pigeon-hole**

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**Remark.** When $P$ is not a chain, there exists a pair $x, y$ with

$$0 \leq h(y) - h(x) < 1$$

**Remark.** Kahn and Saks show that for any such pair, we always have

$$\frac{3}{11} < \text{Prob}[x < y] < \frac{8}{11}$$
Let $x$ and $y$ be incomparable elements of $P$ with $0 \leq h(y) - h(x) \leq 1$. Define

\begin{align*}
a_i &= \mathbb{P}[h_L(y) - h_L(x) = i] \quad \text{and} \\
b_i &= \mathbb{P}[h_L(x) - h_L(y) = i]
\end{align*}
Linear Constraints (1)

Let $x$ and $y$ be incomparable elements of $P$ with $0 \leq h(y) - h(x) \leq 1$. Define

$$a_i = \text{Prob}\left[h_L(y) - h_L(x) = i\right] \quad \text{and}$$

$$b_i = \text{Prob}\left[h_L(x) - h_L(y) = i\right]$$

Then,

$$\text{Prob}[x < y] = \sum_{i \geq 1} a_i$$
Linear Constraints (2)

\[ 1 = \sum_{i \geq 1} a_i + \sum_{i \geq 1} b_i \]

\[ 0 \leq \sum_{i \geq 1} ia_i - \sum_{i \geq 1} ib_i \leq 1 \]

\[ a_1 = b_1 \]

\[ a_2 + b_2 \leq a_1 + b_1 \]

\[ a_{i+1} \leq a_i + a_{i+2} \text{ and } b_{i+1} \leq b_i + b_{i+2} \]
Remark. These linear constraints are not enough, since the optimal solution is still:

\[ \text{Prob}[x < y] = 1 \]
The Alexandrov/Fenchel Inequalities

Let $K_0$ and $K_1$ be convex bodies in $\mathbb{R}^d$, and let $K_\lambda = (1 - \lambda)K_0 + \lambda K_1$. Then there exist unique numbers $a_0, a_1, \ldots, a_d$ so that the volume of $K_\lambda$ is given by:

$$\text{Vol}(K_\lambda) = \sum_{i=0}^{d} \binom{d}{i} a_i (1 - \lambda)^{d-i} \lambda^i$$
The Alexandrov/Fenchel Inequalities

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$$\text{Vol}(K_\lambda) = \sum_{i=0}^{d} \binom{d}{i} a_i (1 - \lambda)^{d-i} \lambda^i$$

Furthermore, the sequence $\{a_i : 0 \leq i \leq d\}$ is log-concave, i.e.,

$$a_{i+1}^2 \geq a_ia_{i+2} \quad \text{for} \quad i = 0, 1, \ldots, d - 2.$$
Height Sequences are Log-Concave

Notation. Let $x \in P$ and let $h_i = \text{Prob}[h_L(x) = i]$. 
Height Sequences are Log-Concave

**Notation.** Let \( x \in P \) and let \( h_i = \text{Prob}[h_L(x) = i] \).

**Theorem. [Stanley]** For any poset \( P \) and any \( x \in P \), the height sequence \( \{h_i : 1 \leq i \leq |P|\} \) is log-concave, i.e.,

\[
h_{i+1}^2 \geq h_i h_{i+2} \quad \text{for} \quad i = 0, 1, \ldots |P| - 2.
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$$h_{i+1}^2 \geq h_i h_{i+2} \quad \text{for} \quad i = 0, 1, \ldots, |P| - 2.$$

**Remark.** No combinatorial proof of this result is known.
Differential Height Sequences are Log-Concave

Remark. Kahn and Saks extended Stanley’s technique to show that \{a_i : i > 0\} and \{b_i : i > 0\} are log-concave, i.e.,

\[ a_{i+1}^2 \geq a_i a_{i+2} \quad \text{and} \quad b_{i+1}^2 \geq b_i b_{i+2} \]
Differential Height Sequences are Log-Concave

Remark. Kahn and Saks extended Stanley’s technique to show that \( \{a_i : i > 0\} \) and \( \{b_i : i > 0\} \) are log-concave, i.e.,

\[
    a_i^{2i+1} \geq a_i a_{i+2} \quad \text{and} \quad b_i^{2i+1} \geq b_i b_{i+2}
\]

Remark. Adding these non-linear constraints to the linear ones listed previously yields the desired inequality:

\[
    \frac{3}{11} \leq \text{Prob}[x < y] \leq \frac{8}{11}
\]
Improving the Pigeon Hole

**Theorem. [Felsner and Trotter]** Let $x$ and $y$ be distinct points in a poset $P$ with $|h(y) - h(x)| \leq 2/3$. Then

$$\frac{1}{3} \leq \text{Prob}[x < y] \leq \frac{2}{3}$$
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**Definition.** For a finite poset $P$, let $\gamma(P)$ denote the minimum value of $|h(y) - h(x)|$ taken over all pairs $x, y \in P$. 
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**Definition.** For a finite poset \( P \), let \( \gamma(P) \) denote the minimum value of \( |h(y) - h(x)| \) taken over all pairs \( x, y \in P \).

**Remark.** [Saks, 85] There exists a poset \( P \) with \( \gamma(P) \sim 0.8657 \).
Could $\frac{3}{11} - \frac{8}{11}$ be Tight

Perhaps the $\frac{1}{3} - \frac{2}{3}$ conjecture is false and $\frac{3}{11} - \frac{8}{11}$ is the right answer. This would require that for every $\epsilon > 0$, we can find a poset $P$ with $\gamma(P) > 1 - \epsilon$. 
Could $3/11 - 8/11$ be Tight

Perhaps the $1/3 - 2/3$ conjecture is false and $3/11 - 8/11$ is the right answer. This would require that for every $\epsilon > 0$, we can find a poset $P$ with $\gamma(P) > 1 - \epsilon$.

Can this happen? And even if it does, can we use another line of reasoning to improve the Kahn/Saks bound?
Progress on the $\frac{1}{3}–\frac{2}{3}$ Conjecture

**Theorem.** The $\frac{1}{3}–\frac{2}{3}$ conjecture holds for any poset $P$ which satisfies any one of the following conditions:
Progress on the $1/3$–$2/3$ Conjecture

**Theorem.** The $1/3$–$2/3$ conjecture holds for any poset $P$ which satisfies any one of the following conditions:

1. $P$ has width 2 (Linial).
**Progress on the 1/3–2/3 Conjecture**

**Theorem.** The 1/3–2/3 conjecture holds for any poset $P$ which satisfies any one of the following conditions:

1. $P$ has width 2 (Linial).

2. $P$ is a semiorder (Brightwell).
Progress on the $1/3-2/3$ Conjecture

**Theorem.** The $1/3-2/3$ conjecture holds for any poset $P$ which satisfies any one of the following conditions:

1. $P$ has width 2 (Linial).
2. $P$ is a semiorder (Brightwell).
3. For every $x \in P$, there are at most 5 elements incomparable with $x$ (Brightwell and Wright).
Theorem. The $1/3$–$2/3$ conjecture holds for any poset $P$ which satisfies any one of the following conditions:

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4. $\gamma(P) \leq 2/3$ (Felsner and Trotter).
Progress on the $1/3 - 2/3$ Conjecture

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1. $P$ has width 2 (Linial).
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3. For every $x \in P$, there are at most 5 elements incomparable with $x$ (Brightwell and Wright).
4. $\gamma(P) \leq 2/3$ (Felsner and Trotter).
5. $P$ has height 2 (Fishburn, Gehrlein and Trotter).
A Conjecture on Large Width

**Conjecture. [Kahn and Saks]** For every $\epsilon > 0$, there exists an integer $w$ so that if the width of $P$ is at least $w$, then there exists a distinct pair $x, y$ in $P$ with

\[
\frac{1}{2} - \epsilon < \text{Prob}[x < y] < \frac{1}{2} + \epsilon
\]
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Theorem. [Komlós] There exists a function $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \to \infty$ and $f(n) = o(n)$ so that for every $\epsilon > 0$, there exists an integer $n$ so that if $P$ has $n$ elements and at least $f(n)$ maximal elements, then there exists a distinct pair $x, y$ in $P$ with

$$\frac{1}{2} - \epsilon < \text{Prob}[x < y] < \frac{1}{2} + \epsilon$$
Just How Important is “Finite”
Theorem. [Brightwell and Trotter] The 1/3–2/3 conjecture is \textit{FALSE} for infinite posets. In fact, there exists a countably infinite poset $P$ satisfying:

1. $P$ has width 2;
2. $P$ is a semiorder;
3. For every $x \in P$, there are at most 2 elements incomparable with $x$;
4. $\gamma(P) = 1$; and
5. For every incomparable pair $x$, $y$,

$$\Pr[x < y] \in \left\{\frac{5 - \sqrt{5}}{10}, \frac{5 + \sqrt{5}}{10}\right\}$$
Theorem. [Felsner and Trotter, 96] There exists a constant $\delta > 0$ so that if $P$ is any poset which is not a chain, then $P$ contains a distinct pair $x, y$ so that

$$\frac{3}{11} + \delta < \text{Prob}[x < y] < \frac{8}{11} - \delta$$
Theorem. [Felsner and Trotter, 96]  There exists a constant $\delta > 0$ so that if $P$ is any poset which is not a chain, then $P$ contains a distinct pair $x, y$ so that

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Remark.  We didn’t bother to calculate $\delta$ since it was very very small!
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Remark. We didn’t bother to calculate $\delta$ since it was very very small!

Remark. This research made extensive use of computation in both the discovery and proof modes.
Conjecture. [Felsner and Trotter] Let $x$, $y$ and $z$ be three distinct points in a poset $P$. For positive integers $i$, $j$, define

$$a(i, j) = \text{Prob}[h_L(y) - h_L(x) = i, \ h_L(z) - h_L(y) = j]$$
The Cross Product Conjecture

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$$a(i, j) = \text{Prob}[h_L(y) - h_L(x) = i, \ h_L(z) - h_L(y) = j]$$

Then

$$a(i, j)a(i + 1, j + 1) \leq a(i, j + 1)a(i + 1, j)$$
The Cross Product Conjecture

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Then

$$a(i, j)a(i + 1, j + 1) \leq a(i, j + 1)a(i + 1, j)$$

**Theorem.** [Brightwell, Felsner and Trotter, 97] The Cross Product Conjecture holds when $i = j = 1$, i.e.,

$$a(1, 1)a(2, 2) \leq a(1, 2)a(2, 1).$$
Improving Kahn/Saks

Theorem. [Brightwell, Felsner and Trotter] Let $P$ be a countable poset which is not a chain. If there exists an integer $k$ so that any element of $P$ is incomparable with at most $k$ others, then there exist distinct $x, y$ with

$$\frac{5 - \sqrt{5}}{10} \leq \text{Prob}[x < y] \leq \frac{5 + \sqrt{5}}{10}$$
**Improving Kahn/Saks**

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$$\frac{5 - \sqrt{5}}{10} \leq \text{Prob}[x < y] \leq \frac{5 + \sqrt{5}}{10}$$

$$\frac{3}{11} = .272727\ldots$$

$$\frac{5 - \sqrt{5}}{10} = .27639\ldots$$
Kahn’s Problem

**Problem.** Let $x$ and $y$ be distinct points in a poset $P$ and suppose that $|D[x] \cup D[y]| = n$. Is it true that

$$\max\{h(x), h(y)\} \geq n - 1$$

where $h(x)$ and $h(y)$ are some functions associated with the poset $P$. 

![Diagram](image)
Observations

Remark.  Kahn's conjecture is true when $P = D[x] \cup D[y]$. This follows from the fact that

$$\max \{ \text{Prob}[h_L(x) = n, \text{Prob}[h_L(y) = n] \} \geq \frac{1}{2}$$
Remark. Kahn’s conjecture is true when $P = D[x] \cup D[y]$. This follows from the fact that

$$\max\{\text{Prob}[h_L(x) = n, \text{Prob}[h_L(y) = n]\} \geq \frac{1}{2}$$

The argument depends on the log-concavity property of the height sequence.
Observations

Remark. *Kahn's conjecture is true when* \( P = D[x] \cup D[y] \). *This follows from the fact that*

\[
\max\{\text{Prob}[h_L(x) = n], \text{Prob}[h_L(y) = n]\} \geq \frac{1}{2}
\]

*The argument depends on the log-concavity property of the height sequence.*

Remark. *When*

\[ D[x] \cup D[y] \subsetneq P \]

*there seems to be nothing which would prevent* \( \max\{h(x), h(y)\} \) *from being far below* \( n \).*
Error Analysis on Log-Concavity

Let \( \{h_i : 1 \leq i \leq n\} \) be the height sequence of a point \( x \) in a poset \( P \). Consider the log-concave inequality

\[
h_{i+1}^2 \geq h_i h_{i+2}.
\]
Error Analysis on Log-Concavity

Let $\{h_i : 1 \leq i \leq n\}$ be the height sequence of a point $x$ in a poset $P$. Consider the log-concave inequality

$$h_{i+1}^2 \geq h_i h_{i+2}.$$

1. When is the inequality tight?
Error Analysis on Log-Concavity

Let \( \{h_i : 1 \leq i \leq n \} \) be the height sequence of a point \( x \) in a poset \( P \). Consider the log-concave inequality

\[
h_i^2 \geq h_i h_{i+2}.
\]

1. When is the inequality tight?

2. When it is not tight, what is the magnitude of the minimum error term as a function of \( n \)?
Correlation: The XYZ Theorems

Theorem. [Shepp] Let $x$, $y$ and $z$ be distinct points in a poset $P$. Then

$$\Pr(x > y) \Pr(x > z) \leq \Pr(x > y, x > z).$$
Correlation: The XYZ Theorems

Theorem. [Shepp]  Let $x$, $y$ and $z$ be distinct points in a poset $P$. Then

$$\text{Prob}[x > y] \text{Prob}[x > z] \leq \text{Prob}[x > y, x > z].$$

Theorem. [Fishburn]  Let $\{x, y, z\}$ be a 3-element antichain in a poset $P$. Then

$$\text{Prob}[x > y] \text{Prob}[x > z] < \text{Prob}[x > y, x > z].$$
Correlation: The XYZ Theorems

Theorem. [Shepp] Let \( x, y \) and \( z \) be distinct points in a poset \( P \). Then

\[
\text{Prob}\{x > y\} \text{ Prob}\{x > z\} \leq \text{Prob}\{x > y, x > z\}.
\]

Theorem. [Fishburn] Let \( \{x, y, z\} \) be a 3-element antichain in a poset \( P \). Then

\[
\text{Prob}\{x > y\} \text{ Prob}\{x > z\} < \text{Prob}\{x > y, x > z\}.
\]

Remark. Both proofs require the Ahlswede/Daykin four functions theorem.
The Ahlswede/Daykin Four Functions Theorem

**Theorem. [Ahlswede and Daykin]** Let $L$ be a distributive lattice. For sets $X$, $Y$ and a function $f$, let

1. $f(X) = \sum_{x \in X} f(x)$.

2. $X \land Y = \{x \land y : x \in X, y \in Y\}$.

3. $X \lor Y = \{x \lor y : x \in X, y \in Y\}$.

Let $\alpha$, $\beta$, $\gamma$ and $\delta$ be four functions mapping $L$ to the non-negative reals. If

$$\alpha(x)\beta(y) \leq \gamma(x \lor y)\delta(x \land y)$$

for all $x, y \in L$, then

$$\alpha(X)\beta(Y) \leq \gamma(X \lor Y)\delta(X \land Y)$$

for all $X, Y \subseteq L$. 
Fishburn’s Lemma

Lemma. [Fishburn] Let $A$ and $B$ be down-sets in a poset $P$ with $|A| = n$, $|B| = m$ and $|A \cap B| = k$. Then

$$|\mathcal{E}(A)| |\mathcal{E}(B)| \binom{n + m}{n} \leq |\mathcal{E}(A \cup B)| |\mathcal{E}(A \cap B)| \binom{n + m}{n + m - k}$$
Fishburn’s Lemma

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\[|\mathcal{E}(A)||\mathcal{E}(B)|\left(\frac{n + m}{n}\right) \leq |\mathcal{E}(A \cup B)||\mathcal{E}(A \cap B)|\left(\frac{n + m}{n + m - k}\right)\]

Remark. Fishburn’s argument also uses the Ahlswede/Daykin theorem.
A Combinatorial Approach to Correlation

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In 1999, Brightwell and Trotter gave a combinatorial proof of Fishburn’s lemma by providing an explicit injection between two sets of appropriate sizes.

They then extended this approach to give combinatorial proofs for the XYZ correlation results of Shepp and Fishburn.

These new arguments do not use the Ahlswede/Daykin theorem or any of its variant forms.
Some modest progress to report:

Suppose $x$ is point in a poset $P$ so that $x$ is incomparable with exactly two other points of $P$. Then there are three non-zero terms in the height sequence of $x$. If these terms are $h_i$, $h_{i+1}$ and $h_{i+2}$, then it is possible to prove that

$$h_{i+1}^2 \geq h_i h_{i+2}$$

via an entirely combinatorial argument.
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via an entirely combinatorial argument.

However, the proof does require the Ahlswede/Daykin four functions theorem.