

## Section 6.2

### \* Motivation and review

a) What does it mean for a set of vectors to be linearly independent?

\* for two vectors:  ; not parallel

\* for  $n$  vectors:  $\sum_{i=1}^n d_i \vec{x}_i = \vec{0} \Leftrightarrow d_1 = d_2 = \dots = d_n = 0$

eg:  $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  are linearly independent.

$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$  are not linearly independent ( $0x_1 + 0x_2 + 3x_3 = \vec{0}$ )

b) What is the span of a set of vectors

$\text{Span}(\vec{v}_1, \dots, \vec{v}_n)$  is the set of all linear combinations of the vector  $\vec{v}_1, \dots, \vec{v}_n$ . This is a subspace.

### \* Uniqueness

$$(t-2)y'' + 3y = t, \quad y(0) = 0, \quad y'(0) = 1$$

$$x_1 = y, \quad x_2 = y' : \quad x_1' = y' = x_2 \quad \text{and} \quad x_2' = y'' = \frac{-3}{t-2}y + \frac{t}{t-2} = \frac{-3}{t-2}x_1 + \frac{t}{t-2}$$

$$\Rightarrow \vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{-3}{t-2} & 0 \end{pmatrix}}_P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{t}{t-2} \end{pmatrix}}_q$$

$P$  and  $q$  are continuous on  $(-\infty, 2)$  or  $(2, +\infty)$

The initial data is for  $t=0$ : we have an unique solution of the initial value problem on  $(-\infty, 2)$ .

### \* Linear independence of functions and the Wronskian

a)  $y_1 = e^t, \quad y_2 = e^{-2t}, \quad y_3 = 3e^t - 2e^{-2t}$

We have  $y_3 = 3y_1 - 2y_2$ :  $y_1, y_2, y_3$  are not linearly independent.

$$b) \psi_1 = t, \psi_2 = t^2, \psi_3 = 1-2t^2$$

$$\text{Assume that } c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3 = 0$$

$$\text{then: } c_1 \psi_1' + c_2 \psi_2' + c_3 \psi_3' = 0 \quad \text{and} \quad c_1 \psi_1'' + c_2 \psi_2'' + c_3 \psi_3'' = 0.$$

$$\text{Therefore we have: } \underbrace{\begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1' & \psi_2' & \psi_3' \\ \psi_1'' & \psi_2'' & \psi_3'' \end{pmatrix}}_A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

$$A = \begin{pmatrix} t & t^2 & 1-2t^2 \\ 1 & 2t & -4t \\ 0 & 2 & -4 \end{pmatrix}, \quad \text{if } \det A \neq 0 \text{ then } \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and  $\psi_1, \psi_2, \psi_3$  are linearly independent.

$$|A| = \begin{vmatrix} t & t^2 & 1-2t^2 \\ 1 & 2t & -4t \\ 0 & 2 & -4 \end{vmatrix} \stackrel{R_2 \leftrightarrow R_3 - tR_3}{=} \begin{vmatrix} t & t^2 & 1-2t^2 \\ 1 & 0 & 0 \\ 0 & 2 & -4 \end{vmatrix} = -1 \begin{vmatrix} t^2 & 1-2t^2 \\ 2 & -4 \end{vmatrix} = 4t^2 + 2(1-2t^2)$$

$\therefore |A| = 2 \neq 0$ :  $\psi_1, \psi_2$  and  $\psi_3$  are linearly independent.

\* Linear independence of Vector Functions, their Wronskian:

$$\vec{\psi}_1(t) = \begin{pmatrix} t \\ 1-t \\ 0 \end{pmatrix}, \quad \vec{\psi}_2(t) = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix}, \quad \vec{\psi}_3(t) = \begin{pmatrix} t \\ 3-t \\ 2t \end{pmatrix}$$

They are linearly independent if  $c_1 \vec{\psi}_1 + c_2 \vec{\psi}_2 + c_3 \vec{\psi}_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

$$\text{or } \begin{pmatrix} \vec{\psi}_1 & \vec{\psi}_2 & \vec{\psi}_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \Rightarrow c_1 = c_2 = c_3 = 0$$

$$\text{or } W[\vec{\psi}_1, \vec{\psi}_2, \vec{\psi}_3](t) \neq 0.$$

$$W[\vec{\psi}_1, \vec{\psi}_2, \vec{\psi}_3](t) = \begin{vmatrix} t & 0 & t \\ 1-t & 1 & 3-t \\ 0 & t & 2t \end{vmatrix} = t(2t-3t+t^2) = 0 + t(t-t^2)$$

$$= t(t^2-t) + t(t-t^2) = 0$$

$\therefore \psi_1, \psi_2$  and  $\psi_3$  are linearly dependent.

## \* Fundamental set of solutions: Vector functions: Example

$\vec{x}_1(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  &  $\vec{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$  form a fundamental set of solutions for the system:

$$\vec{x}' = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \vec{x} = A \vec{x}$$

Indeed:  $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leadsto \vec{x}_1(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is a solution.

$A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \leadsto \vec{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$  is a solution.

and  $W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} e^{5t} & -e^{-t} \\ e^{5t} & e^{-t} \end{vmatrix} = 2e^{4t} \neq 0 \quad (\forall t).$

