Section 2.7: Example 1

\[(x^2 + y^2) \, dx + (x^2 - xy) \, dy = 0\]

\[\Rightarrow (x^2 + y^2) + (x^2 - xy) \frac{dy}{dx} = 0\]

\[M(x,y) = x^2 + y^2 \text{ and } N(x,y) = x^2 - xy \text{ are both homogeneous of degree 1.}\]

Therefore the equation is homogeneous.

Thus, we do the change of variables \(U = \frac{y}{x}\) (when \(x \neq 0\)).

Then, \(\frac{dy}{dx} = \frac{d}{dx} (ux) = u + x \frac{du}{dx}\)

The equation becomes:

\[(x^2 + u^2 x^2) + (x^2 - u x^2) (u + x \frac{du}{dx}) = 0\]

We can simplify by \(x\) (recall that \(x \neq 0\)):

\[(1+u^2) + (1-u)(u + x \frac{du}{dx}) = 0\]

or \(1-u) x \frac{du}{dx} = -(1+u^2) - (1-u)u = -(1+u)\)

This is a separable equation: if \(u \neq -1\):

\[- \frac{(1-u)}{1+u} \, du = \frac{x}{dx}\]

But \(- \frac{(1-u)}{1+u} = \frac{u-1}{1+u} = \frac{(u+1)-2}{1+u} = 1 - \frac{2}{1+u}\)

Therefore, by integration:

\[U - 2 \ln |1+u| = \ln |x| + C\]

Or, for \(y\):

\[\frac{y}{x} - 2 \ln \left| \frac{y}{x} + 1 \right| = \ln |x| + C\]

(implicit form of the solution)