

Section 2.7 : Example 1

$$(x^2 + y^2) dx + (x^2 - xy) dy = 0$$

$$\rightarrow (x^2 + y^2) + (x^2 - xy) \frac{dy}{dx} = 0$$

$M(x,y) = x^2 + y^2$ and $N(x,y) = x^2 - xy$ are both homogeneous of degree 2.

Therefore the equation is homogeneous.

Thus, we do the change of variables $u = \frac{y}{x}$ (when $x \neq 0$)

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx}(uy) = u + x \frac{du}{dx}$$

The equation becomes:

$$(x^2 + u^2 x^2) + (x^2 - ux^2)(u + x \frac{du}{dx}) = 0$$

($x^2 + u^2 x^2$) + ($x^2 - ux^2$) ($u + x \frac{du}{dx}$) = 0 (recall that $x \neq 0$):

We can simplify by x (recall that $x \neq 0$):

$$(1+u^2) + (1-u)(u + x \frac{du}{dx}) = 0$$

$$(1+u^2) + (1-u)u + (1-u)x \frac{du}{dx} = - (1+u)$$

$$\text{or } (1-u)x \frac{du}{dx} = - (1+u^2) - (1-u)u = - (1+u)$$

This is a separable equation: if $u \neq -1$:

$$-\frac{(1-u)}{1+u} du = \frac{x}{dx}$$

$$\text{But } -\frac{(1-u)}{1+u} = \frac{u-1}{1+u} = \frac{(u+1)-2}{1+u} = 1 - \frac{2}{1+u}$$

Therefore, by integration:

$$u - 2 \ln |1+u| = \ln |x| + C$$

Or, for y :

$$\boxed{\frac{y}{x} - 2 \ln \left| \frac{y}{x} + 1 \right| = \ln |x| + C}$$

(implicit form of the solution)