Theorem

If \( x = x(t) \) and \( y = y(t) \) are two differentiable functions of \( t \)
and \( \Psi = \Psi(x(t), y(t)) \).
Then \( \Psi \) is differentiable and

\[
\frac{d\Psi}{dt} = \frac{\partial\Psi}{\partial x} \frac{dx}{dt} + \frac{\partial\Psi}{\partial y} \frac{dy}{dt}
\]

Recall:

1) derivative: \( \frac{df}{dt}(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} \)

2) partial derivative:

\[
\frac{\partial g}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{g(x_0 + h, y_0) - g(x_0, y_0)}{h}
\]

(And the equivalent definition for \( \frac{\partial g}{\partial y}(x_0, y_0) \))

Proof

By definition: \( \frac{d\Psi}{dt}(t_0) = \lim_{h \to 0} \frac{\Psi(x(t_0 + h), y(t_0 + h)) - \Psi(x(t_0), y(t_0))}{h} \)

Problem: If we want to apply the definitions of derivatives or

partial derivatives to simplify this, we need just "one \( h \)"

in the numerator.

Trick 1: Subtract \( \Psi(x(t_0 + h), y(t_0)) \)

\[
\frac{d\Psi}{dt}(t_0) = \lim_{h \to 0} \left( \frac{\Psi(x(t_0 + h), y(t_0 + h)) - \Psi(x(t_0), y(t_0))}{h} + \frac{\Psi(x(t_0), y(t_0)) - \Psi(x(t_0), y(t_0))}{h} \right)
\]

If the limit of both parts of the sum is finite, we

can split the limit and we obtain:
\[
\frac{d\psi}{dt}(t_0) = \lim_{h \to 0} \left( \frac{\psi(x(t_0 + h), y(t_0 + h)) - \psi(x(t_0), y(t_0))}{h} \right) \sim I
\]

\[
+ \lim_{h \to 0} \left( \frac{\psi(x(t_0), y(t_0)) - \psi(x(t_0), y(t_0))}{h} \right) \sim II
\]

\[= I + II\]

Let us deal with the first part (the second part works the same way, it is even easier).

**Trick 2:** multiply and divide by \( y(t_0 + h) - y(t_0) \)

\[
I = \lim_{h \to 0} \left( \frac{\psi(x(t_0 + h), y(t_0 + h)) - \psi(x(t_0), y(t_0))}{y(t_0 + h) - y(t_0)} \right) \times \frac{y(t_0 + h) - y(t_0)}{h}
\]

If the limit of both parts of the product is finite, we can split again the limit:

\[
I = \lim_{h \to 0} \left( \frac{\psi(x(t_0 + h), y(t_0 + h)) - \psi(x(t_0), y(t_0))}{y(t_0 + h) - y(t_0)} \right) \times \lim_{h \to 0} \left( \frac{y(t_0 + h) - y(t_0)}{h} \right)
\]

For the first part, we make the change of variables:

\[x = t_0 + h \quad (h \to 0 \text{ means } x \to t_0)\]

Then:

\[
\lim_{h \to 0} \left( \frac{\psi(x(t_0 + h), y(t_0 + h)) - \psi(x(t_0), y(t_0))}{y(t_0 + h) - y(t_0)} \right) = \lim_{x \to t_0} \left( \frac{\psi(x(k), y(k)) - \psi(x(k), y(k))}{y(k) - y(t_0)} \right)
\]

\[= \frac{d\psi}{dy}(x(t_0), y(t_0))\]

**Conclusion:**  
\[I = \frac{\partial\psi}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt}(t_0)\]

With the same computations, we prove that:

\[\Pi = \frac{\partial\psi}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt}(t_0)\]
Because all functions are differentiable here, those quantities are finite and we were allowed to perform the two splittings in the limits.

Finally, we have
\[
\frac{d\psi(t_0)}{dt} = \mathcal{I} + \mathcal{II} = \frac{\partial \psi(x(t_0), y(t_0))}{\partial x} \frac{dx}{dt} + \frac{\partial \psi(x(t_0), y(t_0))}{\partial y} \frac{dy}{dt}
\]

**Corollary (our case for exact DE)**

If \( \psi = \psi(x, y(x)) \)

Then \( \frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} \)

**Proof:**

This is the previous theorem with \( t = x \).

In this case: \( \frac{dx}{dt} = \frac{dx}{dx} = 1 \).