Homogeneous linear systems with constant coefficients and eigenvectors

We consider the first order homogeneous linear system with constant coefficients:

\[ \frac{dx}{dt} = A \cdot x, \quad \text{where} \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}. \]

\[ \text{if} \; n=1 \]

We have just one equation of the form \( y' = ay \).

Using that this equation is separable or using an integrating factor, we prove that the solution is:

\[ y(t) = y(0) \cdot e^{at}. \]

\[ \text{if} \; n \geq 2 \]

If we want to apply the same method, we need to compute \( e^{tA} \), but \( tA \) is a matrix!

We define the exponential of a matrix by:

\[ e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \]

With this definition, we can check that

\[ e^{tA}' = \frac{d}{dt} \left( \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{(tA)^{k+1}}{k!} = \sum_{k=0}^{\infty} \frac{t^{k+1}}{k!} A^k \]

by the change of variable \( j = k + 1 \):

\[ (e^{tA})' = \left( \sum_{j=1}^{\infty} \frac{t^j}{j!} A^j \right) A = e^{tA} \cdot A. \]

Therefore: The solution of this equation \( \frac{dx}{dt} = A \cdot x \)

is \( x(t) = e^{tA} \cdot x(0) \).

\[ \text{Problem: How do we compute } e^{tA}? \]

(We the formula, we need to compute \( A^k, \forall k \geq 0 \).

\[ \text{too hot} \]

for one matrix

\[ \text{except if:} \]
Solution: it's easy to compute \( A^k \) if \( A \) is diagonal.

**DIAGONALIZATION**

\( A \in \mathbb{R}^{n \times n} \), if we have \( n \) independent eigenvectors \( \vec{v}_1, \ldots, \vec{v}_n \) associated to the eigenvalues \( \lambda_1, \ldots, \lambda_n \), then

for \( P = (\vec{v}_1, \ldots, \vec{v}_n) \) and the diagonal matrix \( D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \),

we have:

\[ A = PDP^{-1} \]

Let us prove then that the solution is thus:

\[ \vec{x}(t) = e^{tA} \vec{x}(0) = Pe^{tD}P^{-1} \vec{x}(0) \]

(1)

To prove this, we have two methods:

* Method 1

We introduce the change of variables \( \vec{\varphi} = P^{-1} \vec{x} \)

(For fact: we don't have to compute \( P^{-1} \))

Then:

\[ \frac{d\vec{\varphi}}{dt} = \frac{d}{dt} \left( P^{-1} \vec{x} \right) \]

\[ = P^{-1} \frac{d}{dt} \vec{x} \]

\[ = P^{-1} A \vec{x} \]

\[ = P^{-1}PDP^{-1} \vec{x} \]

\[ = D \vec{\varphi} \]

Therefore the solution is given by \( \vec{\varphi}(t) = e^{tD} \vec{\varphi}(0) \)

and using that \( \vec{x}(t) = P \vec{\varphi}(t) \) (and \( \vec{\varphi}(0) = P^{-1} \vec{x}(0) \))

we obtain \( \vec{x}(t) = P \vec{\varphi}(t) = Pe^{tD}P^{-1} \vec{x}(0) \)
By definition and using the diagonalization:

\[ e^{HA} = e^{p(HT)p^{-1}} = \sum_{k=0}^{\infty} \frac{(p(HT)p^{-1})^k}{k!} \]

But \( (p(HT)p^{-1})^k = (p(HT)p^{-1})^{\frac{k}{k}} \) (k times)

\[ = p(HT)^k p^{-1} \]

Therefore:

\[ e^{HA} = \sum_{k=0}^{\infty} p \left( \frac{(HT)^k}{k!} \right) p^{-1} = \sum_{k=0}^{\infty} \frac{(HT)^k}{k!} = p e^{TD} p^{-1} \]

and

\[ x(t) = e^{HA} x(0) = p e^{TD} p^{-1} x(0) \]

Now we have to compute \( e^{TD} \)

\( D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix} \) is diagonal, thus

\[ D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_n^k \end{pmatrix} \]

Therefore:

\[ e^{TD} = \sum_{k=0}^{\infty} \frac{(TD)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_n^k \end{pmatrix} \]

(by linearity)

\[ e^{TD} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \end{pmatrix} \]

We computed the exponential of a matrix, yeah! 😊

And now:

\[ x(t) = p e^{TD} p^{-1} x(0) \]

\[ = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \]

(it is almost over!)
Now we can remark that:
\[ \text{Pe}^{t D} = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} e^{dt} & 0 & \cdots & 0 \\ 0 & e^{dt} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{dt} \end{pmatrix} = \begin{pmatrix} e^{dt} v_1 & \cdots & e^{dt} v_n \end{pmatrix}. \]

Moreover,
\[ \text{P}^{-1} \dot{x}(0) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \]
is a constant vector.

Therefore, finally:
\[ \dot{x} = c_1 e^{dt} v_1 + c_2 e^{dt} v_2 + \cdots + c_n e^{dt} v_n \]
is the general solution of \( \frac{dx}{dt} = Ax \) when \( A \)
is diagonalizable, where:
- \( v_1, \ldots, v_n \) are \( n \) independent eigenvectors of \( A \);
- \( \lambda_1, \ldots, \lambda_n \) are the \( n \) corresponding eigenvalues;
- \( c_1, \ldots, c_n \) are constants.