Homogeneous linear systems with constant coefficients and eigenvectors

We consider the first order homogeneous linear system with constant coefficients:

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \text{where } \vec{x} \in \mathbb{R}^n, \ A \in \mathbb{R}^{n\times n}.$$

* if $n=1$

We have just one equation of the form $y' = ay$.
Using that this equation is separable or using an integrating factor, we prove that the solution is:

$$y(t) = y(0)\, e^{at}.$$

* if $n \geq 2$

If we want to apply the same method, we need to compute e^{tA}, but tA is a matrix!

We define the exponential of a matrix by:

$$\boxed{e^{tA} = \sum_{k\geq 0} \frac{(tA)^k}{k!}} \qquad \left(e^B = \sum_{k\geq 0} \frac{B^k}{k!}, \text{ with } B = tA\right)$$

With this definition, we can check that

$$(e^{tA})' = \frac{d}{dt}\left(\sum_{k\geq 0} \frac{t^k A^k}{k!}\right) = \sum_{k\geq 0} \frac{k\, t^{k-1} A^k}{k!} = \left(\sum_{k\geq 1} \frac{t^{k-1}}{(k-1)!} A^{k-1}\right) A$$

(this term is 0 if $k=0$)

by the change of variable $j = k-1$: $(e^{tA})' = \left(\sum_{j\geq 0} \frac{t^j}{j!} A^j\right) A = e^{tA} \cdot A$

Therefore: The solution of this equation $\frac{d\vec{x}}{dt} = A\vec{x}$ is $\boxed{\vec{x}(t) = e^{tA}\,\vec{x}(0)}$

⚠ **Problem:** How do we compute e^{tA} ?
(we the formula, we need to compute $A^k, \forall k \geq 0$)

☹ too hard for any matrix except if...

Solution: it's easy to compute A^k if A is diagonal

$\Rightarrow$ **DIAGONALIZATION**

$A \in \mathbb{R}^{n\times n}$, if we have n independent eigenvectors $\vec{v_1}, \dots, \vec{v_n}$ associated to the eigenvalues $d_1, \dots, d_n$, then for $P = (\vec{v_1} \cdots \vec{v_n})$ and the diagonal matrix $D = \begin{pmatrix} d_1 & & (0) \\ & \ddots & \\ (0) & & d_n \end{pmatrix}$, we have:

$$\boxed{A = PDP^{-1}}$$

Let us prove then that the solution is thus:

$$\boxed{\vec{x}(t) = e^{tA}\vec{x}(0) = Pe^{tD}P^{-1}\vec{x}(0)} \quad (1)$$

To prove this, we have two methods:

* Method 1

We introduce the change of variables $\vec{y} = P^{-1}\vec{x}$

(fun fact: we don't have to compute P^{-1})

Then:

$$\frac{d\vec{y}}{dt} = \frac{d}{dt}\left(P^{-1}\vec{x}\right) \quad \text{(P^{-1} is a constant matrix)}$$

$$= P^{-1}\frac{d}{dt}\vec{x} \quad \left(\frac{d}{dt}\vec{x} = A\vec{x}\right)$$

$$= P^{-1}A\vec{x} \quad (A = PDP^{-1})$$

$$= \underbrace{P^{-1}P}_{=I_n} D \underbrace{P^{-1}\vec{x}}_{=\vec{y}}$$

$$= D\vec{y}$$

Therefore the solution is given by $\vec{y}(t) = e^{tD}\vec{y}(0)$

and using that $\vec{x}(t) = P\vec{y}(t)$ (and $\vec{y}(0) = P^{-1}\vec{x}(0)$)

we obtain $\vec{x}(t) = P\vec{y}(t) = Pe^{tD}P^{-1}\vec{x}(0)$ ⊠

* Method 2

By definition and using the diagonalization:

$$e^{tA} = e^{tPDP^{-1}} = e^{P(tD)P^{-1}} = \sum_{k\geq 0} \frac{(P(tD)P^{-1})^k}{k!}$$

But $(PtDP^{-1})^k = (PtDP^{-1})(PtDP^{-1})P \dots P^{-1}(PtDP^{-1})$ (k times)

$$= P(tD)^k P^{-1}$$

($P^{-1}P = I_n$, $P^{-1}P = I_n$, $P^{-1}P = I_n$)

Therefore: $$e^{tA} = \sum_{k\geq 0} P\left(\frac{(tD)^k}{k!}\right)P^{-1}$$

$$= P\left(\sum_{k\geq 0}\frac{(tD)^k}{k!}\right)P^{-1} = Pe^{tD}P^{-1}$$

and $\vec{x}(t) = e^{tA}\vec{x}(0) = Pe^{tD}P^{-1}\vec{x}(0)$ ☒.

Now we have to compute e^{tD}

$D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}$ is diagonal, thus $D^k = \begin{pmatrix} \lambda_1^k & & (0) \\ & \ddots & \\ (0) & & \lambda_n^k \end{pmatrix}$.

Therefore: $$e^{tD} = \sum_{k\geq 0}\frac{(tD)^k}{k!} = \sum_{k\geq 0}\frac{t^k}{k!}\begin{pmatrix} \lambda_1^k & & (0) \\ & \ddots & \\ (0) & & \lambda_n^k \end{pmatrix}$$

(by linearity) $$= \begin{pmatrix} \sum_{k\geq 0}\frac{t^k\lambda_1^k}{k!} & & (0) \\ & \ddots & \\ (0) & & \sum_{k\geq 0}\frac{t^k\lambda_n^k}{k!} \end{pmatrix}$$

$$\boxed{e^{tD} = \begin{pmatrix} e^{t\lambda_1} & & (0) \\ & \ddots & \\ (0) & & e^{t\lambda_n} \end{pmatrix}}$$

We computed the exponential of a matrix, yeah! ☺

And now: $\vec{x}(t) = Pe^{tD}P^{-1}\vec{x}(0)$ and $P = (\vec{v}_1 \dots \vec{v}_n)$

$$= (\vec{v}_1 \dots \vec{v}_n)\begin{pmatrix} e^{t\lambda_1} & & (0) \\ & \ddots & \\ (0) & & e^{t\lambda_n} \end{pmatrix}P^{-1}\vec{x}(0)$$

(it is almost over!)

Now we can remark that:

* $Pe^{tD} = (\vec{v_1} \dots \vec{v_n}) \begin{pmatrix} e^{\lambda_1 t} & & (0) \\ & \ddots & \\ (0) & & e^{\lambda_n t} \end{pmatrix} = \left(e^{\lambda_1 t}\vec{v_1} \dots e^{\lambda_n t}\vec{v_n} \right)$.

* $P^{-1}\vec{x}(0) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is a constant vector.

Therefore, finally:

$$\vec{X} = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2} + \dots + c_n e^{\lambda_n t}$$ is

the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ when A is diagonalizable, where:

- $\vec{v_1} \dots \vec{v_n}$ are n independent eigenvectors of A
- $\lambda_1 \dots \lambda_n$ are the n corresponding eigenvalues
- $c_1, \dots, c_n$ are constant.