

3.2: Wronskian and linear independence

Motivation: $\vec{x}' = A \vec{x}$, $A \in \mathbb{R}^{2 \times 2}$

λ_1, λ_2 are the eigenvalues, \vec{v}_1, \vec{v}_2 the eigenvectors.

→ general solution should be (see 3.2):

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2.$$

Can we solve an IVP?

IF $\vec{x}(t_0) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ (initial data)

$$\vec{x}(t_0) = c_1 e^{\lambda_1 t_0} \vec{v}_1 + c_2 e^{\lambda_2 t_0} \vec{v}_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

IF $\vec{v}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$

$$\vec{x}(t_0) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t_0} x_{11} & c_2 e^{\lambda_2 t_0} x_{12} \\ c_1 e^{\lambda_1 t_0} x_{21} & c_2 e^{\lambda_2 t_0} x_{22} \end{pmatrix}$$

$$A \rightarrow \begin{pmatrix} e^{\lambda_1 t_0} x_{11} & e^{\lambda_2 t_0} x_{12} \\ e^{\lambda_1 t_0} x_{21} & e^{\lambda_2 t_0} x_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

is this matrix invertible?

$$|A| = e^{(\lambda_1 + \lambda_2)t_0} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = \underbrace{e^{(\lambda_1 + \lambda_2)t_0}}_{\text{always positive}} \underbrace{\begin{vmatrix} \vec{v}_1 & \vec{v}_2 \end{vmatrix}}_{\neq 0?}$$

↓
ok if $\lambda_1 \neq \lambda_2$

• Definition

Wronskian

Given two functions $\vec{x}_1(t) = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$, $\vec{x}_2(t) = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$

The determinant

$$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 \end{vmatrix} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

is named the Wronskian of $\vec{x}_1(t)$ and $\vec{x}_2(t)$

• Example:

$$x_1(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_2(t) = e^{2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} e^t & 2e^{2t} \\ e^t & 3e^{2t} \end{vmatrix} = 3e^{3t} - 2e^{3t} = e^{3t}$$

• Definition

Linearly independent

$\vec{x}_1(t)$ and $\vec{x}_2(t)$ are linearly independent ~~on~~
on interval of time if

$W[\vec{x}_1, \vec{x}_2](t) \neq 0$ everywhere on this interval