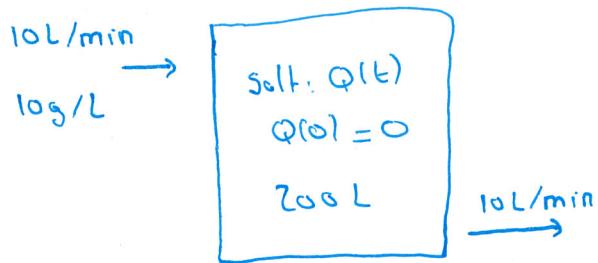


Quiz 1, D1, D2

1)



$$\frac{dQ}{dt} = (\text{rate of salt in}) - (\text{rate of salt out}) = 10 \times 10 - \frac{10 Q(t)}{200}$$

$$\Rightarrow \begin{cases} \frac{dQ(t)}{dt} = 100 - \frac{1}{20} Q(t) \\ Q(0) = 0 \end{cases}$$

This is a first order linear and separable differential equation.

What should we expect for $Q(t)$ as t goes to infinity?

method 1: the concentration of salt should be the same as the one of the mixture coming in: 10 g/L .

Therefore, we should have $Q(t) \approx 10 \cdot 200 = 2000 \text{ g}$.

method 2: ~~the concentration of salt should converge to a constant~~

The quantity $Q(t)$ should converge to a constant

$$\Rightarrow \frac{dQ(t)}{dt} = 0 \Rightarrow 100 - \frac{1}{20} Q(t) = 0 \Rightarrow Q(t) = 2000 \text{ g}$$

$$2) t > 0 \quad \begin{cases} y' + \frac{1}{t} y = e^{t^2} \\ y(1) = e^1 \end{cases}$$

This is a first order linear differential equation (not separable).

Integrating factor: $\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$.

$$\Rightarrow (ty(t))' = te^{t^2}$$

$$\Rightarrow t(y(t)) = \int te^{t^2} dt = \frac{1}{2} e^{t^2} + C$$

By integration:

$$\therefore y(t) = \frac{1}{t} \left(\frac{1}{2} e^{t^2} + C \right)$$

$$y(1) = 1 \Rightarrow 1 = \frac{1}{2} e^1 + C$$

$$\boxed{C = \frac{1}{2} e^1}$$

Finally: $y(t) = \frac{1}{2t} (e^{t^2} + e^t)$

3) $\left\{ \begin{array}{l} \frac{dy}{dt} = \sin t \cdot (y)(3-y) \\ y(\frac{s\pi}{2}) = 1 \end{array} \right.$

This is a first order nonlinear separable differential equation.
 If $y \neq 0$ and $y \neq 3$ we have: ($y=0$ and $y=3$ are two constant solutions)

$$\frac{dy}{y(3-y)} = \sin t dt$$

$$\text{But } \frac{1}{y(3-y)} = \frac{A}{y} + \frac{B}{3-y} = \frac{(B-A)y + 3A}{(3-y)y} \xrightarrow{\substack{A+B=0 \\ A=V3}} \frac{1}{y(3-y)} = \frac{1}{3} \left(\frac{1}{y} + \frac{1}{3-y} \right)$$

$$\left(\frac{1}{y} + \frac{1}{3-y} \right) dy = 3 \sin t dt$$

$$\ln |y| - \ln |3-y| = -3 \cos t + C$$

By integration: $\ln \left| \frac{y}{3-y} \right| = -3 \cos t + C$
 $\left| \frac{y}{3-y} \right| = C e^{-3 \cos t}$

Here: $y(\frac{s\pi}{2}) = 1 \in (0, 3)$, therefore: $0 < y(t) < 3, \forall t$.

$$\Rightarrow |y| = y \text{ and } |3-y| = 3-y$$

$$\Rightarrow \frac{y}{3-y} = C e^{-3 \cos t}$$

$$y = 3C e^{-3 \cos t} - 4C e^{-3 \cos t}$$

$$\therefore y(t) = \frac{3C e^{-3 \cos t}}{1 + C e^{-3 \cos t}}$$

$$y(\frac{s\pi}{2}) = 1 \Rightarrow 1 = \frac{3C}{1+C} \Rightarrow 1+C = 3C \therefore C = \sqrt{2}$$

Finally: $y(t) = \frac{\sqrt{2} e^{-3 \cos t}}{1 + \sqrt{2} e^{-3 \cos t}} = \frac{3}{2 e^{3 \cos t} + 1}, y \text{ is defined everywhere}$